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The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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A CLASS OF MEAN VALUE FUNCTIONS*

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and National Bureau of Standards

1. Introduction. The two problems discussed in the note by G. Pólya in this issue of the MONTHLY suggest the following two problems, of which the first at least is well known, concerning a given finite set of positive values $(a) \equiv (a_1, a_2, \cdots, a_n)$.

PROBLEM I. *Find the value x which minimizes the sum of the squares of the errors,* $\sum_{j=1}^{n}(a_j - x)^2$.

PROBLEM II. *Find the value y which minimizes the sum of the squares of the relative errors,* $\sum_{j=1}^{n}[(a_j - y)/y]^2$.

Solutions are readily obtained by considering first and second derivatives. The solutions are $x = \sum_{j=1}^{n} a_j/n$, which is the arithmetic mean of (a), and

$$y = \sum_{j=1}^{n} a_j^2 \Big/ \sum_{k=1}^{n} a_k, \tag{1}$$

which is a weighted arithmetic mean of (a) with each a_j as its own weight.

We shall investigate (1) as an instance of a continuum of mean value functions, and shall compare these means with the standard means $\mathfrak{M}_t(a)$.

2. The mean of order t. For a set of positive values $(a) \equiv (a_1, a_2, \cdots, a_n)$, the mean of order t, $-\infty \leqq t \leqq +\infty$, is defined [4, p. 54; 3, p. 12] by

$$\mathfrak{M}_{-\infty}(a) = \min\,(a), \qquad \mathfrak{M}_{+\infty}(a) = \max\,(a), \qquad \mathfrak{M}_0(a) = \Big(\prod_{j=1}^{n} a_j\Big)^{1/n};$$

$$\mathfrak{M}_t(a) = \Big(\sum_{j=1}^{n} a_j^t/n\Big)^{1/t}, \quad \text{otherwise.}$$

The harmonic, geometric, and arithmetic means are the special cases $t = -1, 0, +1$, respectively.

We have $\mathfrak{M}_{-t}(a) = 1/\mathfrak{M}_t(1/a)$, and, for positive constants k, $\mathfrak{M}_t(ka) = k\mathfrak{M}_t(a)$.

As is well known, $\mathfrak{M}_t(a)$ is a continuous strictly increasing function of t, $-\infty \leqq t \leqq +\infty$, unless all the a_j are equal; $\mathfrak{M}_t(a)$ is a continuous increasing function of a_k, $k = 1, 2, \cdots, n$; and $\mathfrak{M}_t(a)$ satisfies Minkowski's inequality,

$$\mathfrak{M}_t(a + b) \leqq \mathfrak{M}_t(a) + \mathfrak{M}_t(b), \qquad t \geqq 1, \tag{2}$$

the sign of equality holding for $+\infty > t > 1$ if and only if (a) and (b) are proportional; the inequality sign in (2)is reversed for $t \leqq 1$.

3. Definition. For a set of positive values $(a) \equiv (a_1, a_2, \cdots, a_n)$, we define

* The preparation of this paper was sponsored in part by the Office of Naval Research.

$\mathfrak{N}_t(a)$, $-\infty \leqq t \leqq +\infty$, by

$$\mathfrak{N}_{-\infty}(a) = \min(a), \qquad \mathfrak{N}_{+\infty}(a) = \max(a);$$

$$\mathfrak{N}_t(a) = \sum_{j=1}^{n} a_j^t \Big/ \sum_{k=1}^{n} a_k^{t-1}, \text{ otherwise.}$$

Then $\mathfrak{N}_t(a)$ is a weighted arithmetic mean of (a), with a_j^{t-1} as weight for a_j, $j=1, 2, \cdots, n$.

The function (1) is the special case $t=2$. The harmonic and arithmetic means are the cases $t=0$, 1, respectively. For $n=2$, the geometric mean is the case $t=1/2$.

We have $\mathfrak{N}_{-t}(a)=1/\mathfrak{N}_{t+1}(1/a)$, and, for positive constants k, $\mathfrak{N}_t(ka)=k\mathfrak{N}_t(a)$.

In the theory of asymptotic values it is shown [4, p. 78; 3, p. 62] that

$$\lim_{t\to-\infty} \mathfrak{N}_t(a) = \mathfrak{N}_{-\infty}(a), \qquad \lim_{t\to+\infty} \mathfrak{N}_t(a) = \mathfrak{N}_{+\infty}(a),$$

so that $\mathfrak{N}_t(a)$ is a continuous function of t, $-\infty \leqq t \leqq +\infty$; and clearly $\mathfrak{N}_t(a)$ is a continuous function of the a_k.

4. Comparison of $\mathfrak{N}_t(a)$ and $\mathfrak{M}_t(a)$. For $-\infty < t < +\infty$ we have

$$\mathfrak{N}_t(a) = \frac{[\mathfrak{M}_t(a)]^t}{[\mathfrak{M}_{t-1}(a)]^{t-1}} = \left[\frac{\mathfrak{M}_t(a)}{\mathfrak{M}_{t-1}(a)}\right]^{t-1} \mathfrak{M}_t(a),$$

whence it follows that $\mathfrak{N}_t(a) \geqq \mathfrak{M}_t(a)$ *for* $1<t<+\infty$, *and* $\mathfrak{N}_t(a) \leqq \mathfrak{M}_t(a)$ *for* $-\infty<t<1$, *the signs of equality holding if and only if all the* a_j *are equal. At* $t=-\infty$, 1, $+\infty$, *we have* $\mathfrak{N}_t(a)=\mathfrak{M}_t(a)$.

For negative values $-t$ *we have the stronger result that*

$$\mathfrak{N}_{-t}(a) = 1/\mathfrak{N}_{t+1}(1/a) \leqq 1/\mathfrak{M}_{t+1}(1/a) = \mathfrak{M}_{-t-1}(a).$$

5. $\mathfrak{N}_t(a)$ as function of t. By means of Cauchy's inequality [4, p. 54; 3, p. 16] it has been shown [3, p. 243] that for positive integers p we have $\mathfrak{N}_p(a) \leqq \mathfrak{N}_{p+1}(a)$, so that $\mathfrak{N}_p(a)$ is a non-decreasing function of the integer p. We now extend this result as follows.

THEOREM 1. *The function* $\mathfrak{N}_t(a)$ *is a non-decreasing function of* t, $-\infty \leqq t \leqq +\infty$, *and is strictly increasing unless all the* a_j *are equal.*

Proof. We have

$$\left(\sum_{j=1}^{n} a_j^{t-1}\right)^2 d\mathfrak{N}_t(a)/dt = \sum_{j=1}^{n} a_j^{t-1} \sum_{k=1}^{n} a_k^t \log a_k - \sum_{j=1}^{n} a_j^t \sum_{k=1}^{n} a_k^{t-1} \log a_k$$

$$= \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} a_j^{t-1} a_k^{t-1} (a_j - a_k)(\log a_j - \log a_k),$$

which is positive unless each term vanishes.

An alternative proof, suggested by G. Pólya, is as follows. By Cauchy's inequality we have $d^2 \log (\sum_{j=1}^n a_j^t)/dt^2 \geqq 0$, the sign of equality holding if and only if all the a_j are equal. Hence $\log (\sum_{j=1}^n a_j^t)$ is a convex function of t and therefore, for $t_2 > t_1$, we have

$$\log\left(\sum_{j=1}^n a_j^{t_2}\right) - \log\left(\sum_{j=1}^n a_j^{t_2-1}\right) \geqq \log\left(\sum_{j=1}^n a_j^{t_1}\right) - \log\left(\sum_{j=1}^n a_j^{t_1-1}\right).$$

6. $\mathfrak{M}_t(a)$ as function of the a_k. We have

$$\left(\sum_{j=1}^n a_j^{t-1}\right)^2 \partial \mathfrak{M}_t(a)/\partial a_k = a_k^{t-2} \sum_{j=1}^n a_j^{t-1}[t a_k - (t-1)\mathfrak{M}_t(a)]. \tag{3}$$

Hence *for* $0 \leqq t \leqq 1$, $\mathfrak{M}_t(a)$ *is a strictly increasing function of* a_k. Further, while precise conditions for each t can be obtained from (3), $\mathfrak{M}_t(a)$ is an increasing function of a_k for $t > 1$ provided $a_k > \mathfrak{M}_t(a)$, and for $t < 0$ provided $a_k < \mathfrak{M}_t(a)$.

7. Minkowski's inequality. We shall establish the following result.

THEOREM 2. *Let* $(a) \equiv (a_1, a_2, \cdots, a_n)$ *and* $(b) \equiv (b_1, b_2, \cdots, b_n)$ *be sets of positive values. Then*

$$\mathfrak{M}_t(a+b) \leqq \mathfrak{M}_t(a) + \mathfrak{M}_t(b), \qquad 1 \leqq t \leqq 2, \tag{4}$$

and

$$\mathfrak{M}_t(a+b) \geqq \mathfrak{M}_t(a) + \mathfrak{M}_t(b), \qquad 0 \leqq t \leqq 1, \tag{5}$$

the signs of equality holding if and only if $t = 1$ *or* (a) *is proportional to* (b).

Proof. For the function

$$\phi(\lambda) \equiv \mathfrak{M}_t[\lambda a + (1-\lambda)b], \qquad 0 \leqq \lambda \leqq 1,$$

we have

$$\begin{aligned}\left(\sum_{j=1}^n A_j^{t-1}\right)^3 d^2\phi/d\lambda^2 = {} & (t-1)\Bigg\{2\Bigg[\sum_{j=1}^n A_j^t \sum_{k=1}^n A_k^{t-1} \sum_{l=1}^n A_l^{t-3} B_l^2 \\ & - \sum_{j=1}^n A_j^t \left(\sum_{k=1}^n A_k^{t-2} B_k\right)^2\Bigg] - t\Bigg[\sum_{j=1}^n A_j^t \sum_{k=1}^n A_k^{t-1} \sum_{l=1}^n A_l^{t-3} B_l^2 \\ & + 2\sum_{j=1}^n A_j^{t-1} \sum_{k=1}^n A_k^{t-2} B_k \sum_{l=1}^n A_l^{t-1} B_l - \left(\sum_{j=1}^n A_j^{t-1}\right)^2 \sum_{k=1}^n A_k^{t-2} B_k^2 \\ & - 2\sum_{j=1}^n A_j^t \left(\sum_{k=1}^n A_k^{t-2} B_k\right)^2\Bigg]\Bigg\} \\ = {} & (t-1)(2P - tQ),\end{aligned} \tag{6}$$

where $A_j=\lambda a_j+(1-\lambda)b_j$, $B_j=a_j-b_j$, and P and Q are the indicated expressions in brackets.

By Cauchy's inequality we have $P\geqq 0$. Further, first by the inequality between the arithmetic and geometric means and then by Cauchy's inequality, we have

$$P-Q=\Big(\sum_{j=1}^{n}A_j^{t-1}\Big)^2\sum_{k=1}^{n}A_k^{t-2}B_k^2+\sum_{j=1}^{n}A_j^{t}\Big(\sum_{k=1}^{n}A_k^{t-2}B_k\Big)^2$$

$$-2\sum_{j=1}^{n}A_j^{t-1}\sum_{k=1}^{n}A_k^{t-2}B_k\sum_{l=1}^{n}A_l^{t-1}B_l$$

$$\geqq 2\sum_{j=1}^{n}A_j^{t-1}\sum_{k=1}^{n}A_k^{t-2}B_k\Big\{\Big[\sum_{l=1}^{n}A_l^{t-2}B_l^2\sum_{m=1}^{n}A_m^{t}\Big]^{1/2}-\sum_{l=1}^{n}A_l^{t-1}B_l\Big\}\geqq 0.$$

Thus for $0\leqq t\leqq 2$ we have $2P-tQ\geqq 0$, whence, by (6), $\phi(\lambda)$ is a convex function of λ for $1\leqq t\leqq 2$, and a concave function of λ for $0\leqq t\leqq 1$. Hence $2\phi(1/2)\leqq\phi(0)+\phi(1)$ for $1\leqq t\leqq 2$, and $2\phi(1/2)\geqq\phi(0)+\phi(1)$ for $0\leqq t\leqq 1$; then (4) and (5) follow.

The conditions for the signs of equality to hold in the inequalities used in the proof reduce to the conditions expressed in the theorem.

An alternative method of proof for the interval $1\leqq t\leqq 2$ was suggested by R. Bellman, who has kindly supplied the details [1]. A third method for this interval, suggested by G. Forsythe, is included in the next section because of its geometric aspects.

8. An alternative proof. By Minkowski's lemma [2, p. 23], if a positive vector function $\|x\|$ satisfies $\|kx\|=k\|x\|$ for all $k>0$, and if the "unit sphere" $\|x\|=1$ is convex, then this function satisfies $\|x+y\|\leqq\|x\|+\|y\|$. For under the hypotheses we have

$$\|x+y\|=[\|x\|+\|y\|]\cdot\left\|\frac{\|x\|}{\|x\|+\|y\|}\,\frac{x}{\|x\|}+\frac{\|y\|}{\|x\|+\|y\|}\,\frac{y}{\|y\|}\right\|\leqq\|x\|+\|y\|.$$

For the function $\|a\|\equiv\mathfrak{N}_t(a)$, $1\leqq t\leqq 2$, let $\|a\|=\|b\|=1$, with $a_j>0$ and $b_j>0, j=1, 2, \cdots, n$. Then a_j^t is a convex function, and a_j^{t-1} a concave function, of a_j, whence, as observed by W. Seidel, for $0\leqq\lambda\leqq 1$ we have the following relationship:

$$\|\lambda a+(1-\lambda)b\|=\frac{\sum_{j=1}^{n}[\lambda a_j+(1-\lambda)b_j]^t}{\sum_{j=1}^{n}[\lambda a_j+(1-\lambda)b_j]^{t-1}}\leqq\frac{\lambda\sum_{j=1}^{n}a_j^t+(1-\lambda)\sum_{j=1}^{n}b_j^t}{\lambda\sum_{j=1}^{n}a_j^{t-1}+(1-\lambda)\sum_{j=1}^{n}b_j^{t-1}}=1,$$

and (4) follows from Minkowski's lemma.

For example, with $t=2$, $n=2$, $x>0$, $y>0$, the "unit sphere" $(x^2+y^2)/(x+y)=1$ is the portion of the circle $(x-1/2)^2+(y-1/2)^2=1/2$ which lies in the first

quadrant; this is convex. But the boundary would not be convex if we included all quadrants: there are *real* sets (a) and (b) for which $\mathfrak{M}_2(|a+b|)>\mathfrak{M}_2(|a|)+\mathfrak{M}_2(|b|)$.

9. Other values of t. Clearly the inequalities (4) and (5) hold for $t=+\infty$ and $t=-\infty$, respectively. But it is easy to show that for any t satisfying $2<t<+\infty$ there is an $\epsilon>0$ such that for $(a)=(1, \epsilon, \cdots, \epsilon)$ and $(b)=(1, \epsilon^2, \cdots, \epsilon^2)$ the inequality (4) does not hold. Similarly for any t satisfying $-\infty<t<0$ there is an $M>0$ such that the inequality (5) does not hold for $(a)=(1, M, \cdots, M)$, $(b)=(1, M^2, \cdots, M^2)$. Nevertheless we have the following result supplied by W. Seidel.

THEOREM 3. *If the components of (b) are all equal, $(b)=(b, b, \cdots, b)$, then*

$$\mathfrak{M}_t(a+b) \leqq \mathfrak{M}_t(a)+\mathfrak{M}_t(b), \qquad 1<t<+\infty, \tag{7}$$

and

$$\mathfrak{M}_t(a+b) \geqq \mathfrak{M}_t(a)+\mathfrak{M}_t(b), \qquad -\infty<t<1, \tag{8}$$

the signs of equality holding if and only if all components of (a) are equal.

Proof. For the function $f(x)=\sum_{j=1}^n(a_j+x)^t/\sum_{k=1}^n(a_k+x)^{t-1}$, we have

$$\frac{df}{dx}=\frac{t\left[\sum_{j=1}^n(a_j+x)^{t-1}\right]^2-(t-1)\sum_{j=1}^n(a_j+x)^t\sum_{k=1}^n(a_k+x)^{t-2}}{\left[\sum_{j=1}^n(a_j+x)^{t-1}\right]^2}$$

It follows from Cauchy's inequality that for $t>1$ we have $df/dx\leqq t-(t-1)=1$, whence $f(b)\leqq f(0)+b$, which is (7). The sign of equality holds if and only if the sets $(a_j+b)^{t/2}$ and $(a_j+b)^{t/2-1}$ are proportional, that is, if and only if all the a_j are equal. We obtain (8) similarly.

10. Generalizations. The above results hold also for weighted means,

$$\mathfrak{M}_t(a, p)\equiv\frac{\sum_{j=1}^n p_j a_j^t}{\sum_{k=1}^n p_k a_k^{t-1}}, \qquad p_j>0,$$

for simply or doubly infinite sequences $(a)\equiv(a_1, a_2, \cdots)$ or $(a)\equiv(\cdots a_{-1}, a_0, a_1, \cdots)$, and for integral means

$$\mathfrak{M}_t(f, p) \equiv \frac{\int_a^b p(x)[f(x)]^t dt}{\int_a^b p(x)[f(x)]^{t-1} dt}, \qquad p(x) > 0.$$

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1. Richard Bellman, Some elementary inequalities, U. S. Airforce Project Rand, D-484, The Rand Corporation, Santa Monica ,Calif.
2. T. Bonnesen und W. Fenchel, Theorie der Konvexen Körper, Springer, Berlin, 1934.
3. G. H. Hardy, J. E. Littlewood, G. Pólya, Inequalities, Cambridge, England, 1934.
4. G. Pólya und G. Szegö, Aufgaben und Lehrsätze aus der Analysis, vol. I, Springer, Berlin, 1925.

ERRATA

The following errata in recent volumes have been called to the attention of the editors:

Edmund Pinney, *Vibration Modes of Tapered Beams*, vol. 54.

p. 393, formula (10), change sign of both μ's.

p. 393, formula (13), change sign of μ's in $\beta\nu - \alpha$ column.

p. 393, formula (15), replace $x^{\mu/2}$ by $x^{-\mu/2}$.

p. 394, formula (17), replace left member of equation by

$$xF_{\mu+1}(A, B, C, D; x) + F_{\mu-1}(A, B, -C, -D; x).$$

P. M. Bailey, abstract of the paper *A general finite difference summation formula*, vol. 55.

p. 658, replace the displayed equations by

$$\sum_{x=\alpha}^{\alpha+(n-1)r}{}_r\ \phi(x) = \sum_{x=0}^{(h/r)-1}{}_1\ \Delta_n f(\alpha + xr)$$

$$\sum_r\ \phi(x) = \sum_{k=0}^{(h/r)-1}{}_1\ f(x + kr) + C.$$

Victor Thebault, *Concerning Two Classes of Remarkable Perfect Square Pairs*, vol. 56.

p. 444, (ii), replace $x^2 = 2\ \overline{46}\ 4\ \overline{46}$ by $x^2 = 4\ \overline{46}\ 4\ \overline{46}$.

Problem 4224 [1946, 537], vol. 56.

p. 348. line 17, replace the summation limits $j=1$ to n by $j=0$ to $c-1$.

JAMES HUME AND EXPONENTS

C. B. BOYER, Brooklyn College

A widely-used work of reference, Cajori's *History of Mathematical Notations*, reports that "Hume took an important step in his edition of *L'algèbre de Viète* (Paris, 1636), in which he wrote A^{iii} for A^3. Except for the use of Roman numerals one has here the notation used by Descartes in 1637 in his *La géométrie.*" [1] Further on Cajori makes this same assertion more explicitly: "In 1636 James Hume brought out an edition of the algebra of Vieta, in which he introduced a superior notation, writing down the base and elevating the exponent to a position above the regular line and a little to the right . . . Thus, this Scotsman, residing in Paris, had almost hit upon the exponential symbolism which has become universal through the writings of Descartes . . . Hérigone and Hume almost hit upon the scheme of Descartes. The only difference was, in one case, the position of the exponent, and, in the other, the exponent written in Roman numerals. Descartes expressed the exponent in Arabic numerals and assigned it an elevated position. Where Hume would write $5a^{iv}$ and Hérigone would write 5a4, Descartes wrote $5a^4$." [2] Years after reading these sentences, I had occasion to refer to the work of Hume in question and was struck by the absence of elevated exponents. It would appear that Cajori had been misled by a passage from the *Oeuvres de Descartes* [3] in which the use of "exposants marqués en chiffres romains" is traced to Hume's *algèbre de Viète.* Wieleitner, too, jumped to the erroneous conclusion that Hume's exponents were elevated, for he refers to the same treatise of 1636 as one in which "römische Ziffern als wirkliche Exponenten benutzt wurden, z. B. in der Gleichung A^{III}—AAH égal à X." [4] Hume did indeed use Roman numerals as indices of powers. He wrote that "Aij signifie que la grandeur Aij est quarré, ou bien le quarré de la grandeur Aj"; in the same way "Aiij signifie le cube de la grandeur Aj"; and "Ajv signifie le quarré de quarré de Aj"; and similarly for higher powers [5]. This notation is used throughout, so that the equation which now would be written as $x^4-a^3x=b^4$ appears [6] in the form Ajv−Diij en Aj est egal à Zjv. Occasional unevenness of the type results in the placing of some of the exponents (*logarymes*) slightly above or below the letter to which they apply; but there obviously is no intentional elevation as in the case of Cartesian exponents in *La géométrie.* Until new evidence is presented, it appears that Descartes, not Hume, was the first mathematician to adopt exponents in the modern sense, namely, as "a symbol written above another symbol and on the right."

References

1. Florian Cajori, *A history of mathematical notations* (2 vols., Chicago, 1928–1929), I, 204. In my paper on "Fractional indices, exponents, and powers," *National Mathematics Magazine*, XVIII (1943), 81–86, the error of Cajori unfortunately is repeated. See also V. B. Bly, "A history of the exponent," *Bulletin of the Kansas Association of Teachers of Mathematics*, XXIII (1949), 20–23.

2. Cajori, *op. cit.*, I, 345–346.

3. Ed. by Charles Adam and Paul Tannery (12 vols. and supplement, Paris, 1897–1913), V, 504–512. See especially page 504.

4. Heinrich Wieleitner, *Geschichte der Mathematik* (2 vols., Leipzig, 1908–1921), vol. II, part 1, page 4.

5. *Algèbre de Viète, d'une methode nouvelle, claire, et facile* (Paris, 1636), page 3.

6. *Ibid.*, page 550.

A THEOREM ON SETS OF COPRIME INTEGERS

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1. Introduction. Let n, s, r be integers such that $n \geqq 1$, $2 \leqq r \leqq s$, and denote by $N_{s,r}(n)$ the number of representations (order being regarded as relevant) of n as the sum of s positive integers such that the highest common factor of any r of them is equal to 1. The object of this note is to investigate the asymptotic behavior of $N_{s,r}(n)$ when s, r are fixed and $n \to \infty$.

It will be convenient to write

$$\sum \mathfrak{C} \mid f(m, n, \cdots)$$

in place of

$$\sum_{\mathfrak{C}} f(m, n, \cdots);$$

here $\mathfrak{C}$ stands for the set of conditions which define the range of summation. The highest common factor of $n_1, \cdots, n_s$ will be denoted by $(n_1, \cdots, n_s)$.

The case $r = s$ is dealt with very easily. Writing

$$N_{s,s}(n; d) = \sum \begin{matrix} n = n_1 + \cdots + n_s \\ (n_1, \cdots, n_s) = d \end{matrix} \mid 1,$$

so that, in particular, $N_{s,s}(n; 1) = N_{s,s}(n)$, we have

$$N_{s,s}(n; d) = \sum \begin{matrix} n/d = \nu_1 + \cdots + \nu_s \\ (\nu_1, \cdots, \nu_s) = 1 \end{matrix} \mid 1 = N_{s,s}\left(\frac{n}{d}\right).$$

Hence

$$\sum_{d \mid n} N_{s,s}\left(\frac{n}{d}\right) = \sum n = n_1 + \cdots + n_s \mid 1 = M_s(n),$$

where

$$M_s(n) = \frac{(n-1)(n-2)\cdots(n-s+1)}{(s-1)!} = \sum_{0\leqq i\leqq s-1} A_i n^{s-1-i}.$$

Therefore, by Möbius's inversion formula,

$$\begin{aligned} N_{s,s}(n) &= \sum_{d\mid n} \mu(d) M_s\left(\frac{n}{d}\right) = \sum_{d\mid n} \mu(d) \sum_{0\leqq i\leqq s-1} A_i \left(\frac{n}{d}\right)^{s-1-i} \\ &= \sum_{0\leqq i\leqq s-1} A_i n^{s-1-i} \sum_{d\mid n} \frac{\mu(d)}{d^{s-1-i}} \\ &= \sum_{0\leqq i\leqq s-1} A_i n^{s-1-i} \prod_{p\mid n}\left(1 - \frac{1}{p^{s-1-i}}\right) \\ &= \frac{n^{s-1}}{(s-1)!} \prod_{p\mid n}\left(1 - \frac{1}{p^{s-1}}\right) + O(n^{s-2}). \end{aligned}$$

It is also easy to see that for $s=2$, $n>1$, the error term in this formula is identically equal to zero.

The case $2\leqq r<s$ is more interesting. We shall write

$$a_k^{(s,r)} = (-1)^{r+k-1}\binom{s}{r+k} \sum_{0\leqq i\leqq r-1} (-1)^i \binom{r+k}{i} \qquad (0 \leqq k \leqq s-r),$$

$$\mathrm{X}_{s,r}(p) = a_0^{(s,r)} p^{s-r-1} + a_1^{(s,r)} p^{s-r-2} + \cdots + a_{s-r-1}^{(s,r)},$$

$$\mathfrak{S}_{s,r}(n) = \prod_{p\mid n}\left(1 - \frac{\mathrm{X}_{s,r}(p)}{p^{s-1}}\right) \prod_{p\mid n}\left(1 - \frac{\mathrm{X}_{s,r}(p) + a_{s-r}^{(s,r)}}{p^{s-1}}\right).$$

The following result will be proved.

THEOREM. *Let $2\leqq r<s$. Then, as $n\to\infty$,*

$$N_{s,r}(n) = \frac{n^{s-1}}{(s-1)!}\mathfrak{S}_{s,r}(n) + O\left(\frac{n^{s-1}}{\log^{r-1} n \log\log n}\right),$$

where the O-constant depends at most on s.

Furthermore, the true order of magnitude of $N_{s,r}(n)$ is n^{s-1}.

2. Notation. The letter x denotes a positive number; all other small letters denote integers, which are positive unless the contrary is stated. The letter p is reserved for primes. As usual $\pi(x)$ denotes the number of primes not exceeding x.

The O-notation refers to the passage $n\to\infty$ unless otherwise stated, and the O-constants depend at most on s.

We shall frequently write $a\equiv b(\cdot m)$ as an abbreviation for $a\equiv b \pmod m$.

If $a_1, \cdots, a_s$ are non-negative integers, then $[a_1, \cdots, a_s]$ denotes the number of zeros among them.

For $s\geqq 1$, $n\geqq 1$, any p, we write

$$\tau_s(n, p) = \sum_{\substack{0 < \lambda_1, \cdots, \lambda_s < p \\ \lambda_1 + \cdots + \lambda_s \equiv n(\cdot p)}} \Bigg| 1.$$

For $2 \leqq r < s$, $n \geqq 1$, any p, we write

$$t_{s,r}(n, p) = \sum_{\substack{0 \leqq \lambda_1, \cdots, \lambda_s < p \\ \lambda_1 + \cdots + \lambda_s \equiv n(\cdot p); \quad [\lambda_1, \cdots, \lambda_s] < r}} \Bigg| 1.$$

3. Lemmas.

LEMMA 1.

$$\tau_s(n, p) = \begin{cases} \dfrac{(p-1)^s - (-1)^s}{p} & \text{if } p \nmid n, \\ \dfrac{(p-1)^s + (-1)^s(p-1)}{p} & \text{if } p \mid n. \end{cases}$$

Proof.

$$\tau_1(n, p) = \sum_{\substack{0 < \lambda < p \\ \lambda \equiv n(\cdot p)}} \Bigg| 1 = \begin{cases} 1 & \text{if } p \nmid n, \\ 0 & \text{if } p \mid n. \end{cases}$$

Hence the lemma is true for $s = 1$. Assume next that it is true for some $s \geqq 1$. Then

$$\begin{aligned} \tau_{s+1}(n, p) &= \sum_{0<\lambda<p} \sum_{\substack{0 < \lambda_1, \cdots, \lambda_s < p \\ \lambda_1 + \cdots + \lambda_s \equiv n - \lambda(\cdot p)}} \Bigg| 1 \\ &= \sum_{0<\lambda<p} \tau_s(n - \lambda, p). \end{aligned}$$

If $p \nmid n$, then

$$\left.\begin{array}{l} p \nmid (n - \lambda) \text{ for } p - 2 \text{ values} \\ p \mid (n - \lambda) \text{ for one value} \end{array}\right\} \text{of } \lambda \text{ in the range } 0 < \lambda < p.$$

Hence

$$\begin{aligned} \tau_{s+1}(n, p) &= (p-2)\frac{(p-1)^s - (-1)^s}{p} + \frac{(p-1)^s + (-1)^s(p-1)}{p} \\ &= \frac{(p-1)^{s+1} - (-1)^{s+1}}{p}. \end{aligned}$$

If $p \mid n$, then $p \nmid (n - \lambda)$ for $0 < \lambda < p$, and so

$$\tau_{s+1}(n, p) = (p-1)\frac{(p-1)^s - (-1)^s}{p} = \frac{(p-1)^{s+1} + (-1)^{s+1}(p-1)}{p}.$$

This completes the proof of Lemma 1.

LEMMA 2.

$$\frac{t_{s,r}(n,p)}{p^{s-1}} = \begin{cases} 1 - \dfrac{X_{s,r}(p)}{p^{s-1}} & \text{if } p \nmid n, \\ 1 - \dfrac{X_{s,r}(p) + a_{s-r}^{(s,r)}}{p^{s-1}} & \text{if } p \mid n. \end{cases}$$

Proof. We have

$$t_{s,r}(n,p) = \sum_{0 \le i \le r-1} \sum_{\substack{0 \le \lambda_1, \cdots, \lambda_s < p; \quad [\lambda_1, \cdots, \lambda_s] = i \\ \lambda_1 + \cdots + \lambda_s \equiv n (\cdot p)}} 1$$

$$= \sum_{0 \le i \le r-1} \binom{s}{i} \tau_{s-i}(n,p), \tag{1}$$

and the required result now follows by a straightforward calculation on making use of Lemma 1.

LEMMA 3. *There exist positive constants C_1, C_2 (depending at most on s) such that, for $2 \le r < s$, $n \ge 1$,*

$$C_1 < \mathfrak{S}_{s,r}(n) < C_2.$$

Proof. By Lemma 1 we see that $\tau_s(n,p) \ge 0$ always, and $\tau_s(n,p) > 0$ unless $p=2$, $2 \mid n$, $2 \nmid s$ or $p=2$, $2 \nmid n$, $2 \mid s$. Hence, for $2 \le r < s$, $n \ge 1$, we have, using (1),

$$t_{s,r}(n,p) \ge \tau_s(n,p) + s\tau_{s-1}(n,p) > 0.$$

Furthermore, by Lemma 2,

$$\frac{t_{s,r}(p)}{p^{s-1}} = 1 + O\left(\frac{1}{p^r}\right) \qquad (p \to \infty), \tag{2}$$

and

$$\mathfrak{S}_{s,r}(n) = \prod_p \frac{t_{s,r}(p)}{p^{s-1}}. \tag{3}$$

The assertion therefore follows.

4. Proof of the theorem. Let x be a function of n, to be fixed later, such that $x \to \infty$, as $n \to \infty$, and

$$s \prod_{p \le x} p < n.$$

Denote by $p_1, \cdots, p_k$ the primes not exceeding x.

Let $N^{(1)} = N^{(1)}(n, s, r)$ be the number of representations of n as the sum of s positive integers, no r of which are divisible by any $p \le x$. Furthermore, let $N^{(2)} = N^{(2)}(n, s, r)$ be the number of representations of n as the sum of s positive

integers, at least r of which are divisible by some $p>x$.

We clearly have

$$0 \leqq N^{(1)} - N_{s,r}(n) \leqq N^{(2)}. \tag{4}$$

Moreover,

$$\begin{aligned} N^{(2)} &\leqq \sum_{p>x} \sum \left| \begin{array}{l} n_1 + \cdots + n_s = n \\ \text{At least } r \text{ among } n_1, \cdots, n_s \text{ are divisible by } p \end{array} \right. 1 \\ &\leqq \binom{s}{r} \sum_{p>x} \sum \left| \begin{array}{l} n_1 + \cdots + n_s = n \\ n_1 \equiv \cdots \equiv n_r \equiv 0 (\cdot p) \end{array} \right. 1 \\ &= O\Big(\sum_{p>x} \sum \nu_1 p + \cdots + \nu_r p + \nu_{r+1} + \cdots + \nu_s = n \Big| 1 \Big) \\ &= O\Big(\sum_{p>x} \frac{n^r}{p^r} n^{s-1-r} \Big) = O\Big(n^{s-1} \sum_{p>x} \frac{1}{p^r} \Big). \end{aligned}$$

But, using the well-known estimate* $\pi(x) = O(x/\log x)$, we have

$$\begin{aligned} \sum_{p>x} \frac{1}{p^r} &= \sum_{n>x} \frac{\pi(n) - \pi(n-1)}{n^r} \leqq \sum_{n>x} \pi(n) \Big(\frac{1}{n^r} - \frac{1}{(n+1)^r} \Big) \\ &= \sum_{n>x} O\Big(\frac{n}{\log n} \Big) O\Big(\frac{1}{n^{r+1}} \Big) = O\Big(\frac{1}{x^{r-1} \log x} \Big). \end{aligned}$$

Therefore

$$N^{(2)} = O\Big(\frac{n^{s-1}}{x^{r-1} \log x} \Big),$$

and so, by (4),

$$N_{s,r}(n) = N^{(1)} + O\Big(\frac{n^{s-1}}{x^{r-1} \log x} \Big). \tag{5}$$

To estimate $N^{(1)}$ we observe that if $n = n_1 + \cdots + n_s$ and p is any given prime, then fewer than r among $n_1, \cdots, n_s$ are divisible by p if and only if there exist numbers $\lambda_1, \cdots, \lambda_s$ such that

$$0 \leqq \lambda_i < p, \qquad n_i \equiv \lambda_i \pmod{p} \qquad (1 \leqq i \leqq s),$$
$$[\lambda_1, \cdots, \lambda_s] < r, \qquad \lambda_1 + \cdots + \lambda_s \equiv n \pmod{p}.$$

Hence we have

* For a proof see, for instance, G. H. Hardy and E. M. Wright, An introduction to the theory of numbers (Second edition, Oxford, 1945), §22.7.

$$N^{(1)} = \sum_{\substack{0 \leqq \lambda_{ij} < p_j;\ \lambda_{1j} + \cdots + \lambda_{sj} \equiv n(\cdot p_j) \\ [\lambda_{1j}, \cdots, \lambda_{sj}] < r;\ (1 \leqq i \leqq s;\ 1 \leqq j \leqq k)}} \quad (6)$$

$$\left| \sum_{\substack{n_1 + \cdots + n_s = n \\ n_i \equiv \lambda_{ij}(\cdot p_j)\ (1 \leqq i \leqq s;\ 1 \leqq j \leqq k)}} \right| 1.$$

Here the inner sum J is equal to

$$J = \sum_{\substack{n_1 + \cdots + n_s = n \\ n_i \equiv \mu_i(\cdot p_1 \cdots p_k)\ (1 \leqq i \leqq s)}} \left| 1, \right.$$

where the μ's depend on the λ's. Since clearly $p_1 \cdots p_k \mid (n - \mu_1 - \cdots - \mu_s)$ and $0 \leqq \mu_1, \cdots, \mu_s < p_1 \cdots p_k$, we have

$$J = \frac{1}{(s-1)!}\left(\frac{n - \mu_1 - \cdots - \mu_s}{p_1 \cdots p_k}\right)^{s-1} + O\left\{\left(\frac{n}{p_1 \cdots p_k}\right)^{s-2}\right\}$$

$$= \frac{1}{(s-1)!}\left(\frac{n}{p_1 \cdots p_k}\right)^{s-1} + O\left\{\left(\frac{n}{p_1 \cdots p_k}\right)^{s-2}\right\}.$$

Hence, by (6),

$$N^{(1)} = \frac{n^{s-1}}{(s-1)!} \prod_{p \leqq x} \frac{t_{s,r}(n, p)}{p^{s-1}} + O\left(n^{s-2} \prod_{p \leqq x} \frac{t_{s,r}(n, p)}{p^{s-2}}\right). \quad (7)$$

But

$$t_{s,r}(n, p) \leqq \sum_{0 \leqq \lambda_1, \cdots, \lambda_s < p} \left| 1 \leqq p^{s-1}, \right. \quad (8)$$

and therefore, by (7), (8), (2), and (3),

$$N^{(1)} = \frac{n^{s-1}}{(s-1)!} \mathfrak{S}_{s,r}(n) \left\{1 + O\left(\frac{1}{x^{r-1} \log x}\right)\right\} + O\left(n^{s-2} \prod_{p \leqq x} p\right). \quad (9)$$

But, as is well known,*

$$\sum_{p \leqq x} \log p \leqq 3x,$$

so that

$$\prod_{p \leqq x} p \leqq e^{3x}.$$

Hence, by (9), (5), and Lemma 3,

* Hardy and Wright, *op. cit.*, §22.6.

$$N_{s,r}(n) = \frac{n^{s-1}}{(s-1)!}\mathfrak{S}_{s,r}(n) + O\left(\frac{n^{s-1}}{x^{r-1}\log x}\right) + O(n^{s-2}e^{3x}).$$

The choice of x is still at our disposal. It is advantageous to take $x = \frac{1}{4}\log n$; with this choice of x we obtain

$$N_{s,r}(n) = \frac{n^{s-1}}{(s-1)!}\mathfrak{S}_{s,r}(n) + O\left(\frac{n^{s-1}}{\log^{r-1} n \log\log n}\right).$$

This is our first assertion and the second now follows immediately by Lemma 3.

5. **The conjugate problem.** In conclusion it may be of some interest to mention the conjugate problem of the problem discussed above. This consists in the determination of an asymptotic formula for the number $F(x) = F(x; r; k_1, \cdots, k_s)$ of positive integers $n \leqq x$ which are such that the highest common factor of any r of the integers $n+k_1, \cdots, n+k_s$ is 1. An account of this problem (in a more general form) has previously appeared in this MONTHLY†.

† L. Mirsky, On coprime values taken by given polynomials, this MONTHLY, vol. 55, 1948, pp. 88–89.

A CALCULUS OF FIGURATE NUMBERS AND FINITE DIFFERENCES*

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1. **Introduction.** The purpose of this article is to derive an operational calculus in the field of finite differences, which is based on the properties of generalized figurate numbers. The procedures necessarily yield old as well as new results, but the emphasis will be on methods and results which are believed to be new.

2. **Figurate Numbers** are here generalized to include figurate numbers of negative order as well as the traditional series of positive order [1]. The more general numbers are derived by using 1, 0, 0, $\cdots$ as a generating series, and operating on it by repeated summation for figurate numbers of positive order and by repeated inverse summation for figurate numbers of negative order.

The generating series 1, 0, 0, $\cdots$ is defined as the series of figurate numbers of zero order. In our notation, superscripts will represent orders of figurate numbers, positive or negative, and subscripts 0, 1, 2, $\cdots$ will represent the successive terms. Thus, the $(r+1)$th term in the series of figurate numbers of the nth order will be represented by F_r^n. The operation of repeated summation is, of course, the process of successive cumulative addition which connects the

* Based on a presentation by the author at the Mathematics Colloquium of the University of Pittsburgh, April 24, 1947, through the courtesy of Professor J. S. Taylor, Head of the Department.

adjacent columns, or rows, of Pascal's triangle. Inverse summation, as the term implies, is the operation which reverses or "undoes" the operation of summation. Thus, regardless of the sign of the superscript, the following basic relationships are true:

$$F_r^{n+1} = F_0^n + F_1^n + F_2^n + \cdots + F_r^n \tag{1}$$

$$F_r^{n+1} = F_r^n + F_{r-1}^{n+1}, \qquad r > 0 \tag{2}$$

$$F_0^m = F_0^{-n} = 1. \tag{3}$$

Table 1 presents a short table of figurate numbers of orders ranging from -7 to 7, and terms ranging from 0 to 7.

TABLE 1

r	*Negative Order*							*Positive Order*							
	F_r^{-7}	F_r^{-6}	F_r^{-5}	F_r^{-4}	F_r^{-3}	F_r^{-2}	F_r^{-1}	F_r^0	F_r^1	F_r^2	F_r^3	F_r^4	F_r^5	F_r^6	F_r^7
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7
2	21	15	10	6	3	1	0	0	1	3	6	10	15	21	28
3	−35	−20	−10	−4	−1	0	0	0	1	4	10	20	35	56	84
4	35	15	5	1	0	0	0	0	1	5	15	35	70	126	210
5	−21	−6	−1	0	0	0	0	0	1	6	21	56	126	252	462
6	7	1	0	0	0	0	0	0	1	7	28	84	210	462	924
7	−1	0	0	0	0	0	0	0	1	8	36	120	330	792	1716

In standard combinatorial notation,

$$F_r^n = \binom{n+r-1}{r} = \binom{n+r-1}{n-1} = \binom{r+1+n-2}{n-1} = F_{n-1}^{r+1} \tag{4}$$

$$F_r^{-n} = \binom{-n+r-1}{r} = \binom{-(n-r+1)}{r} = (-1)^r\binom{n}{r} \tag{5}$$

$$= (-1)^r\binom{n}{n-r} = (-1)^n F_{n-r}^{-n}. \tag{6}$$

Conversely,

$$\binom{a}{b} = F_b^{a-b+1} = (-1)^b F_b^{-a} \tag{7}$$

and

$$\binom{-a}{b} = F_b^{-a-b+1} = F_b^{-(a+b-1)} = (-1)^b\binom{a+b-1}{b}. \tag{8}$$

In the notation of the calculus of finite differences $\Delta u_x = u_{x+1} - u_x$. Since, from (2)

$$F_{r+1}^{n-1} = F_{r+1}^{n} - F_{r}^{n}, \tag{9}$$

$$\Delta F_{r}^{n} = F_{r+1}^{n-1}. \tag{10}$$

In particular,

$$\Delta F_0^n = F_1^n - F_0^n = F_1^{n-1}$$
$$\Delta^2 F_0^n = \Delta F_1^{n-1} = F_2^{n-2}$$

and, in general,

$$\Delta^t F_0^n = F_t^{n-t}. \tag{11}$$

Thus, the leading term and leading differences of the series of figurate numbers F_r^n, $(r=0, 1, 2, \cdots)$, are the figurate numbers F_0^n, F_1^{n-1}, F_2^{n-2}, *etc.*, which, as in Table 1, form a downward diagonal to the left, starting with F_0^n. It may be observed that the numbers above such a diagonal do not appear at all when a series of figurate numbers is successively differenced by the repeated operation $\Delta u_x = u_{x+1} - u_x$. Apparently the operation of inverse summation, which we shall represent by the symbol S^{-1} (corresponding to S for summation), is a more general type of difference operation since it produces the numbers above the leading difference diagonal in addition to the leading differences themselves. This suggests the desirability of studying similar patterns derived from other series.

3. Summation and Inverse Summation of Series. When generalized figurate numbers are arranged as in Table 1 any two adjacent columns bear mutually inverse relationships to each other. The series at the right is the summation of the series at the left. The series at the left is the inverse summation of the series at the right. In effect, any one of the columns may be regarded as a generating series from which all of the others are derived.

Analogously, the successive equidistant values of a function u_x, $(x=0, 1, 2, \cdots, r)$, may be treated as a generating series and operated upon by repeated summation and by repeated inverse summation. If u_r represents the general term of the generating series, Su_r the general term of the summation series, and $S^{-1}u_r$ the general term of the inverse summation series, the following relationships are true:

$$S^0 u_r = u_r \tag{12}$$

$$Su_r = u_0 + u_1 + u_2 + \cdots + u_r = \sum_{t=0}^{r} u_t \tag{13}$$

$$S^2 u_r = Su_0 + Su_1 + Su_2 + \cdots + Su_r = \sum_{t=0}^{r} Su_t, \quad \textit{etc.} \tag{14}$$

$$S^{-1}u_r = u_r - u_{r-1}, \qquad r > 0 \tag{15}$$

$$S^{-2}u_r = S^{-1}u_r - S^{-1}u_{r-1}, \qquad r > 0, \quad etc. \tag{16}$$

In particular,

$$S^{-1}u_1 = u_1 - u_0 = \Delta u_0 \tag{17}$$

$$S^{-2}u_2 = S^{-1}u_2 - S^{-1}u_1 = \Delta S^{-1}u_1 = \Delta^2 u_0. \tag{18}$$

Analogous to (11)

$$\Delta^t S^n u_0 = S^{n-t}u_t. \tag{19}$$

When $r=0$, no operation of summation or of inverse summation is performed on u_0. Therefore, analogous to (3),

$$S^m u_0 = S^{-n}u_0 = u_0. \tag{20}$$

Table 2 exhibits basic relationships between the generating series, the summation series and the inverse summation series. It should be noted that the

TABLE 2

x	*Inverse Summation* $S^{-1}u_x$	*Generating Series* S^0u_x	*Summation* Su_x
0	u_0	u_0	u_0
1	u_1-u_0	u_1	u_0+u_1
...	...	...	...
r	u_r-u_{r-1}	u_r	$u_0+u_1+\cdots+u_r$

mutually inverse operations S and S^{-1} are commutative. By contrast, the operation Δ and its inverse Δ^{-1} (or Σ) are not commutative [2].

4. Parallel Leading Differences and Intersecting Leading Differences. Suppose u_x is a rational integral function of the nth degree in x, and that the series of equidistant terms u_x, $(x=0, 1, \cdots, n+1)$, has been operated on by repeated inverse summation. Table 3 summarizes certain results of such operations; that is, it lists parallel leading differences and intersecting leading differences of a

TABLE 3

x	$S^{-(n+1)}u_x$	$S^{-n}u_x$	$S^{-(n-1)}u_x$	...	$S^{-2}u_x$	$S^{-1}u_x$	S^0x_x
0	u_0	u_0	u_0	...	u_0	u_0	u_0
1	d_1		...			Δu_0	u_1
2	d_2		...		$\Delta^2 u_0$		u_2
...	...		...	...			...
$n-1$	d_{n-1}		$\Delta^{n-1}u_0$		...		u_{n-1}
n	d_n	$\Delta^n u_0$			...		u_n
$n+1$	0	$\Delta^n u_0$			...		u_{n+1}
...	0	...			...		...

rational integral function of the nth degree in x. In column $S^{-n}u_x$ the nth differences, starting with $\Delta^n u_0$, are constant. In the successive columns starting with $S^{-(n+1)}u_x$ the $(n+1)$th and higher orders of differences vanish. In general, the terms $d_1, d_2, \cdots, d_n$ in column $S^{-(n+1)}u_x$ are not zero. We designate them the parallel leading differences of the generating series, in order to distinguish them from $\Delta u_0, \Delta^2 u_0, \cdots, \Delta^n u_0$ which we call the intersecting leading differences. In this way, both sets of differences may be derived by repeated inverse summation of the $(n+1)$th order.

The rth parallel leading difference is

$$d_r = S^{-(n+1)}u_r. \tag{21}$$

The rth intersecting leading difference is

$$\Delta^r u_0 = S^{-r}u_r. \tag{22}$$

5. Criss-Cross Multiplication of Series. Having given two series a_x and b_x, $(x=0, 1, \cdots, r)$, a third series $c_x=a_x*b_x$ can be obtained by a process of criss-cross multiplication shown in Table 4. The symbol of criss-cross multiplication

TABLE 4

x	a_x	b_x	$c_x=a_x*b_x$
0	a_0	b_0	$c_0=a_0b_0$
1	a_1	b_1	$c_1=a_0b_1+a_1b_0$
...	...	...	...
r	a_r	b_r	$c=a_r*b_r=\sum_{t=0}^{r} a_t b_{r-t}$

$(*)$ is somewhat suggestive of the operation which it represents. The symbol is taken from the theory of the Laplace Transformation where it has an analogous significance in connection with certain continuous functions [3]. The operation of criss-cross multiplication obviously conforms to the commutative law; that is, $a_r*b_r=b_r*a_r$. It is easily seen that the operation of criss-cross multiplication conforms to the distributive law; that is, $a_r*(b_r+c_r)=a_r*b_r+a_r*c_r$. We shall now demonstrate that it also conforms to the associate law.

Proof:

$$(a_r*b_r)*c_r = (a_0*b_0)c_r + (a_1*b_1)c_{r-1} + \cdots + (a_{r-1}*b_{r-1})c_1 + (a_r*b_r)c_0. \tag{23}$$

Multiplying out and regrouping the terms, we obtain

$$(a_r*b_r)*c_r = a_0(b_r*c_r) + a_1(b_{r-1}*c_{r-1}) + \cdots + a_r(b_0*c_0) \tag{24}$$

$$= a_r*(b_r*c_r). \tag{25}$$

$$(a_r*b_r)*c_r = b_0(a_r*c_r) + b_1(a_{r-1}*c_{r-1}) + \cdots + b_r(a_0*c_0) \tag{26}$$

$$= b_r*(a_r*c_r). \tag{27}$$

6. Theorems on Criss-Cross Multiplication, Figurate Numbers, and Summation of Series.

THEOREM 1. *Repeated summation of a series n times is equivalent to criss-cross multiplication of the series by the nth order of figurate numbers.* That is,

$$S^n u_r = u_r * F_r^n \tag{28}$$

$$= u_0 F_r^n + u_1 F_{r-1}^n + \cdots + u_{r-1} F_1^n + u_r F_0^n \tag{29}$$

$$= u_0 \binom{n+r-1}{r} + u_1 \binom{n+r-2}{r-1} + \cdots + u_{r-1} \binom{n}{1} + u_r \binom{n-1}{0} \tag{30}$$

$$= u_0 \binom{n+r-1}{n-1} + u_1 \binom{n+r-2}{n-1} + \cdots + u_{r-1} \binom{n}{n-1} + u_r \binom{n-1}{n-1}. \tag{31}$$

An equivalent result has been given by Sheppard [4] with a reference to Elderton [5].

THEOREM 2. *Repeated inverse summation of a series n times is equivalent to criss-cross multiplication of the series by the $(-n)$th order of figurate numbers. That is,*

$$S^{-n} u_r = u_r * F_r^{-n} \tag{32}$$

$$= u_0 F_r^{-n} + u_1 F_{r-1}^{-n} + \cdots + u_{r-1} F_1^{-n} + u_r F_0^{-n} \tag{33}$$

$$= u_0 (-1)^r \binom{n}{r} + u_1 (-1)^{r-1} \binom{n}{r-1} + \cdots + u_r \binom{n}{0} \tag{34}$$

$$= u_r - \binom{n}{1} u_{r-1} + \binom{n}{2} u_{r-2} - \cdots + (-1)^r \binom{n}{r} u_0. \tag{35}$$

An equivalent result has been given by Dwyer [6].

THEOREM 3. *Criss-cross multiplication of figurate numbers of order r (positive or negative) by figurate numbers of order s (positive or negative) results in figurate numbers of order $r+s$.*

(36) That is, $F_m^r * F_m^s = F_m^{r+s}$

(37) Corollary: $F_m^r * F_m^s * F_m^{-t}$ *etc.* $= F_m^{r+s-t \text{ etc.}}$.

THEOREM 4. *If $u_x(x=0, 1, 2, \cdots, n)$ is a rational integral function of the nth degree in x, the rth parallel leading difference is*

$$d_r = S^{-(n+1)}u_r = u_r * F_r^{-(n+1)} \tag{38}$$

and the rth intersecting leading difference is

$$\Delta^r u_0 = S^{-r}u_r = u_r * F_r^{-r} = d_r * F_r^{n-r+1}. \tag{39}$$

THEOREM 5. *Let u_x and v_x represent the successive terms of two series for $x=0, 1, 2$, etc., and let $u_r * v_r$ represent the general term of their criss-cross product. If the first series is operated on by repeated summation of the mth order, and the second series by repeated inverse summation of the nth order, then the general term of the criss-cross product of the two new series is*

$$S^m u_r * S^{-n} v_r = (F_r^m * u_r) * (F_r^{-n} * v_r) = F_r^{m-n} * (u_r * v_r) = S^{m-n}(u_r * v_r). \tag{40}$$

THEOREM 6. *If in the preceding formula $n=m$, it becomes*

$$S^m u_r * S^{-m} v_r = F_r^0 * (u_r * v_r) = u_r * v_r. \tag{41}$$

This last result is equivalent to Dwyer's Successive Cumulation Theorem [6], which he applied to the development of new techniques for the computation of moments of statistical frequency distributions.

Proofs of theorems 1 to 6 present no unusual features, and are omitted here.

7. Applications. To derive a general formula for the $(r+1)$th term in the kth order of repeated summation of the series $u_x(x=0, 1, \cdots, r)$, a rational integral function of the nth degree in x:

In Table 3, any column operated upon by repeated summation and inverse summation will produce the entire field of numbers. Therefore, any column operated upon by criss-cross multiplication with figurate numbers will produce the entire field. If we choose the column which contains the parallel leading differences of the original series, that is, if we apply criss-cross multiplication by figurate numbers to the series $u_0, d_1, d_2, \cdots, d_n$, then any term in any column can be represented by an expression of not more than $n+1$ terms involving the parallel leading differences and figurate numbers.

Applying the foregoing principles, the desired term is $S^k u_r$, and

$$S^k u_r = u_r * F_r^k = (d_r * F_r^{n+1}) * F_r^k = d_r * F_r^{k+n+1} \tag{42}$$

$$= u_0 F_r^{k+n+1} + d_1 F_{r-1}^{k+n+1} + \cdots + d_n F_{r-n}^{k+n+1} \tag{43}$$

$$= u_0 \binom{k+n+r}{r} + d_1 \binom{k+n+r-1}{r-1} + \cdots + d_n \binom{k+r}{r-n} \tag{44}$$

$$= u_0 \binom{k+n+r}{k+n} + d_1 \binom{k+n+r-1}{k+n} + \cdots + d_n \binom{k+r}{k+n}. \tag{45}$$

The parallel leading differences d_1, d_2, *etc.* which are required in applying the above formulas may be obtained either by repeated inverse summation or by criss-cross multiplication of the given series by the series of figurate numbers of order $-(n+1)$, that is, by $F^{-(n+1)}$. It may be observed that the combinatorial expressions represent figurate numbers of constant order $(k+n+1)$. Such expressions are simpler to evaluate numerically than the binomial coefficients of constant order which appear in corresponding formulas in terms of the intersecting leading differences Δ, Δ^2, Δ^3, *etc.*

8. Numerical Examples. A few numerical examples will illustrate the application of the summation formula described in the preceding section.

Example (a): Derive an expression for the sum of the cubes of the first r odd numbers. Here, $u_x=(2x+1)^3$, the expression is of the third degree, and the parallel leading differences may be obtained by criss-cross multiplication by figurate numbers of the fourth negative order, that is, by F_x^{-4}, as follows:

x	$u_x=(2x+1)^3$	F_x^{-4}	$d_x=u_x * F_x^{-4}$
0	1	1	1
1	27	−4	23
2	125	6	23
3	343	−4	1

In this case, $k=1$; $n=3$; $u_0=1$; $d_1=23$; $d_2=23$; $d_3=1$; and the required summation ends with the term u_{r-1}. Substituting,

$$Su_{r-1} = \binom{r+3}{4} + 23\binom{r+2}{4} + 23\binom{r+1}{4} + \binom{r}{4}$$

$$= \frac{(r+3)(r+2)(r+1)r}{24} + \frac{23(r+2)(r+1)r(r-1)}{24}$$

$$+ \frac{23(r+1)r(r-1)(r-2)}{24} + \frac{r(r-1)(r-2)(r-3)}{24}.$$

When this is multiplied out, it reduces to

$$1^3 + 3^3 + \cdots + (2r-1)^3 = r^2(2r^2-1).$$

Example (b): Find polynomial expressions for the rth term and the sum of r terms of the series -6, -3, 10, 39, 90, 169, $\cdots$.

Since, in this case, inspection does not reveal the degree of the general term, the parallel differences are formed by repeated inverse summation, as follows:

x	u_x	$S^{-1}u_x$	$S^{-2}u_x$	$S^{-3}u_x$	$S^{-4}u_x=d_x$
0	−6	−6	−6	−6	−6
1	−3	3	9	15	21
2	10	13	10	1	−14
3	39	29	16	6	5
4	90	51	22	6	0
5	169	79	28	6	0

From the above difference table, since terminal zeros appear in column $S^{-4}u_x$, we assume that the desired polynomial is of the third degree. The term desired is u_{r-1}, since that is the rth term of the series $u_0, u_1, u_2, \cdots, u_{r-1}$. Substituting in formula (45), $k=0$; $n=3$; $u_0=-6$; $d_1=21$; $d_2=-14$; $d_3=5$; and for the rth term of the series

$$S^0u_{r-1} = -6\binom{r+2}{3} + 21\binom{r+1}{3} - 14\binom{r}{3} + 5\binom{r-1}{3}$$

$$= r^3 - r^2 - r - 5.$$

Similarly, for the sum of the first r terms,

$$Su_{r-1} = -6\binom{r+3}{4} + 21\binom{r+2}{4} - 14\binom{r+1}{4} + 5\binom{r}{4}$$

$$= r(3r^3 + 2r^2 - 9r - 68)/12.$$

9. Vandermonde's Theorem. Using the notation $r^{(k)}=r(r-1)\cdots(r-k+1)$, the identity known as Vandermonde's Theorem or the factorial binomial theorem [7] may be written

$$(r+s)^{(k)} = r^{(k)} + \binom{k}{1}r^{(k-1)}s + \binom{k}{2}r^{(k-2)}s^{(2)} + \cdots \tag{46}$$

which is analogous in form to the binomial expansion

$$(r+s)^k = r^k + \binom{k}{1}r^{k-1}s + \binom{k}{2}r^{k-2}s^2 + \cdots. \tag{47}$$

The factorial binomial theorem may be proved starting with Theorem 3, as follows:

$$F_k^{-r-s} = F_k^{-r} * F_k^{-s} \tag{48}$$

$$(49)\quad (-1)^k\binom{r+s}{k} = F_k^{-r}F_0^{-s} + F_{k-1}^{-r}F_1^{-s} + \cdots + F_0^{-r}F_k^{-s}$$

$$= (-1)^k\binom{r}{k}\binom{s}{0} + (-1)^{k-1}\binom{r}{k-1}(-1)^1\binom{s}{1}$$

$$+ \cdots + \binom{r}{0}(-1)^k\binom{s}{k}.$$

Eliminating $(-1)^k$ from both sides of the equation,

$$(50)\quad \binom{r+s}{k} = \binom{r}{k}\binom{s}{0} + \binom{r}{k-1}\binom{s}{1} + \cdots + \binom{r}{0}\binom{s}{k}, \text{ or}$$

$$(51)\quad \frac{(r+s)^{(k)}}{k!} = \frac{r^{(k)}s^{(0)}}{k!0!} + \frac{r^{(k-1)}s^{(1)}}{(k-1)!1!} + \cdots + \frac{r^{(0)}s^{(k)}}{0!k!}.$$

Multiplying both sides by $k!$ we obtain

$$(52)\quad (r+s)^{(k)} = r^{(k)}s^{(0)} + \binom{k}{1}r^{(k-1)}s^{(1)} + \binom{k}{2}r^{(k-2)}s^{(2)} + \cdots + r^{(0)}s^{(k)}.$$

10. A Problem in Binomial Coefficients. The properties of generalized figurate numbers can be applied to solve Problem 4189 in the MONTHLY, February, 1946, Vol. 53, No. 2, page 103.
Prove that

$$\sum_{r=0}^{m}(-1)^r\binom{m+k+a}{r+k+a}\binom{r+k}{k} = \binom{m+a-1}{m}, \qquad a \geqq 0.$$

The expression

$$(-1)^r\binom{m+k+a}{r+k+a} = (-1)^r\binom{m+k+a}{m-r} = (-1)^m F_{m-r}^{-(m+k+a)}.$$

The expression

$$\binom{r+k}{k} = \binom{r+k}{r} = F_r^{k+1}.$$

Therefore, the original expression is equivalent to

$$(-1)^m \cdot \sum_{r=0}^{m} F_{m-r}^{-(m+k+a)} \cdot F_r^{k+1} = (-1)^m F_m^{-(m+k+a)} * F_m^{k+1}$$

$$= (-1)^m F_m^{-(m+a-1)} = \binom{m+a-1}{m}.$$

The result is independent of the restriction $a \geqq 0$. The above proof may be compared with the proof in the MONTHLY, June–July, 1947, page 347.

11. Summation of Product Series. Theorem 6 can be applied to find the sum of a product series such as $u_r v_0$, $u_{r-1}v_1$, $\cdots$, $u_0 v_r$, where v_x is a polynomial of degree n. The sum of the series is evidently

$$u_r * v_r = u_r v_0 + u_{r-1}v_1 + \cdots + u_0 v_r. \tag{53}$$

Introducing figurate numbers of the $(n+1)$th order, positive and negative,

$$u_r * v_r = (F_r^{n+1} * u_r) * (F_r^{-(n+1)} * v_r) \tag{54}$$

$$= (F_r^{n+1} * u_r) * d_r. \tag{55}$$

Changing back from criss-cross multiplication by figurate numbers to repeated summation,

$$u_r * v_r = S^{n+1}u_r * d_r, \tag{56}$$

When the above is expanded, since the function v_x is of the nth degree, the terms involving d_{n+1}, d_{n+2}, *etc.*, will vanish. Therefore, the desired sum

$$u_r * v_r = d_0 S^{n+1}u_r + d_1 S^{n+1}u_{r-1} + \cdots + d_n S^{n+1}u_{r-n}, \tag{57}$$

where

$$d_t = F_t^{-(n+1)} * v_t, \qquad (t = 0, 1, \cdots, n).$$

This result is equivalent to one given by Dwyer [6] for the evaluation of $\sum_0^k u_{a+x}v_{a+x}$.

12. Parallel Leading Differences of Powers of Integers. Although Dwyer used another type of notation, the differencing operation which he applied is identical with the operation of inverse summation as defined in this article. Carrying out his objective of simplifying the computation of statistical moments, Dwyer derived general expressions and recurrence formulas for what, in our terminology, are the parallel leading differences of the powers of integers, which are analogous to the "differences of zero" of actuarial literature. For the powers r^n, $(r=0, 1, 2, \cdots)$, since the function is of the nth degree, Theorem 4 is applicable, and we have

$$d_r = S^{-(n+1)}r^n = F_r^{-(n+1)} * r^n. \tag{58}$$

An equivalent formula, in another connection has been given by Wall [8]. Values of d_1, d_2, $\cdots$ are given in Table 5 from $n=1$ to $n=11$.

TABLE 5

r	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$
1	1	1	1	1	1	1	1	1	1	1	1
2		1	4	11	26	57	120	247	502	1013	2036
3			1	11	66	302	1191	4293	14608	47840	152637
4				1	26	302	2416	15619	88234	455192	2203488
5					1	57	1191	15619	156190	1310354	9738114
6						1	120	4293	88234	1310354	15724248
7							1	247	14608	455192	9738114
8								1	502	47840	2203488
9									1	1013	152637
10										1	2036
11											1
Totals	1	2	6	24	120	720	5040	40320	362880	3628800	39916800
	1!	2!	3!	4!	5!	6!	7!	8!	9!	10!	11!

These values of d may be substituted in Formula 45 to derive expressions for powers of integers and their sums in terms of figurate numbers or in terms of factorial products. From a statistical point of view, r^n turns out to be a weighted average of factorial products, in which the d's are the weights. Thus,

$$r^n = \frac{d_1 r^{(n)} + d_2 (r+1)^{(n)} + \cdots + d_n (r+n-1)^{(n)}}{d_1 + d_2 + \cdots + d_n} \tag{59}$$

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8. H. S. Wall, "On the Expansion of an Integral of Stieltjes," The American Mathematical Monthly, February, 1932, page 98.

quantities are *different*, one of them is *greater* than the right-hand side of (6), that is

$$\max_x \frac{|p-x|}{x} > \frac{b-a}{b+a}. \tag{7}$$

It happens only in the case (4) that (7) goes over into an equation. This proves our theorem in the case in which the value of p lies between a and b. The two remaining (*a priori* unplausible) cases, $p<a$ and $p>b$, can be discussed even more simply.

NOTE ON CAUCHY'S LIMIT THEOREM

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Cauchy's limit theorem states that $\lim_{n\to\infty}(a_1+\cdots+a_n)/n=\lim_{n\to\infty} a_n$ if the second limit exists. We may generalize this theorem as follows.

THEOREM 1. *Let a_n be a sequence of real numbers convergent to A: $a_n\to A$. Let $f_n(\alpha_1, \alpha_2, \alpha_3, \cdots)$ be a sequence of real-valued functions which is defined for any such sequence $(\alpha_1, \alpha_2, \alpha_3, \cdots)$ that, for some positive integer h, $\alpha_\nu=a_\nu$ or A when $\nu\leqq h$ and $\alpha_\nu=a_\nu$ when $\nu>h$. Suppose that such individual a_k is ultimately immaterial to $f_n(a_1, a_2, a_3, \cdots)$ and $f_n(a_1, a_2, a_3, \cdots)$ is ultimately a mean, namely that*

$$\lim_{n\to\infty}[f_n(\alpha_1, \cdots, \alpha_{k-1}, \alpha_k, a_{k+1}, a_{k+2}, \cdots) - f_n(\alpha_1, \cdots, \alpha_{k-1}, A, a_{k+1}, a_{k+2}, \cdots)] = 0 \tag{1}$$

and

$$\min(\alpha_1, \cdots, \alpha_k, a_{k+1}, a_{k+2}, \cdots) + \eta_n \leqq f_n(\alpha_1, \cdots, \alpha_k, a_{k+1}, a_{k+2}, \cdots) \leqq \max(\alpha_1, \cdots, \alpha_k, a_{k+1}, a_{k+2}, \cdots) + \zeta_n \tag{2}$$

for every $(\alpha_1, \alpha_2, \alpha_3, \cdots)$ and for every positive integer k, where $\eta_n\to 0$, $\zeta_n\to 0$. Then

$$f_n(a_1, a_2, a_3, \cdots)\to A.$$

For we may write

$$\begin{aligned} f_n(a_1, a_2, a_3, \cdots) - A = {} & f_n(a_1, a_2, a_3, \cdots) - f_n(A, a_2, a_3, \cdots) \\ & + f_n(A, a_2, a_3, \cdots) - f_n(A, A, a_3, \cdots) \\ & \vdots \\ & + f_n(A, \cdots, A, a_k, a_{k+1}, \cdots) \\ & - f_n(A, \cdots, A, a_{k+1}, a_{k+2}, \cdots) \\ & + f_n(A, \cdots, A, a_{k+1}, a_{k+2}, \cdots) - A, \end{aligned}$$

hence (1) implies that

$$\overline{\lim_{n\to\infty}} \left| f_n(a_1, a_2, a_3, \cdots) - A \right| \leqq \overline{\lim_{n\to\infty}} \left| f_n(A, \cdots, A, a_{k+1}, a_{k+2}, \cdots) - A \right|.$$

And (2) implies that the right member of the above inequality tends to zero as $k\to\infty$. Thus the left member of the inequality is equal to zero, and the theorem is proved.

Similarly we have the following result.

THEOREM 2. *Let a_n be a sequence of complex numbers convergent to A, and let $f_n(\alpha_1, \alpha_2, \alpha_3, \cdots)$ be a sequence of complex-valued functions which is defined for any such sequence $(\alpha_1, \alpha_2, \alpha_3, \cdots)$ that, for some positive integer h, $\alpha_\nu = a_\nu$ or A when $\nu \leqq h$, and $\alpha_\nu = a_\nu$ for $\nu > h$ or $\alpha_\nu = A$ for $\nu > h$. Suppose that each individual a_k is ultimately immaterial to $f_n(a_1, a_2, a_3, \cdots)$ and $f_n(\alpha_1, \alpha_2, \alpha_3, \cdots)$ is ultimately continuous, namely that condition* (1) *is satisfied together with the condition*

$$\lim_{k\to\infty} \overline{\lim_{n\to\infty}} \left| f_n(\alpha_1, \cdots, \alpha_k, a_{k+1}, a_{k+2}, \cdots) - f_n(\alpha_1, \cdots, \alpha_k, A, A, \cdots) \right| = 0 \tag{2'}$$

for each $(\alpha_1, \alpha_2, \alpha_3, \cdots)$ and k. Then

$$f_n(a_1, a_2, a_3, \cdots) = f_n(A, A, A, \cdots) + o(1).$$

It is to be observed that conditions (1) and (2) are also *necessary* for $a_n\to A$ to imply $f_n(a_1, a_2, a_3, \cdots)\to A$; and that (1) and (2′) form also necessary conditions for $a_n\to A$ to imply the conclusion of Theorem 2.

The necessity of (1) is evident; for, if $(a_1, a_2, a_3, \cdots)$ has limit A then $(\alpha_1, \cdots, \alpha_{k-1}, A, a_{k+1}, a_{k+2}, \cdots)$ has the same limit. And (2) follows immediately from the relations:

$$(\alpha_1, \cdots, \alpha_k, a_{k+1}, a_{k+2}, \cdots) \to A, \qquad f_n(\alpha_1, \cdots, \alpha_k, a_{k+1}, a_{k+2}, \cdots) \to A.$$

A similar argument proves the necessity of (2′).

We note also that Theorems 1 and 2 hold for $A = \pm\infty$ as well as for finite A if we substitute A_n for A in conditions (1), (2), and in the conclusion of Theorem 2 together with the definition of $f_n(\alpha_1, \alpha_2, \alpha_3, \cdots)$, where A_n is any sequence (real in the case of Theorem 1) such that $A_n\to A$. The necessity of the new conditions thus obtained is still only for finite A, however.

The well-known generalizations (due to Jensen, Stolz, Toeplitz and Silverman) of Cauchy's theorem lay stress on the linear form of $f_n(a_1, a_2, a_3, \cdots)$. (See K. Knopp, *Theory and Application of Infinite Series*, English translation by R. C. Young, pp. 70–76.) The above extensions contain these as particular cases and seem sometimes more convenient to apply.

Silverman's theorem reads: If

$$a_n \to 0 \quad \text{or} \quad \sum_{\nu=1}^{\infty} t_{n\nu} \to 1,$$

then

$$\sum_{\nu=1}^{\infty} t_{n\nu}a_\nu \to A,$$

provided that $a_n \to A$ and that

(a) $$t_{nk} = o(1) \qquad \text{for } k = 1, 2, 3, \cdots,$$

(b) $$\sum_{\nu=1}^{\infty} |t_{n\nu}| = O(1).$$

This immediately follows from Theorem 2, since (a) implies (1), (b) implies (2′).

By making use of Theorem 2, it is also readily seen that a generalization of Knopp's theorem on bilinear form (*op. cit.* p. 73) runs as follows.

If $(p_{n1}, p_{n2}, p_{n3}, \cdots)$ *is a permutation of* $(1, 2, 3, \cdots)$ *for each* n, *then*

$$\sum_{\nu=1}^{\infty} t_{n\nu}a_\nu b_{p_{n\nu}} \to AB \quad \textit{when} \quad a_n \to A,\ b_n \to B\ (A, B \textit{ finite}),$$

provided that $\sum_{\nu=1}^{\infty} t_{n\nu} \to 1$, *and that* (a) *and* (b) *are satisfied together with the condition*

(a′) $$\lim_{n\to\infty} t_{n\nu_{nk}} = 0 \qquad \textit{for } k = 1, 2, 3, \cdots,$$

where ν_{nk} *is the integer value of* ν *for which* $p_{n\nu} = k$.

For, (b) implies that the function

$$f_n(a_1, a_2, a_3, \cdots) = \sum_{\nu=1}^{\infty} t_{n\nu}a_\nu b_{p_{n\nu}}$$

is ultimately continuous, and (a) implies that each a_k is immaterial to it; hence we have

$$f_n(a_1, a_2, a_3, \cdots) = \sum_{\nu=1}^{\infty} t_{n\nu}Ab_{p_{n\nu}} + o(1) = Ag_n(b_1, b_2, b_3, \cdots) + o(1),$$

where

$$g_n(b_1, b_2, b_3, \cdots) = \sum_{\nu=1}^{\infty} t_{n\nu}b_{p_{n\nu}}.$$

Now (b) implies that $g_n(b_1, b_2, b_3, \cdots)$ is ultimately continuous and (a′) implies that each b_k is immaterial to it; hence we also have

$$g_n(b_1, b_2, b_3, \cdots) = \sum_{\nu=1}^{\infty} t_{n\nu}B + o(1) = B + o(1).$$

Therefore $f_n(a_1, a_2, a_3, \cdots) = AB + o(1)$, and the statement is proved.

To illustrate Theorem 1, let a_{ij}^n $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, n)$ be new

notations for $a_1, a_2, a_3, \cdots, a_{n^2}$ in any order, and let

$$|\overset{n}{a}_{ij}|^+ = \sum_p \overset{n}{a}_{1p_1}\overset{n}{a}_{2p_2}\cdots\overset{n}{a}_{np_n},$$

where the summation extends over all the permutations of $(1, 2, \cdots, n)$. Then we have

$$\frac{1}{n}(|\overset{n}{a}_{ij}|^+)^{1/n} \to \frac{1}{e}A \qquad \text{if } a_n \to A,\ a_n \geqq 0.$$

For, Theorem 1 (with $\eta_n = \zeta_n = 0$) gives

$$\left(\frac{1}{n!}|\overset{n}{a}_{ij}|^+\right)^{1/n} \to A,$$

and

$$\lim_{n\to\infty}\left(\frac{1}{n!}\right)^{1/n}(n+1) = \lim_{n\to\infty}\left[\left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\cdots\left(\frac{n+1}{n}\right)^n\right]^{1/n}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e.$$

Finally, the author should like to express his indebtedness and gratitude to the referee for invaluable criticisms and suggestions.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

After February 1, 1950 all material for this department should be sent to C. B. Allendoerfer, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

A PARADOX RELATING TO MATHEMATICAL INDUCTION

R. G. ALBERT, Providence, R. I.

The axiom of mathematical induction as included in the set of axioms of Peano which characterize the system of natural numbers asserts:

If S be a set of natural numbers containing 1 and containing the successor x' of every natural number x which S contains, then S is the set of *all* natural numbers.

The following proof by mathematical induction (by Landau), that every natural number except 1 has an immediate predecessor, suggested an interesting

paradox which may aid in gaining an insight into this important method of proof:

Let x' denote the successor of x (i.e., $x+1$).

Let S be the set of natural numbers including 1 and including all n.n.'s u such that there exists a n.n. x with $x'=u$. Then, obviously, S contains 1. If S contains $u=k$, then S contains $u=k'$, since $x'=k'$ has the evident solution $x=k$. Hence S is the set of all n.n.'s. Since S was defined to be the set comprised of 1 and of all n.n.'s u which have predecessors, it follows that every n.n. except 1 has an immediate predecessor.

Now, examine the following proof which seems to trace the same lines:

We shall try to show that every n.n. except 1 equals its own successor.

Let S be the set of n.n.'s containing 1 and containing all n.n.'s u such that $u=u'$ (u' denotes $u+1$, as before). S contains 1. If S contains $u=k$, then $k=k'$. Since every n.n. has a unique successor (Axiom 2, Peano), then $k'=(k')'$. The latter verifies $u=u'$ for $u=k'$. Thence, S contains $u=k'$. Thus, S is the set of all n.n.'s. Since S was defined to contain 1 and all n.n.'s equal to their own successors, it follows that every n.n. with the possible exception of 1 is its own successor.

This procedure is of course generalizable and seems to suggest that many theorems or formulas which are known to be false can be "proved" for all n.n.'s except 1, by this device of consigning 1 to S a priori. It appears as if tossing 1 in bodily from the start is a kind of circumvention of the intention in the induction hypothesis, raising doubts regarding the validity of arbitrary inclusion of 1.

It has been of interest to me to note how many of my students have failed to "see through" the error in the last proof and why the first proof is still valid.

Of course, the error in the second demonstration lies in the step:

If S contains $u=k$, then $k=k'$.

From S containing $u=k$, follows: *Either* $k=1$ *or* $k=k'$. And in the first of these alternative cases, we can no longer assert that S contains k', since neither qualification for admission to S is satisfied by k', i.e., $1'\neq 1$ and $1'\neq(1')'$.

In the first proof, S containing k *did* imply *invariably* that S contained k', for $x'=k'$ does in all cases have a solution, $x=k$.

A NOTE ON TAYLOR'S THEOREM

C. L. SEEBECK, JR., University of Alabama

Although there are many proofs of Taylor's Theorem in mathematical literature, most of these are difficult for a not-too-advanced Calculus student to master. The following proof requires no knowledge of infinite series, or of convergence. It is direct, readily illustrated geometrically, and provides the remainder using only elementary or easily explained theorems.

Let $f(x)$ be a function continuous with its derivatives through order $n+1$ in a suitable neighborhood of $x=a$. Moreover, assume $f^{[k]}(a)$ may be evaluated for

$k=0, 1, 2, \cdots, n$. We desire the value of $f(x)$ at a point in the given neighborhood. Since the evaluation process may call for operations other than $(+, -, \times, \div)$, we first approximate its value by means of

$$P(x) = \sum_{k=0}^{n} c_k(x - a)^k,$$

a polynomial. We choose c_k so that $P(x)$ will be identified with $f(x)$ as closely as possible when $x=a$, or such that

$$P^{[k]}(a) = f^{[k]}(a), \qquad k = 0, 1, 2, \cdots, n, \tag{1}$$

a condition that implies

$$c_k = \frac{f^{[k]}(a)}{k!}.$$

Now let $R(x)$ be the amount by which the approximation fails.

$$R(x) = f(x) - P(x). \tag{2}$$

Equation (1) implies

$$R^{[k]}(a) = 0, \qquad k = 0, 1, 2, \cdots, n. \tag{3}$$

Differentiating equation (2) n times gives

$$R^{[n]}(x) = f^{[n]}(x) - P^{[n]}(x).$$

But since $P(x)$ is a polynomial of degree n, its nth derivative is a constant,

$$P^{[n]}(x) = P^{[n]}(a) = f^{[n]}(a), \quad \text{and}$$
$$R^{[n]}(x) = f^{[n]}(x) - f^{[n]}(a).$$

By the law of the mean

$$R^{[n]}(x) = f^{[n+1]}(\theta)(x - a), \qquad \theta \text{ between } a \text{ and } x. \tag{4}$$

Let M and m be respectively the maximum and minimum values of $f^{[n+1]}(\theta)$ in the interval (a, x). Then if $a \leqq t \leqq x$,

$$m(t - a) \leqq R^{[n]}(t) \leqq M(t - a).$$

It is clear geometrically that if an ordinate of one curve is never greater than the corresponding ordinate of a second curve in a given interval, then the area under the first curve cannot exceed the area under the second for this interval. Accordingly, we integrate this inequality from a to x and obtain

$$\tfrac{1}{2}m(x - a)^2 \leqq R^{[n-1]}(x) \leqq \tfrac{1}{2}M(x - a)^2.$$

Continuity of the $(n+1)$st derivative of $f(t)$ now implies

$$R^{[n-1]}(x) = f^{[n+1]}(\theta_1)\frac{(x-a)^2}{2}, \qquad \theta_1 \text{ between } a \text{ and } x.* \tag{5}$$

If $x \leqq t \leqq a$, the inequalities are reversed and (5) is again true.

Repeated integration of the inequalities from a to x will produce

$$R(x) = \frac{f^{[n+1]}(\phi)}{(n+1)!}(x-a)^{n+1}, \qquad \phi \text{ between } a \text{ and } x.$$

Equation (2) now gives

$$f(x) = P(x) + R(x) \tag{6}$$

which is Taylor's Theorem with remainder.

It may be noted that the question of convergence has been completely sidestepped. $P(x)$ is a good approximation of $f(x)$ whenever the least upper bound for $|R(x)|$ is as small as desired. When n is finite, equation (6) is identically true in x, and the question of the series representing the function does not arise.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 896. *Proposed by F. J. Duarte, Caracas, Venezuela*

If p is an odd prime show that

$$[1\cdot 3\cdot 5\cdots(p-2)]^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$

E 897. *Proposed by W. R. Ransom, Tufts College*

A cube is divided into equal parts by the plane of a hexagon whose vertices are midpoints of its edges. In what ratio is a cube divided by a parallel plane that divides an edge in the ratio $a:(1-a)$?

E 898. *Proposed by N. S. Mendelsohn, University of Manitoba*

A pack of N cards is disposed of as follows. The top card is placed at the bottom of the deck, the next card is discarded, the third card is placed at the bottom, the fourth discarded, *etc.*, this process being carried on until there is only

* This follows more directly and more rigorously from equation (4) by the first law of the mean for integrals if this is available.

one card left. Let $f(N)$ be the position of this card, from the top, in the original arrangement of the deck. Prove that

$$f(N) = 2N - 2^{[\log_2 N]+1} + 1,$$

where $[x]$ denotes, as usual, the largest integer not greater than x.

E 899. *Proposed by E. R. Bowersox, Chicago, Illinois*

Construct an equilateral triangle given the distances of a point in its plane from the three vertices.

E 900. *Proposed by N. A. Court, University of Oklahoma*

If four spheres may be divided into two pairs so that the circle common to one pair is cospherical with the circle common to the second pair, then the circle determined by any two of the given spheres is cospherical with the circle determined by the remaining two spheres.

E 849 [1949, 552]. Correction

In both the title and the solution of this problem read " pedal" for "cevian."

SOLUTIONS

A Curious Representation of Integers

E 861 [1949, 262]. *Proposed by Vern Hoggatt, College of Puget Sound, and Leo Moser, University of Manitoba*

Let a be any positive number different from 1 and let p be any integer greater than 3. Show that every integer may be expressed by using p a's and a finite number of operator symbols used in high school texts.

Solution by the Proposers. It is easily verified that

$$\pm \log_{\log \sqrt{a}\, a} \log \sqrt{\sqrt{\cdots \sqrt{a}}}\, a = \pm n, \tag{1}$$

$$\pm \log_{(a+a)/a} \log \sqrt{\sqrt{\cdots \sqrt{a}}}\, a = \pm n, \tag{2}$$

$$(a - a) + (a - a) = (a - a) + \log_a \log_a a = 0, \tag{3}$$

where, in (1) and (2), there are n radical signs in the iterated set. These formulas solve the problem for $p=4$ and 5. If $p>5$, then add to the appropriate left member a suitable number of terms of the form $(a-a)$.

Hoggatt has also shown that every integer can be expressed by using the first $m>3$ positive integers, repetitions allowed, and a finite number of operator symbols used in high school texts.

Extreme Rectangles About a Parallelogram

E 862 [1949, 262]. *Proposed by R. E. Horton, Los Angeles City College*

Find the rectangles of greatest and least area which can be circumscribed about a given parallelogram.

Solution by Roger Lessard, École Polytechnique, Montreal. Let a and b, $a \geqq b$, be the sides of the parallelogram P, and $\alpha \leqq \pi/2$ the angle between them. Let β be the acute angle made with a by a side of the rectangle passing through the vertex of a non-acute angle of P. Then we must have $0 \leqq \beta \leqq \alpha$. Denote the areas of the rectangle and the parallelogram by A_R and A_P respectively. Then

$$\begin{aligned} A(\beta) &\equiv A_R - A_P \\ &= a^2 \sin\beta \cos\beta + b^2 \sin(\alpha - \beta)\cos(\alpha - \beta) \\ &= (1/2)a^2 \sin 2\beta + (1/2)b^2 \sin 2(\alpha - \beta), \end{aligned} \tag{1}$$

whence

$$dA/d\beta = a^2 \cos 2\beta - b^2 \cos 2(\alpha - \beta) \tag{2}$$

and

$$d^2A/d\beta^2 = -2a^2 \sin 2\beta - 2b^2 \sin 2(\alpha - \beta) = -4A < 0.$$

Since there is no turning point minimum, the least area will be at one end of the interval $(0, \alpha)$. Now

$$A(0) = (1/2)b^2 \sin 2\alpha \leqq (1/2)a^2 \sin 2\alpha = A(\alpha).$$

Therefore

$$\min A_R = (a + b\cos\alpha)b \sin\alpha.$$

Setting $dA/d\beta = 0$ we find

$$\tan(2\beta - \alpha) = [(a^2 - b^2)/(a^2 + b^2)] \cot\alpha.$$

Therefore, since $2\beta - \alpha \leqq \alpha$, we have a turning point maximum only if

$$(a^2 - b^2)/(a^2 + b^2) \leqq \tan^2\alpha, \tag{3}$$

in which case we find

$$\max A_R = ab \sin\alpha + [a^4 - 2a^2b^2 \cos 2\alpha + b^4]^{1/2}/2.$$

If (3) is not satisfied then (2) is positive and (1) is continuously increasing in the interval $(0, \alpha)$, and the greatest area is obtained at $\beta = \alpha$. Thus, in this case, we have

$$\max A_R = (b + a\cos\alpha)a \sin\alpha.$$

Also solved by D. H. Browne, Sam Kravitz, N. D. Lane, C. S. Ogilvy, I. W. Thompson, and the proposer.

Let the parallelogram be $ABCD$, $AB \leqq BC$, and the circumscribed rectangle $PQRS$, such that A lies on PQ, B on QR, C on RS, S on SP. Let O be the common center of the parallelogram and rectangle. Browne showed that a turning point maximum exists if $\measuredangle ABO - \measuredangle OAB < \pi/2$, and that in this case $\measuredangle OBR = \measuredangle RCO$. A euclidean construction of this maximum rectangle can be made, and

we have

$$\max A_R = (AC)(BD)\sin^2(135^\circ - BOC/2).$$

The Generalized Steiner-Lehmus Problem

E 863 [1949, 263]. *Proposed by W. O. Pennell, Exeter, N. H.*

If $\alpha>0$, $\beta>0$, $\alpha+\beta<\pi$, $0<k<1$, then $\alpha=\beta$ is a necessary and sufficient condition for

$$\sin\alpha\sin(k\alpha+\beta) = \sin\beta\sin(k\beta+\alpha).$$

I. *Solution by Mary Payne, Michigan State College.* If $\alpha=\beta$, the identity is obviously true without any restrictions on k or on the size of α and β.

If the identity is true, we may differentiate it twice with respect to k to obtain

$$-\alpha^2\sin\alpha\sin(k\alpha+\beta) = -\beta^2\sin\beta\sin(k\beta+\alpha).$$

Dividing this result by the identity, we have

$$\beta^2 = \alpha^2 \quad \text{or} \quad \beta = \pm\alpha,$$

and if $\alpha>0$, $\beta>\alpha$, the truth of the identity implies $\alpha=\beta$ with no restrictions on k and no further restrictions on α and β.

Since the identity is evidently true for $\alpha=\pm\beta$, we may state the following more general theorem: A necessary and sufficient condition that the identity

$$\sin\alpha\sin(k\alpha+\beta) = \sin\beta\sin(k\beta+\alpha)$$

hold is that $\alpha=\pm\beta$.

II. *Solution by N. D. Lane, St. Andrew's College, Aurora, Ontario.* Evidently, if $\alpha=\beta$, the equation is satisfied. Suppose that $\alpha>\beta$. Multiplying by 2 we get

$$-\cos(k\alpha+\beta+\alpha)+\cos(k\alpha-\alpha+\beta) = -\cos(k\beta+\alpha+\beta)+\cos(k\beta-\beta+\alpha),$$

or

$$\cos(k\beta+\alpha+\beta)-\cos(k\alpha+\beta+\alpha) = \cos(k\beta-\beta+\alpha)-\cos(k\alpha-\alpha+\beta),$$

whence

$$\sin[(k+2)(\beta+\alpha)/2]\sin[k(\beta-\alpha)/2] = \sin[k(\beta+\alpha)/2]\sin[(2-k)(\alpha-\beta)/2].$$

Now if $\alpha>\beta$, the left side is negative, while the right side is positive, and we have a contradiction.

Also solved by B. B. Dressler, L. M. Kelly, Norman Miller, C. S. Ogilvy, and the proposer.

Editorial Note. Consider a triangle ABC with base angles at A and B denoted by α and β respectively. Let AD and BE be drawn so that $\sphericalangle BAD = k\alpha$ and $\sphericalangle ABE = k\beta$. Then, by the law of sines applied to triangles ABE and ABD,

$$\sin \alpha / \sin (k\beta + \alpha) = BE/AB,$$
$$\sin \beta / \sin (k\alpha + \beta) = AD/AB.$$

Therefore, if $BE = AD$,

$$\sin \alpha \sin (k\alpha + \beta) = \sin \beta \sin (k\beta + \alpha),$$

and, by the proposed problem, $\alpha = \beta$, and triangle ABC is isosceles. If $k = 1/2$, then AD and BE are the angle bisectors of α and β and we have the Steiner-Lehmus theorem: "If the bisectors of the base angles of a triangle are equal, then the triangle is isosceles."

A Well Known Identity

E864 [1949, 263]. *Proposed by N. S. Mendelsohn, University of Manitoba*

Prove that

$$\sum_{n=1}^{r} 1/n = \sum_{n=1}^{r} (-1)^{n+1}(1/n)\binom{r}{n}.$$

Solution by Leo Moser, University of Manitoba. Evaluate

$$\int_0^1 \frac{1 - x^r}{1 - x}\, dx$$

directly to obtain the left side of the given expression. Then evaluate the same integral after making the substitution $x = 1 - u$ to obtain the right side of the given expression.

Also solved by W. D. Berg, W. G. Brady, D. H. Browne, Paul Carnahan, Ragnar Dybvik, B. F. Hadnot, R. Hamming, Vern Hoggatt, M. S. Klamkin, Roger Lessard, Norman Miller, S. T. Parker, C. F. Pinzka, C. M. Sandwick, C. W. Trigg, and the proposer.

Editorial Note. Many references were sent in for this problem. See, for example, prob. 38, p. 6, Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*; prob. 21, p. 393, Hardy, *Pure Mathematics*; prob. 23, p. 169, Hall and Knight, *Higher Algebra*; p. 327, Jordan, *Calculus of Finite Differences*; prob. 18, v. 2, p. 19, Chrystal, *Algebra*; prob. 117(a), p. 270, Knopp, *Infinite Series*; p. 104, ser. 2, v. 1, 1891, *Mathesis*; prob. 1245, Jan. 1933, *School Science and Mathematics*; problems 4130 [1945, 527] and E 460 [1941, 700], this MONTHLY. For other problems in the MONTHLY dealing with sums involving the binomial coefficients see E 205, E 445, E 670, 3625, 3701, 3748, and for other problems dealing with the sum of a finite number of terms of the harmonic series see E 46, E 53, E 520, E 819, 3652, 4267.

Equivalent Concurrent Sections of a Tetrahedron

E 865 [1949, 263]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find a point such that planes drawn through this point parallel to the faces of a tetrahedron cut the opposite trihedrals in equivalent triangles. Express the common area of these triangles in terms of the areas of the faces of the tetrahedron.

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. Denote the areas of the faces of the tetrahedron by A_i $(i=1, 2, 3, 4)$, the corresponding altitudes by H_i, and the perpendicular distances from the corresponding vertices to the corresponding parallel sections by h_i. Let V be the volume of the tetrahedron and A the common area of the parallel sections. Then

$$\Sigma(H_i - h_i)A_i = 3V = H_iA_i, \qquad h_i^2/H_i^2 = A/A_i.$$

Therefore

$$3V = \Sigma\{H_iA_i - H_i(AA_i)^{1/2}\} = 12V - \Sigma H_i(AA_i)^{1/2},$$

or

$$A^{1/2} = 9V/\Sigma H_iA_i^{1/2} = 3/\Sigma A_i^{-1/2}.$$

Also

$$h_i = 3H_iA_i^{-1/2}/\Sigma A_i^{-1/2}.$$

Also solved by N. D. Lane and the proposer.

The analogous problem for the plane was proposed by J. Neuberg as question 30, p. 148, 1881, *Mathesis*. For this case we have

$$2/k = 1/a + 1/b + 1/c,$$

where a, b, c are the sides of the triangle and k is the common length of the concurrent lines which are drawn parallel to the sides of the triangle. This, and the result of the given problem, suggested to Lane for the corresponding problem of an n-dimensional simplex the formula

$$n/k = \Sigma 1/a,$$

where k^{n-1} is the common content of the $(n-1)$-dimensional cells parallel to the $(n-1)$-dimensional cell "faces" of the simplex, and a^{n-1} is the content of a "face."

A Lattice Game

É 866 [1949, 338]. *Proposed by L. J. Burton, Bryn Mawr College*

Players A and B take turns, beginning with A, each marking a previously unmarked unit line segment joining two points with integral coordinates at a unit distance in a plane.

(a) Prove that B can prevent A from ever marking all the line segments in the perimeter of any closed polygon.

(b) Prove that if $P_1:(x_1, y_1)$ and $P_2:(x_2, y_2)$ are any two fixed points with integral coordinates such that $|P_1P_2| > 1$, then B can prevent A from joining P_1 and P_2 by a broken line consisting of segments marked by A.

Solution by the Proposer. (a) B simply prevents A from ever making an L with two of his segments. Whenever A draws a vertical segment, B completes the L by drawing the proper horizontal segment, and whenever A draws a horizontal segment, B completes the L by drawing the proper vertical segment. Of course every closed polygon made of unit segments joining points with integral coordinates has at least one L.

(b) Draw the line P_1P_2. By interchanging the axes if necessary, we may assume that P_1P_2 is not vertical. Whenever A marks a segment lying along P_1P_2 (if this is possible), B marks another segment lying along the line, which will be possible since $|P_1P_2| > 1$. When A draws a segment with at least one vertex above P_1P_2, B completes the inverted backward L at a vertex above P_1P_2 if the slope of $P_1P_2 \geqq 0$, and completes the inverted L at a vertex above P_1P_2 if the slope < 0. If no vertex of A's segment is above P_1P_2 and if at least one vertex is below P_1P_2, B completes the backward L at a vertex below P_1P_2 if the slope $\geqq 0$, and completes the L at a vertex below P_1P_2 if the slope < 0.

It is clear that if A leaves P_1P_2 he must use an angle of the kind drawn to reach P_2, but B prevents him from making such an angle. It is fairly evident that the case when A's segment crosses P_1P_2 causes no difficulty, because if A ever gets above P_1P_2 on his route, he can never get down to the line again.

A Limit Given by John Wallis

E 867 [1949, 338]. *Proposed by Walter Fleming, Fort Hays Kansas State College*

Find

$$\lim_{n\to\infty} n^{-p} \sum_{j=1}^{n} j^{p-1}.$$

Solution by S. T. Thompson, Tacoma, Washington. The definition of a Riemann integral shows that the limit is $1/p$ if $p \geqq 1$. If $0 < p < 1$ the corresponding integral is improper but converges to $1/p$, and the limit is still $1/p$. The familiar p-series of elementary calculus shows that the limit is $+\infty$ if $p \leqq 0$.

Also solved by Philip Anselone, W. D. Berg, Louis Berkofsky, A. B. Boggs, D. H. Browne, Paul Carnahan, K. L. Cooke, B. B. Dressler, Maurice Dunn, Vern Hoggatt, Philip Kirmser, N. D. Lane, Roger Lessard, A. E. Livingston, Joseph Rosenbaum, and the proposer. Many of these solutions were incomplete.

As Azriel Rosenfeld pointed out, this limit was established by John Wallis in his *Opera Mathematica* in 1656. See, *e.g.*, *Scripta Mathematica*, vol. IX, no. 4, p. 241. Boggs located the problem in Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, probs. 20 and 22, p. 39, vol. 1.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4375. *Proposed by N. J. Fine, University of Pennsylvania*

Let $((x))=x-[x]-\frac{1}{2}$. Prove that the sums

$$\sum_{n=1}^{m}((2^n x+\tfrac{1}{2}))$$

are uniformly bounded.

4376. *Proposed by Victor Thébault, Tennie, Sarthe, France*

For a tetrahedron $ABCD$ let (A) be the sphere with center A and radius equal to the altitude AA', and let the tangent planes to (A) through the edges BC, CD, DB intersect in the point A_1. Let B_1, C_1, D_1 be analogously defined. If $ABCD$ is orthocentric show that AA_1, BB_1, CC_1, DD_1 are concurrent at the isogonal conjugate of the orthocenter, and that otherwise AA_1, BB_1, CC_1, DD_1 constitute a hyperbolic group of lines.

4377. *Proposed by Paul Brock, Reeves Instrument Company, New York City*

Consider two non-decreasing sequences of n positive integers, $(n<M)$, the integers being chosen at random. What is the most probable number of integers in the sequence containing the maximum integer, each of which is larger than the maximum of the second sequence?

4378. *Proposed by H. W. Smith, Oklahoma Agricultural and Mechanical College.*

Determine

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{n}{}_nC_k/(2k+1)}{\sum_{k=0}^{n+1}{}_{n+1}C_k/(2k+1)},$$

where the ${}_nC_k$ are the binomial coefficients.

4379. *Proposed by Watson Fulks, University of Minnesota*

If

(1) $\phi(t)$ is defined in I: $m \leqq t \leqq M$,

(2) for any a, b in I, $a \neq b$,

$$\phi\left(\frac{a+b}{2}\right) < \frac{\phi(a)+\phi(b)}{2},$$

(3) p_i are non-negative numbers, $i=1, 2, 3, \cdots, n$,

(4) $t_i \in I$, not all t_i equal,

it has been shown that

$$\phi\left(\frac{\sum_{i=1}^{n} p_i t_i}{\sum_{i=1}^{n} p_i}\right) \leqq \frac{\sum_{i=1}^{n} p_i \phi(t_i)}{\sum_{i=1}^{n} p_i}.$$

See Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, v. I, p. 53, problem 74.

Show that the relation is true without the equality sign.

SOLUTIONS

Orthocentric Tetrahedron, Minimum Property

4251 [1947, 286]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In an orthocentric tetrahedron the lines joining the symmedian points (Lemoine points) of the faces to the midpoints of the corresponding altitudes, are concurrent at a point such that the sum of the squares of its distances to the planes of the faces is a minimum.

Note. The problem is solved by the following result due to Robert Bouvaist which appeared in *Mathesis*, t. LVI, 1947, p. 113.*

Let (Γ) be a quadric inscribed in the tetrahedron $ABCD$; let its points of contact with the faces BCD, CDA, DAB, ABC be A', B', C', D'. We restrict our attention to the case for which AA', BB', CC', DD' are concurrent at P. If P varies on AA' the one-parameter family of quadrics (Γ) includes the points A and A' and the conic (γ_1) inscribed in BCD at the feet of the cevians of A'. Hence the locus of the centers of the pencil of quadrics is the line M_1O_1 joining the midpoint M_1 of AA' to the center O_1 of (γ_1). Thus the four lines M_1O_1, M_2O_2, M_3O_3, M_4O_4 pass through the center of (Γ).

If $ABCD$ is orthocentric the point P is the orthocenter, and we have the following proposition: In an orthocentric tetrahedron the lines joining the symmedian points (Lemoine points) of each face to the midpoint of the correspond-

* Translation by W. E. Byrne, Virginia Military Institute.

ing altitude pass through the first Lemoine point of the tetrahedron.[1]

It is known that in a triangle the conic tangent to the three sides at the feet of the altitudes has as its center the Lemoine point of the triangle. Likewise in an orthocentric tetrahedron the quadric tangent to the faces at the feet of the altitudes has as its center the first Lemoine point of the tetrahedron.[2]

The Four Liars

4288 [1948, 165]. *Proposed by J. W. Campbell, University of Alberta.*

In his Messenger Lecture on Probability (A. S. Eddington, *New Pathways in Science*, 1935, p. 121) Eddington referred to the following problem: If A, B, C, D each speak the truth once in three times (independently), and A affirms that B denies that C declares that D is a liar, what is the probability that D was telling the truth? He used the Exclusion Method of Solution and arrived at the numerical result 25/71.

Prove that the correct probability is 13/41, and that this is also the probability that each of A, B, and C told the truth.

I. *Solution by Bart Park, Michigan College of Mining and Technology.* Of the 81 possible combinations of truth and lie, the only ones which are compatible with the statement, "A affirms that B denies that C declares that D is a liar," are those which contain an even number of lies. Thus, if D tells the truth, all the others must tell the truth or exactly two of them must lie. These combinations or paths may be represented as follows:

T-T-T-T,	T-L-L-T,	L-T-L-T,	L-L-T-T.
(1 way)	(4 ways)	(4 ways)	(4 ways)

Similarly, if D tells a lie the following paths are possible:

L-L-L-L,	T-T-L-L,	L-T-T-L,	T-L-T-L.
(16 ways)	(4 ways)	(4 ways)	(4 ways)

Therefore the probability that D was telling the truth is 13/41. Exactly similar reasoning applies to A, B and C.

II. *Solution by G. N. Garrison and Theodore Hailperin, Lehigh University.* The following notations will be used:

α	for	A speaks the truth about B's statement
β	for	B speaks the truth about C's statement
γ	for	C speaks the truth about D's statement
δ	for	D speaks the truth about an event X
$\overline{E}$	for	the contrary event to E
EF	for	the event "E and F"
$E \vee F$	for	the event "E or F"

[1] The first Lemoine point of a tetrahedron is a point such that the sum of the squares of its distances to the faces of the tetrahedron is a minimum.

[2] R. Bouvaist et V. Thébault, Comptes-Rendus de l'Académie des Sciences, Paris, 1943, pp. 418–419.

The probabilities of $\alpha, \beta, \gamma, \delta$ are denoted by a, b, c, d; and $\bar{x}$ is used for $1-x$.

The event "C declares that D lied" may be expressed as

$$\gamma\bar{\delta} \vee \bar{\gamma}\delta,$$

for the event will happen if D lies and C speaks the truth about D's statement, or if D speaks the truth and C lies. The event "B denies that C declares that D lied" then is

$$\overline{\beta(\gamma\bar{\delta} \vee \bar{\gamma}\delta)} \vee \bar{\beta}(\gamma\bar{\delta} \vee \bar{\gamma}\delta)$$

or, by the calculus of logic,

$$\beta\gamma\delta \vee \beta\bar{\gamma}\bar{\delta} \vee \bar{\beta}\gamma\bar{\delta} \vee \bar{\beta}\bar{\gamma}\delta.$$

Finally "A affirms that B denies that C declares that D lied" becomes

$$\alpha(\beta\gamma\delta \vee \beta\bar{\gamma}\bar{\delta} \vee \bar{\beta}\gamma\bar{\delta} \vee \bar{\beta}\bar{\gamma}\delta) \vee \bar{\alpha}\overline{(\beta\gamma\delta \vee \beta\bar{\gamma}\delta \vee \bar{\beta}\gamma\bar{\delta} \vee \bar{\beta}\bar{\gamma}\delta)}$$

which is

$$\alpha\beta\gamma\delta \vee \alpha\beta\bar{\gamma}\bar{\delta} \vee \alpha\bar{\beta}\gamma\bar{\delta} \vee \alpha\bar{\beta}\bar{\gamma}\delta \vee \bar{\alpha}\bar{\beta}\gamma\delta \vee \bar{\alpha}\beta\bar{\gamma}\delta \vee \bar{\alpha}\beta\gamma\bar{\delta} \vee \bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}.$$

We note that this expression is invariant under the cyclical substitution $(\alpha\beta\gamma\delta)$, since the first and last members of the alternation are invariant and the remaining six include all of the possible cases where two of the events are denied; we also note that these alternatives are mutually exclusive and consist of the conjunction of independent events. Thus the probability of the event is

$$abcd + ab\bar{c}\bar{d} + a\bar{b}cd + a\bar{b}\bar{c}d + \bar{a}\bar{b}cd + \bar{a}b\bar{c}d + \bar{a}bc\bar{d} + \bar{a}\bar{b}\bar{c}\bar{d},$$

and the probability that this event and δ happened is obtained by deleting those terms containing $\bar{d}$. Thus the *à posteriori* probability of δ on the hypothesis that "A affirms that $B \cdots$" is

$$\frac{abcd + a\bar{b}\bar{c}d + \bar{a}\bar{b}cd + \bar{a}b\bar{c}d}{abcd + ab\bar{c}\bar{d} + a\bar{b}cd + a\bar{b}\bar{c}d + \bar{a}\bar{b}cd + \bar{a}b\bar{c}d + \bar{a}bc\bar{d} + \bar{a}\bar{b}\bar{c}\bar{d}}.$$

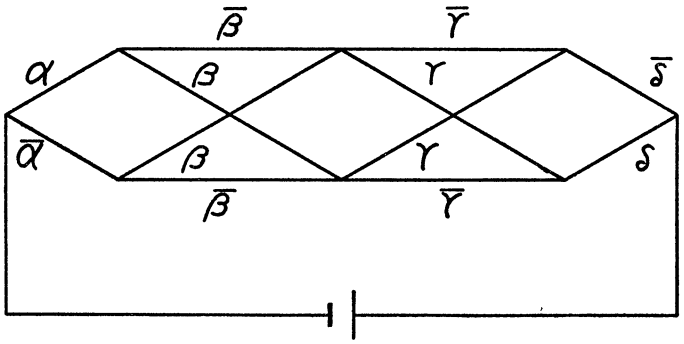
If we put $a=b=c=d=1/3$, we obtain 13/41. Clearly this is also the *à posteriori* probability of α, β, and γ because of the previously mentioned symmetry.

Also solved by Karl Itkin, John Riordan, and the Proposer.

Editorial Note. Riordan cites the following problem as one to which the same analysis applies. Let α, β, γ, and δ be interpreted as switching variables (C. E. Shannon, *A Symbolic Analysis of Relay and Switching Circuits*, Trans. A.I.E.E., 1938), where if γ is a switch, then $\bar{\gamma}$ is its negative, closed when γ is open and vice versa. Given that the circuit shown below is open and that each switch operates with probability 1/3, what is the probability that a specified switch is operated?

All the solvers made the assumption that $\bar{\gamma}$, for example, means "C asserts the opposite of D's statement." Without such assumptions about B and C we

are not sure that either of them made any statement whatever—perhaps A fabricated the whole chain. For discussion and solution of this more general interpretation see (references supplied by C. D. Olds) *Math. Gazette* 1935, pp. 256–257; 1936, pp. 298–308, 309–310. In the last reference, the interpretation is given which justifies Eddington's original result 25/71.



Twisted Curves

4292 [1948, 253]. *Proposed by R. Goormaghtigh, Bruges, Belgium*

If s and ρ are the arc length and the radius of curvature of a plane curve Γ at a variable point M, and if it be required that s have a constant ratio to the distance of M from a fixed point, then Γ must be a cycloidal curve and

$$\lambda^2 s^2 - \rho^2 = a^2,$$

λ and a being constants.

Prove that, in the case of a twisted curve, the condition is

$$\lambda^2 s^2 - \rho^2 = \left(\int \frac{\rho}{\tau} ds + a \right)^2,$$

τ being the radius of torsion of Γ at M.

Solution by the Proposer. Consider variable axes of coördinates, these being the tangent Mx, the binormal My and the principal normal Mz of Γ at M. Then, α, β, γ being the coördinates of a fixed point,

$$(1) \qquad d\alpha/ds = \gamma/\rho - 1, \quad d\beta/ds = \gamma/\tau, \quad d\gamma/ds = -\alpha/\rho - \beta/\tau,$$

(see Çesàro-Kowalewski, *Natürliche Geometrie*, Teubner, 1901, p. 157, also Lane, *Metric Differential Geometry of Curves and Surfaces*, p. 47, ex. 3) and

$$(2) \qquad \alpha^2 + \beta^2 + \gamma^2 = k^2 s^2.$$

Differentiating (2), taking (1) into account,

$$\alpha = -k^2 s;$$

differentiating again,

$$\gamma = (1 - k^2)\rho,$$

and again

$$\beta = \tau\{k^2 s/\rho - (1 - k^2)d\rho/ds\}.$$

Substitution of these values of α, β, γ into (2) gives a relation which can be put in the form

$$\frac{\rho}{\tau} = \frac{k^2 s/(1 - k^2) - \rho d\rho/ds}{\{k^2 s^2/(1 - k^2) - \rho^2\}^{1/2}}$$

whence the theorem follows immediately, $\lambda^2 = k^2/(1-k^2)$, $-1 < k < 1$.

Summation

4293 [1948, 254]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Evaluate

$$\sum_{n=1}^{\infty} \frac{\left(\frac{3 - \sqrt{5}}{2}\right)^n}{n^3}.$$

Solution by D. A. Darling, Rutgers University. The result is

$$\frac{2}{15}\left\{6\zeta(3) + \pi^2 \log\left(\frac{-1 + \sqrt{5}}{2}\right) - 5\left[\log\left(\frac{-1 + \sqrt{5}}{2}\right)\right]^3\right\},$$

where $\zeta(s)$ is the Riemann zeta function. This is an immediate consequence of a formula due to Spence to the effect that if

$$\phi(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^3}, \qquad |x| \leqq \tfrac{1}{2},$$

then

$$\phi\left(\frac{x}{x - 1}\right) + \phi(x) + \phi(1 - x) - \phi(1)$$
$$= \frac{\pi^2}{6} \log(1 - x) + \frac{1}{6}\{\log(1 - x)\}^2\{\log(1 - x) - 3 \log x\}.$$

See W. Spence, *An essay on the theory of the various orders of logarithmic transcendents*, (1809), p. 28. The formula was discovered independently by Ramanujan. In the *Journal of the London Mathematical Society*, v. 3 (1928), p. 217, G. N. Watson gives an elementary proof based upon the obvious relation

$$\phi(x) = \frac{x}{2}\int_0^1 \frac{\log(1 - u)^2 du}{1 - x + xu},$$

expressing the left-hand side of Spence's formulas as the sum of four such integrals and making elementary transformations.

Letting $x=(3-\sqrt{5})/2$ we see that $(x-1)^2=x$ and $x/(x-1)=x-1$, and putting further $v=1-x$, Spence's formula gives

$$\phi(x)+\phi(v)+\phi(-v)=\phi(1)+\frac{\pi^2}{6}\log v-\frac{5}{6}(\log v)^3.$$

From the series definition it is easy to see that

$$\phi(v)+\phi(-v)=\frac{2}{2^3}\phi(v^2)=\frac{1}{4}\phi(x),$$

and observing finally that $v=(-1+\sqrt{5})/2$, $\phi(1)=\zeta(3)$, we obtain the result announced.

Also solved by the Proposer.

The Identity as a Product of Successive Elements

4300 [1948, 369]. *Proposed by Leo Moser, University of Toronto*

Let $a_1, a_2, \cdots, a_n$ be n, not necessarily distinct, elements of a group of order n. Show that there exist integers p and q, $1 \leqq p \leqq q \leqq n$, such that

$$\prod_{i=p}^{q} a_i = 1.$$

Solution by D. J. Newman, Freshman, College of the City of New York. Consider $1, a_1, a_1a_2, a_1a_2a_3, \cdots, a_1a_2a_3\cdots a_n$. Since there are $n+1$ elements of the group present in this set, we must have at least two of them equal. If $1=\prod_{i=1}^{k}a_i$, then the proposed theorem is proved. Otherwise

$$\prod_{i=1}^{q} a_i = \prod_{i=1}^{v} a_i, \qquad q > v,$$

whence

$$\prod_{i=v+1}^{q} a_i = 1.$$

With $p=v+1$, this completes the proof.

Also solved by Michael Aissen, R. V. Andree, Paul Brock, S. H. Gould, B. A. Hausmann, J. B. Kelly, Arnold Walfisz, and the Proposer. Kelly notes the corollary: From any row of n integers, a block of adjacent integers may always be selected whose sum is divisible by n.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Applied Differential Equations. By F. E. Relton. Blackie and Son, Ltd. Glasgow, 1948. 4+264 pages. 20s.

This book is divided into eleven chapters with the following headings. I. Preliminary notions; hyperbolic functions. II. First order; standard methods. III. The linear equations; constant coefficients. IV. Miscellaneous theorems and methods. V. Simultaneous equations. VI. Fourier series. VII. Partial differential equations. VIII. The method of isoclinals. IX. Numerical methods of solution. X. Equations in three variables. XI. Variable coefficients.

As these headings indicate the contents of the book is essentially that of any good American introductory textbook on differential equations. The arrangement of its material is, however, somewhat different.

The book was written primarily for students of applied science. This is made evident by the numerous applications which are developed in detail throughout the text or are listed among the exercises (given with hints and answers). These include problems in mechanical vibrations, heat flow, and bending of beams. It is also made evident by the emphasis upon the intuitive or formal treatment of the various topics. As, for example in the chapters on Fourier series and on numerical methods, no conditions are stated for the convergence of the series or of the other processes.

Throughout the book the author makes statements which attempt to justify this omission of theorems and proofs. For example, in the preface, he says: " 'Epsilonology' is to the mathematician what J. S. Bach is to the musician; there is an enormous sense of gratification in its essential rightness. It is an exhilarating experience to go right down to the foundations of one's subject; especially if one does not stay there. . . . But I see no justification on that account for expecting those to whose studies mathematics is ancillary to master the philosophical profundities of a limit. It would be as rational to expect the piano-tuner to be interested in the comma of Pythagoras. Geometric intuition is good enough for the applied scientist."

But *is* geometric intuition good enough for the applied scientist? With the increasing degree of accuracy required in modern manufacturing processes, the engineer is (or should be) more conscious than ever of the importance of questions of convergence. Many have learned from experience that it is foolhardy to be ignorant of the restrictions attached to the use of a mathematical process; that is, they realize the importance of knowing what hypotheses are needed to ensure certain conclusions. While it may be conceded that an epsilon-delta proof does not often appeal to an engineer, he nevertheless wishes to know at least the

gist of a proof so that he might understand the reasons for the validity of the theorem. In the experience of the reviewer, neither graduate nor undergraduate engineering students regard a discussion of theorems and proofs as worthless or uninteresting.

In spite of the above criticisms, the reviewer considers the book to be a clear and well-organized exposition of the subject. The reader will particularly enjoy the author's numerous informal remarks, both for their content and language. The book would make good collateral reading in an introductory course on differential equations or, if somewhat supplemented with respect to theorems and proofs, would also be adaptable as a textbook for such a course.

MORRIS MARDEN

Theory of Experimental Inference. By C. W. Churchman. New York, The Macmillan Company, 1948. 11+292 pp. $4.25.

This book attempts "to preach the gospel of modernism to two groups"—the philosophers and the scientists. To the modern philosopher of science, it preaches that "we can no longer conceive of scientific method in the older empirical fashion of 'discrete' observation sentences"; that "we put far more into the process of question-answering than is contained in so much of present-day philosophical writing"; and that "the simplest question of fact in science requires for even an approximation, a judgment of value." (p. vii) To the scientist, the book has tried "to preach the moral that the process of raising a question and proceeding to reply to it, is a process which eventually demands the active cooperation of all fields of research." Moreover, the author has attempted "to show that besides certain statistical considerations, the self conscious experimenter must take into account very general problems concerning the natural universe within which he is solving his special problems." (p. viii) However praiseworthy his endeavor, Churchman has substantially failed in giving clear statement to his brand of "modernism," and his lack of clarity suggests fundamental weaknesses, inaccuracies, and confusions.

The first three chapters are concerned with the nature of statistical tests, the general methodology of inference and certain problems of method. The next six chapters form a backdrop for the author's presentation of his own position, called "experimentalism." One by one, traditional and current positions fall under Churchman's attack. It turns out that, for the author, the fundamental problems of the philosophy of science are: "What can be known?" and "How can it be known?" Or, more specifically, the important problem is how to define "answer" in such a way that answers exist even though (1) "the answering of any question of law presupposes the answering of at least some questions of fact" and (2) "the answering of any questions of fact presupposes the answering of at least some questions of law." (pp. 47, 51, 57)

The remainder of the work is devoted to the definition of *experimentalism* and its applications. Experimentalism is characterized by its fundamental postulate, namely, that "*there exists a formalization of nature, such that stochastic limits exist*

for certain sequences of mathematical functions of the observations which are pertinent to a given question of fact." (p. 178) One wonders whether the complexity of the machinery developed under the name of experimentalism, is warranted by the questions Churchman seeks to answer. The essay is concluded with discussions of chance, loss, risk, and quality control. It is the appeal of the essay "that the sciences, and society in general, collaborate in setting up a controlled science of losses and risks, i.e., a controlled science of ethics." (p. 263)

To the professional philosopher, further objections occur. Churchman seems to have confused methodology of science with epistemology, and (at times) both of these fields with metaphysics. Another serious confusion is the author's repeated claim that science requires ethics and that ethics can be a science. Perhaps the only way to understand his claim is to realize that, throughout, a form of utilitarianism has been tacitly assumed, but surely not scientifically established.

V. G. HINSHAW, JR.

College Algebra by E. A. Cameron and E. T. Browne. New York, Henry Holt and Company, 1949. 10+406 pp. $3.00.

The topics covered in this book are essentially those contained in the standard college algebra text, the first nine of twenty-four chapters containing a thorough review of elementary algebra. The authors have been generally successful in their aim to treat this review material from a more mature point of view than is possible in a high school course. Although some attempt has been made to show that the formal rules of manipulation of algebraic quantities are consequences of the postulates assumed for the number systems, in most instances the common practice of stating rules of procedure is adopted.

The theorems are carefully stated, often in more general form than is customary, and the proofs are concise and accurate.

The theory of equations, including the quadratic equation, is especially well done. Descartes' Rule of Signs and the theorem on rational roots are examples of theorems stated and proved in their general forms. Only second and third order determinants are discussed, the properties being proved in the third order case, and the generalizations to the nth order indicated.

The illustrative examples throughout the book are well chosen and the exercises numerous and well graded. Finally, the printing is excellent, each page presenting an uncrowded appearance which should be inviting to the student and instructor alike.

R. A. BEAUMONT

NEW BOOKS RECEIVED

Le Superficie Algebriche. By Federigo Enriques. Nichola Zanichelli Editore, Bologne, 1949. 14+464 pp. 3000 lire.

Radio and Television Mathematics. By B. Fischer. New York, Macmillan, 1949. 18+484 pp. $6.00.

Transformation Calculus and Electrical Transients. By S. Goldman. New York, Prentice-Hall, 1949. 14+439 pp. $6.25.

Advanced Calculus for Engineers. By F. B. Hildebrand. New York, Prentice-Hall, 1949. 14+594 pp. $6.00.

Teorica Della Capitalizzazione. By F. Insolera. Turin, Giulio Einaudi, 1949. 237 pp. L. 1500.

Statistical Methods in Research. By P. O. Johnson. New York, Prentice-Hall, 1949. 18+377 pp. $5.00.

Rinehart Mathematical Tables. Compiled by H. D. Larsen. Alternate Edition. New York, Rinehart, 1948. 8+160 pp. $1.00.

Analytic Geometry and Calculus. By F. H. Miller. New York, Wiley, 1949. 12+658 pp. $5.00.

Differential Equations. Second Edition. By H. W. Reddick. New York, Wiley, 1949. 10+288 pp. $3.00.

First-year Mathematics for Colleges. By P. R. Rider. New York, Macmillan, 1949. 16+714 pp. $5.00.

Introduction to Algebraic Geometry. By J. G. Semple and L. Roth. Oxford University Press, 1949. 16+446 pp. $7.50.

Modern Algebra. Vol. 1. Translated from the second revised German edition by Fred Blum with revisions and additions by the author, B. L. van der Waerden. New York, Frederick Ungar, 1949. 12+264 pp. $5.50.

Quantum Theory of Fields. By G. Wentzel. Translated from the German by C. Houtermans and J. M. Jauch. New York, Interscience, 1949. 10+224 pp. $6.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1948–49

Pi Mu Epsilon, University of Missouri

The program of the *Missouri Alpha* Chapter of *Pi Mu Epsilon* for the academic year 1948–49 was:

Functions defined by more than one equation, by Dr. G. M. Ewing

Mathematics as a career, by Dr. L. M. Blumenthal

The students' points of view on mathematics courses, by a panel of students

Saturn and the Moon, by Dr. E. S. Hayner of the Astronomy department

Mathematics and vagueness, by Dr. W. D. Oliver, of the Philosophy department.

Other activities of the fraternity were: the annual initiation; the "Problem of the Month" Contest, consisting of a monthly mathematics problem with all prizes awarded to the first person giving the correct solution; and the annual calculus examination with prizes won by William Posladek, William Nichols, and Aldo Linsenbardt.

Newly elected officers are: President, William Nichols; Vice-President, Martin Smith; Secretary, Walter Hickman; Treasurer, Jesse Corder.

Kappa Mu Epsilon, College of St. Francis

The *Illinois Delta* Chapter of *Kappa Mu Epsilon* made a special study this year of the lives and works of the late Cassius J. Keyser and Alfred North Whitehead. Most of the papers and reports centered about these two outstanding scholars. The program was as follows:

Pastures of Wonder, by Keyser, by Sister M. Claudia O.S.F.

The life of Alfred North Whitehead, by Miss Mary Hodor

Cassius J. Keyser, by Miss Florence Kane

Humanization of the teaching of mathematics, by Miss Mary Jean Lafond

Educational ideals most worthy of loyalty, by Miss Lillian Rafac

Nature and Life, by Whitehead, by Sister M. Rita Clare O.S.G.

Whitehead's Introduction to Business Adrift, by Donham, by Miss Anne Hutchings

Aims of education, by Sister M. Perter O.S.F.

Mathematical curriculum, by Sister M. Hilary O.S.F.

The place of the classics in education, by Sister M. Elizbeth O.S.F.

Element in the history of thought, by Miss Jane Rourke

Non-euclidean geometries, by Miss Rita Grogan.

Miss Jane Rourke was elected by the Chapter to speak at a Mathematics Symposium sponsored by the Catholic Colleges of the Chicago area and held at DePaul University. Miss Rourke reviewed the book *Cybernetics* by N. Weiner. The paper was very timely and worth while.

The new officers for 1949–50 are: President, Mary Hodor; Vice-President, Anne Hutchings; Secretary, Margaret Dreska; Treasurer, Rita Grogan; Corresponding Secretary, Sister M. Rita Clare O.S.F.

Mathematics Club, University of Colorado

Members of the *Mathematics Club* of the University of Colorado presented the following talks during 1948–49:

On the origin of some mathematical terms and symbols, by Gideon Culpepper

How to win a free beer (nim), by Gordon Burton

Bromwich's integral, by Roy Reeves

A four number game, by Burrowes Hunt. Mr. Hunt extended the problem to include all real numbers

On p-adic numbers, by Prof. Burton Jones

Generating functions, by Prof. David Hawkins
On Jacobian elliptic functions, by Gideon Culpepper
On the algebra of sets, by David De Vol
On exterior differential forms, by Prof. Wesley Brittin
Cryptography, by Burrowes Hunt
Ruler and/or compasses, by Ralph Merrill
On some mathematical concepts in biophysics, by William Musgrove
Iteration, by Prof. Aubrey Kempner
Continued fraction approximation, by Prof. Burton Jones.

Officers for 1948–49 were: President, David De Vol; Secretary, Gordon Burton.

Sigma Phi Mu, Montclair Teachers College

Papers presented to *Sigma Phi Mu*, of Montclair Teachers College, New Jersey during the year 1948–49 were:

Cryptography, by Dr. Davis
Mathematics in the modern world, by Dr. Frieschower
Unknown quantity or logical system in algebra, by Dr. Fehr.

Movie strips shown were:

Geometry and Nature
Geometry in the Home
Seeing Is Believing.

The social program consisted of our annual Christmas Party with *Kappa Mu Epsilon*, a most enjoyable Probability Night, and our annual May picnic.

Instead of a February meeting here, several members attended the meeting of the *David Eugene Smith Mathematics Club* of Teachers College, Columbia University, at which time Mr. Clifford gave a talk on *Quality control*. Members of the Teachers College returned the visit to help us enjoy Probability Night.

We had a good attendance at all of our meetings. The number present each month averaged about sixty-five.

Officers for the coming year: President, William Koellner; Vice-President, Martin Vallaster; Secretary, Catherine Bruce; Treasurer, Dorothy Ryan.

Mathematics Club, University of Massachusetts

The *Mathematics Club* of the University of Massachusetts was organized on a formal basis at the beginning of the year and held monthly meetings. The following talks were presented to the Club:

Applications of probability, by Jerome Landry
Complex numbers, by Louis Robinson
Aerial navigation simplified, by Deane Beytes
Mathematical induction, by Murray Altsher
The mathematical analysis of a game, by Prof. I. H. Rose
The Ceasaro curve, by Francis Vigneau
Nature of postulational systems, by Janice Rittenburg

Infinite products, by Paul Ritger
The Lorentz transformation, by Robert San Soucie
Pythagorean numbers, by Henry Skillings
Number systems, by Walter Szetela
Mathematical principles of field artillery methods, by Stephen Allen
Symbolic logic, by Arthur Budd
Some entertaining applications of topology, by Prof. A. E. Andersen.

The newly elected officers are: President, Francis Vigneau; Vice-President, Charles Reynolds; Secretary-Treasurer, Grace Hyder; Faculty Advisor, Prof. A. E. Andersen.

Delta Rho, Southern Illinois University

Delta Rho, honorary mathematics fraternity of Southern Illinois University, held two meetings each quarter for the year 1948–49. Among the interesting papers given during the year were:

The four-color problem, by William Staudacher
The life of Pascal, by Joseph Rezetka
The life of LaGrange, by Louis Rowell
Philosophy and mathematics, by Dr. Charles Tenny
Abstract equivalence relations, by Charles Ablett
Involution and evolution, by Miss Ruth Sprankel
Zero, by Emmett Moll
Poly-dimensional manifolds, by Robert Etherton.

The outstanding meeting of the year was the annual Founders' Day banquet at which time nine new members were initiated. The total number initiated this year was 31.

The newly elected officers for the year 1949–50 are: President, William Staudacher; Vice-President, Charles Elliott; Secretary, Richard Vorwald; Treasurer, Imogene Beckemeyer; Program Chairman, Joseph Rezetka.

Pi Mu Epsilon, University of Oklahoma

The *Oklahoma Alpha* chapter of *Pi Mu Epsilon* began its activities for the year with a business meeting, at which time the following officers were elected for the year: Director, Lowell Gregory; Vice-Director, Richard Yeilding; Secretary-Treasurer, Gerald Webster.

The dates, speakers, and their topics for the subsequent meetings are as follows:

The five-color conjecture, by Dr. Arthur Bernhart
The effects of atmospheric disturbances on projectiles, by Otis Spears
Eigenfunctions and limits to the characteristic roots of a matrix, by Lowell Gregory
Cybernetics, by Dr. Carlton Berenda
Galois theory, by Miss Ann Sheldon.

The annual banquet and initiation was held at which Dr. J. Rud Nielsen, Research Professor of Physics, spoke on *The Raman effect.* Prizes consisting of mathematics books were presented to the winners of the Annual Mathematics Contest. These were: First prize, Mr. Miles Maxwell, Jr.; Second prize, Mr. Earl Scott.

The Officers for the year 1949–50 will not be elected until next fall.

Mathematics Club, Butler University

The *Mathematics Club* of Butler University submits the following report for the academic year 1948–49:

The activities included bi-weekly business meetings and programs and three social functions—Fall Wiener Roast, Christmas Party, and Spring Picnic.

The papers presented include:

Mathematical fallacies, by Mrs. Juna L. Beal

Mathematics in art, by Nancy Heemstra

Calculating machines, by Robert Gambill

Book review of Whom the Gods Love, by Mary Ethel Jackson

The abacus, by Clifford Wagoner

The navigation of the battleship Missouri from Norfolk, Virginia to Guantanamo, Cuba, by Dr. Harry E. Crull

Geometric exercises in paper folding, by Donald Reisinger

Mathematics of the automobile, by Max Sayler

The history of mathematics, by Anna White

The philosophy of mathematics, by Elsie Popplewell

Rubber sheet geometry, by Miss Jane Uhrhan

Mathematical literature, by Byron Carmichal.

The officers for the first semester 1949–50 are: President, Robert Gambill; Vice-President, Mary Ethel Jackson; Secretary, Phyllis Stultz; Treasurer, Ann Montgomery.

Kappa Mu Epsilon, University of New Mexico

The following papers were presented to the *New Mexico Alpha* chapter of *Kappa Mu Epsilon* for the year 1948–49:

History of the theory of planetary motion, by Robert Flanagan

The theory and use of the electronic oscilloscope, by Thomas Martin

The morality of science, by Dr. Sherman Smith

Bessel's functions applied to the theory of heat transfer by the use of fins, by Edward Rightley.

The general theory of alignment and contour charts, by Dr. Morris Hendrickson.

Science and art, by Dr. W. P. Albrecht.

Officers elected for 1949–50 are: President, Ross Schmidt; Vice-President, Thomas Wooten; Secretary, George Vincent; Treasurer, Prof. H. P. Rogers; Faculty Sponsor, Prof. Morris Hendrickson.

Mu Alpha Theta, Yeshiva College

The Yeshiva Mathematics Club, *Mu Alpha Theta*, was formed early in 1948. Since its inception, more than thirty papers have been presented to its members by students. In addition to major papers, there were some minor talks, including discussions on a number of problems which have appeared in the MONTHLY.

Other activities of the club included two parties, the presentation of a play at the College Class-Nite, and the planning of a Publication which later developed into a Yeshiva College Academic Journal.

The present officers are: President, William Frank; Vice-President, Seymour Haber; Secretary, Azriel Rosenfeld.

Kappa Mu Epsilon, Hofstra College

Regular monthly meetings were held by the *New York Alpha* Chapter of *Kappa Mu Epsilon* during 1948–49. Speakers and the titles of their papers were as follows:

Nomograms, by Leonard Hinder

Introduction to vector analysis, by Dr. L. F. Ollmann

Projectiles and parabolae, by Frank Hawthorne

Elements of Lewis Carroll, by Dorothea Reiffel

Huygen's relations with France, by Dr. Henri Brugmans of the Department of French

The useful and the useless in mathematics, by Dr. E. R. Stabler

Models and methods, by Lt. Col. Robert Yates of West Point, in which models for the construction of plane curves, diagrams, and linkages were presented.

The chapter members joined the Adelphi College Mathematics Club in hearing two lectures: *Diagrams and models*, by Col. Robert Beard; and *The number e and the law of continuing growth*, by Dr. Howard Fehr of Columbia University.

Two student and two faculty members attended the National Convention at Topeka, Kansas in April. Miss E. Marie Hove was re-elected National Secretary and Dr. L. F. Ollmann was re-elected National Treasurer of Kappa Mu Epsilon.

Pi Mu Epsilon, University of Kansas

The *Kansas Alpha* Chapter of *Pi Mu Epsilon* reports the following activities:

A get-acquainted picnic early in the fall term for all members of the chapter and all members of the mathematics staff.

A tea in January at which 13 new members were initiated.

Annual banquet in May at which Dr. N. W. Storer, of the Physics department, spoke on *Consider the universe.*

Officers for 1949–50 are: Director, Otho M. Rassmussen; Vice-Director, Martha Wagner; Corresponding Secretary, Wealthy Babcock; Recording Secretary, Frances Wolfe; Treasurer, Kathleen O'Donnell; Librarian, Gilbert Ulmer.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

ACKNOWLEDGMENT

The editor of the MONTHLY wishes to make grateful acknowledgment of the services rendered by the following persons who have refereed papers: A. A. Albert, C. B. Allendoerfer, H. T. R. Aude, J. M. Barbour, E. T. Bell, Lipman Bers, L. M. Blumenthal, P. W. Bridgman, J. R. Britton, Herbert Busemann, W. B. Carver, Alonzo Church, C. J. Coe, I. S. Cohen, J. L. Coolidge, A. H. Copeland, H. S. M. Coxeter, P. H. Daus, J. B. Diaz, Arnold Dresden, F. G. Dressel, P. S. Dwyer, J. D. Elder, Wade Ellis, H. W. Eves, C. D. Firestone, J. S. Frame, Abraham Franck, E. T. Frankel, Philip Franklin, G. E. Hay, E. Marie Hove, Ralph Hull, P. M. Hummel, R. D. James, R. A. Johnson, B. W. Jones, P. S. Jones, Edward Kasner, V. L. Klee, Jr., S. C. Kleene, R. E. Langer, H. D. Larsen, C. C. MacDuffee, Morris Marden, N. H. McCoy, Karl Menger, D. C. Murdoch, C. D. Olds, Gordon Pall, W. V. Parker, G. P. Price, G. Y. Rainich, E. D. Rainville, W. T. Reid, G. deB. Robinson, Louise J. Rosenbaum, Arthur Rosenthal, Vera Sanford, Robert Schatten, I. J. Schoenberg, Wladmir Seidel, Samuel Selby, I. M. Sheffer, W. H. Simons, F. C. Smith, Ruth B. Smyth, Pauline Sperry, Robert Stalley, F. H. Steen, Gabor Szegö, H. S. Uhler, Henry Van Engen, R. W. Wagner, R. J. Walker, J. L. Walsh, A. R. Williams, John Williamson, H. E. Wolfe, R. C. Yates, Leo Zippin.

BENJAMIN PEIRCE INSTRUCTORSHIPS AT HARVARD UNIVERSITY

The Mathematics Department of Harvard University expects to have one or two vacancies in the rank of Benjamin Peirce instructor beginning in the fall of 1950. Appointments are for a maximum period of three years. Applications should reach the Department prior to March 15, 1950.

EXECUTIVE DIRECTOR OF THE AMERICAN MATHEMATICAL SOCIETY

The Council and Trustees of the American Mathematical Society take pleasure in announcing the appointment of Dr. H. M. MacNeille as Executive Director of the Society. Dr. MacNeille assumed his duties in the New York office of the Society on November 15, 1949. The position of Executive Director to aid in the administration of the affairs of the Society was created by the Council and approved by the Trustees, on recommendation of a Committee on Reorganization. The members of the Society approved this action of the Council when the new By-Laws were adopted at the Annual Meeting in December, 1948.

Dr. MacNeille was graduated with highest honors from Swarthmore College in 1928. For two years he was Rhodes Scholar from New Jersey at Balliol College, Oxford, and took an honors degree in mathematics at that university. He received the degree of Doctor of Philosophy from Harvard University in 1935, the subject of his dissertation being *Extensions of partially ordered sets*. During the academic year 1935–1936 he was Sterling Fellow at Yale University, from 1936–1938 he was an Instructor of Mathematics at Harvard University, and from 1938 to 1947 he was Professor of Mathematics at Kenyon College. At the close of the Army and Air Force training programs at Kenyon College early in 1944 Dr. MacNeille was appointed Scientific Liaison officer and later head of the London Mission of the Office of Scientific Research and Development. In 1946 he became the Scientific Director of the newly established Office of the Assistant Naval Attache for Research in London. Since January, 1948 he has been Chief of the Fundamental Research Branch of the United States Atomic Energy Commission in Washington.

PERSONAL ITEMS

Professor A. T. Brauer of the University of North Carolina has received one of the three inaugural Science Research Awards of the Oak Ridge Institute of Nuclear Studies for "significant contributions to science in the South."

Dean E. L. Mackie of the University of North Carolina represented the Association at the installation of President A. H. Edens of Duke University on October 22, 1949.

Dr. C. V. Newsom, Assistant Commissioner for Higher Education for the State of New York, was the representative of the Association upon the commemoration of the 125th Anniversary of Rensselaer Polytechnic Institute on October 12–15, 1949.

Professor J. B. Reynolds of Lehigh University served as the representative of the Association at the inauguration of President D. E. Weinland of Moravian Seminary and College for Women on October 22, 1949.

Dr. S. A. Schelkunoff of the Bell Telephone Laboratories was the delegate of the Association at a meeting of the American Standards Association on October 14, 1949.

Assistant Professor Edith R. Schneckenburger of the University of Buffalo represented the Association at the installation of the Very Reverend Juvenal Lalor as president of St. Bonaventure College on September 22, 1949.

Professor J. M. Thomas, Duke University, was appointed to represent the Association at the inauguration of President A. G. Moron of Hampton Institute on October 29, 1949.

Montana State University announces: Assistant Professor Harold Chatland of Ohio State University has been appointed to a professorship; Mr. R. L. Berggren, University of Minnesota, has been appointed to an instructorship.

The Institute for Mathematics and Mechanics of New York University reports: Professor Emil Artin of Princeton University, Professor Hans Rade-

macher of the University of Pennsylvania, and Professor Carl Siegel of The Institute for Advanced Study have been appointed Visiting Professors; Dr. Bernard Friedman, who is a research associate at the Institute, has been promoted to an associate professorship at Washington Square College.

At The Oklahoma Agricultural and Mechanical College the Department of Mathematics is now offering the Ph.D. degree. This expansion is being made with the support of a research contract with the Office of Naval Research. The work of this contract is under the direction of Research Professor Nachman Aronszajn. The following are also participating in this work: Professors A. H. Diamond, O. H. Hamilton, H. W. Smith; Research Assistants M. Arnsten, C. J. Clark, Alan Jennings, L. F. McAuley, G. B. Pedrick, G. D. Pirtle, A. W. Wortham, Arthur Zeichner. The following research divisions are supporting the program: The Research Foundation, The Agricultural Experiment Station, The Division of Engineering Research and Experiment Station and The Oklahoma Power and Propulsion Laboratory.

The University of Buffalo makes the following announcements: Mr. Benjamin Lapidus of the University of Toronto has been appointed to an instructorship; Mr. A. G. Fadell and Mr. L. J. Montzingo have received appointments as teaching fellows.

University of California at Los Angeles reports: Professor J. B. Rosser of Cornell University, who has the position of Director of Research at the Institute for Numerical Analysis, has been appointed Visiting Professor of Mathematics; Assistant Professor F. A. Valentine has been promoted to an associate professorship and has been granted a year's leave of absence for the purpose of further study at the University of Chicago; Dr. L. J. Paige and Dr. J. D. Swift have been promoted to assistant professorships; Dr. P. G. Hodge, Jr. of Brown University has been appointed to an assistant professorship; Dr. G. M. Wing, Cornell University, has been appointed to an instructorship; Assistant Professor E. R. Worthington has retired.

University of South Carolina announces: Adjunct Professor Marguerite Z. Hedberg has been promoted to an associate professorship; Mr. C. W. Huff of Pennsylvania State College has been appointed to an instructorship.

Washington University announces the appointments of Dr. I. I. Hirschman, Harvard University, to an assistant professorship and of Dr. H. Margaret Elliott, Radcliffe College, to an instructorship.

Professor E. P. Adams of Princeton University has retired with the title of Emeritus Professor of Physics.

Professor V. W. Adkisson, chairman of the Department of Mathematics of the University of Arkansas, has been appointed Dean of the Graduate School.

Dr. A. V. Baez, formerly instructor in the Department of Physics at Stanford University, has accepted an appointment as a research physicist at the Cornell Aeronautical Laboratory, Buffalo, New York.

Mr. R. T. Bailey has been appointed to an instructorship at Norwich University.

Brother Ladislaus Balbert, formerly at Christian Brothers College, Memphis, is now an instructor at Colegio San Jose, Bluefield, Nicaragua.

Dr. R. E. Basye has been promoted to an associate professorship at Agricultural and Mechanical College, Texas.

Mr. Leon Benson of Lehigh University is teaching now at Blasdell High School, Buffalo, New York.

Professor H. L. Black of Michigan State College has accepted an appointment as associate professor in the Department of Engineering, Pennsylvania State College.

Associate Professor R. C. Blackwell has been promoted to a professorship at Furman University.

Dr. A. L. Blakers of Princeton University has been appointed to an assistant professorship at Lehigh University.

Dr. J. H. Blau, Massachusetts Institute of Technology, has received an appointment as assistant professor at Pennsylvania State College.

Associate Professor F. E. Bowling has been promoted to the position of Head of Department of Mathematics, Lincoln Memorial University.

Dr. L. E. Boyer has been promoted to the chairmanship of the Department of Mathematics, State Teachers College, Millersville, Pennsylvania.

Dr. R. A. Bradley has been appointed to an assistant professorship at McGill University.

Dean J. W. Branson, who has been serving as acting president of The New Mexico College of Agriculture and Mechanic Arts, has been appointed President.

Miss Jeneva J. Brewer, formerly graduate fellow at the University of Wichita, has been appointed to an instructorship.

Assistant Professor J. B. Brewster of Clemson Agricultural College has been promoted to an associate professorship.

Mr. B. B. Clark, previously graduate student at the University of Michigan, has been appointed to an instructorship at Grinnell College.

Instructor R. L. Coker, Southern College of Optometry, has been appointed Professor of Mathematics and Physics.

Assistant Professor R. H. Cole has been promoted to an associate professorship at the University of Western Ontario.

Assistant Professor Mamie M. Davis, Southeastern Louisiana College, has accepted an instructorship at Mississippi Southern College.

Mr. B. B. Dressler has been promoted to an assistant professorship at Kent State University.

Dr. J. C. Eaves has been appointed to an assistant professorship at the University of Alabama.

Mr. J. L. Ericksen has been appointed to a teaching fellowship at Indiana University.

Assistant Professor Walter Fleming, Fort Hays Kansas State College, is now

Associate Professor of Mathematics at Mankato State Teachers College, Minnesota.

Mr. Charles Franciol is teaching at Gillis High School, Lake Charles, Louisiana.

Dr. M. J. Gottlieb, who has been at the Institute for Advanced Study, has been appointed to an assistant professorship at the University of Chicago.

Miss Alice Graeber, who has been teaching at Dates School, San Francisco, has accepted a position as teacher at Mission High School, San Francisco.

Mr. H. H. Harman, formerly research associate, has been promoted to the position of Chief of the Statistical Research and Analysis Unit, Department of Army.

Mr. J. R. Hodges, teaching fellow at Tulane University, has received an appointment as instructor at Little Rock Junior College.

Mr. Julius Honig has the position of Research Assistant at Long Island College of Medicine.

Mr. G. K. Howe of Bell Aircraft Corporation, Marietta, Georgia, has retired.

Mr. Eugene Isaacson has been appointed to an assistant professorship at New York University.

Dr. John Kronsbein has been appointed Head of the Department of Engineering of Evansville College.

Dr. Paolo Lanzano of the University of Rome has received an appointment as instructor at St. Louis University.

Dr. Lee Lorch, City College of the City of New York, has been appointed to an assistant professorship at Pennsylvania State College.

Professor E. J. Lowry has been promoted to the position of Head of the Department of Mathematics, Hastings College.

Miss Hazel L. Mason, formerly instructor at Mary Hardin-Baylor College, is now a member of the faculty of the Grand Prairie Public Schools, Texas.

Professor J. C. C. McKinsey has returned to the Oklahoma Agricultural and Mechanical College after spending the last two years on leave with the Rand Corporation.

Dr. G. D. Mostow, who has been at the Institute for Advanced Study, has been appointed to an assistant professorship at Syracuse University.

Mr. E. J. Musch has been appointed to an instructorship at the University of Louisville.

Mr. J. A. S. Neilson has been promoted to the chairmanship of the Department of Mathematics, Westmont College.

Associate Professor H. E. Nelson of Gustavus Adolphus College has been appointed to an associate professorship at Augustana College.

Mr. J. A. Nickel is now serving as graduate assistant at Oregon State College.

Dr. Helen K. Nickerson has been promoted to an assistant professorship at Wheaton College, Massachusetts.

Mr. J. V. Pennington, formerly associate director of Southwest Research Institute, San Antonio, has been appointed Technical Director of Drilling Research Inc., Houston.

Associate Professor P. M. Pepper of the University of Notre Dame has been appointed Associate Professor of Industrial Engineering at Ohio State University.

Mr. C. R. Perisho has been promoted to an associate professorship at Nebraska Wesleyan University.

Professor Mary Pettus, Lander College, has been appointed to an assistant professorship at the University of Richmond.

Professor J. C. Polley of Wabash College has been promoted to the position of Head of the Department of Mathematics.

Associate Professor D. W. Pugsley has been promoted to the position of Professor and Head of the Department of Mathematics at Berea College.

Professor H. R. Pyle of Whittier College is now Head of the Department of Mathematics.

Assistant Professor G. E. Reves of The Citadel has been promoted to an associate professorship.

Professor B. D. Roberts, New Mexico Highlands University, has been promoted to the position of Dean and Head of the Department of Mathematics.

Miss Margarita E. Rodríguez, formerly at Montanzas Instituto, Cuba, is teaching at Instituto del Vedado, Cuba.

Assistant Professor D. R. Ryan of Gonzaga University is now Chairman of the Department of Mathematics.

Mr. A. H. Sarno, St. John's College, has been promoted to an assistant professorship and chairmanship of the Department of Mathematics.

Mr. F. W. Saunders has been promoted to a professorship at Coker College.

Mr. J. A. Schumaker of MacMurray College for Women has been appointed to an assistant professorship.

Mr. Whitney Scobert, formerly at University of Oregon, has accepted a position as associate professor at Idaho State College.

Mr. R. C. Seber, formerly graduate assistant at the State University of Iowa, has been appointed to an instructorship at Rockford College.

Mr. D. P. Shore is teaching at Carter-Riverside High School, Fort Worth.

Mr. C. E. Shotwell, who has been an assistant instructor at the University of Missouri, has received an appointment as instructor at the University of the South.

Mr. A. T. Skinner has been appointed to an assistant professorship at Champlain College.

Professor A. A. Smith, previously chairman of the Department of Mathematics of San Angelo College, has been appointed Business Manager of Texas Western College.

Associate Professor Ernst Snapper of the University of Southern California is now Visiting Associate Professor at Princeton University.

Assistant Professor P. I. Speicher has been promoted to an associate professorship at Albright College.

Mr. Isaay Stempnitzky is serving as research assistant in Electrical Engineering at Massachusetts Institute of Technology and teaching fellow at Brandeis University.

Mr. R. F. Steward, formerly teaching assistant at New Jersey College for Women, has been appointed to an instructorship at Virginia Military Institute.

Assistant Professor C. F. Strobel of North Carolina State College has been promoted to an associate professorship.

Mr. J. S. Stubbe of the University of New Hampshire has been appointed to an assistant professorship at Clark University.

Mr. Kaidy Tan is now Tutor at Anglo-Chinese College, Kulangsu, Amoy, China.

Professor Alex Tartler of Drexel Institute has been appointed to an associate professorship at Lafayette College.

Mr. B. P. Taylor is now Head of the Calculation Group, Lofting Department, Douglas Aircraft Company.

Mr. C. M. Terry has been appointed to a fellowship at Iowa State College.

Instructor O. M. Thomas of the United States Naval Academy has been promoted to an assistant professorship.

Mr. C. W. Trigg has been appointed Assistant to the Dean of Instruction, Los Angeles City College.

Dr. W. R. Utz, University of Michigan, has been appointed to an assistant professorship at the University of Missouri.

Miss Frances Weisbecker of Milwaukee-Downer College has been promoted to an assistant professorship.

Assistant Professor D. W. Whitford has been promoted to an associate professorship at Polytechnic Institute of Brooklyn.

Professor P. D. Wilkins of Bates College has been promoted to the position of Head of the Department of Mathematics.

Mr. A. D. Wirshup, formerly an instructor at Multnomah College, has accepted a teaching fellowship at Oregon State College.

Miss Ferna E. Wrestler has been promoted to an assistant professorship at the University of Wichita.

Miss Frances M. Wright has been promoted to an assistant professorship at Triple Cities College of Syracuse University.

Associate Professor A. J. Zanolar has been appointed President of St. Joseph's College, Indiana.

Mr. R. K. Zeigler has been appointed to an associate professorship at Bradley University.

Professor Emeritus S. E. Brasefield of Rutgers University died October 14, 1949.

Professor A. F. Carpenter, formerly executive officer of the Department of Mathematics of the University of Washington, died on October 16, 1949.

Professor Emeritus Peter Field of the University of Michigan, died on September 24, 1949. He was a charter member of the Association.

Professor F. A. Foraker of the University of Pittsburgh, died on June 30, 1949. He was a charter member of the Association.

Associate Professor R. K. Luneburg of the University of Southern California died on August 19, 1949.

Dr. D. A. Rothrock, professor emeritus of Indiana University and a charter member of the Association, died September 2, 1949.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 91 persons have been elected to membership by the Board of Governors on applications duly certified:

J. D. R. BAHNG, Student, St. Norbert College, West DePere, Wis.

ESTHER P. BARNHART, M.A.(Michigan) Instructor, Capital University, Columbus 9, Ohio.

R. D. BARTZ, Student, University of Wisconsin, Milwaukee 3, Wis.

R. W. BEALS, JR., M.Ed.(Alfred) Instructor, Alfred University, N. Y.

R. W. BIRD, B.A.(Roosevelt) Bookkeeper, Hartford Fire Insurance Co., Chicago, Ill.

W. W. BLAKELY, B.A.(Macalester) Grad. Asst., Macalester College, St. Paul, Minn.

I. H. BRUNE, Ph.D.(Ohio State) Asso. Professor, Iowa State Teachers College, Cedar Falls, Iowa.

R. K. BUTZ, B.S.(Colorado A & M) Grad. Student, University of Georgia, Athens, Ga.

W. W. CHAMBERS, M.A.(Colorado State) Instructor, Purdue University, Indianapolis, Ind.

M. H. CLAYTON, M.E.(North Carolina) Instructor, North Carolina State College, Raleigh, N. C.

L. W. COHEN, Ph.D.(Michigan) Professor, Queens College, Flushing, L. I.

J. W. COY, M.A.(New Mexico) Instructor, Michigan State College, East Lansing, Mich.

W. V. CRESSY, Student, University of California, Los Angeles, Calif.

W. F. CRUM, B.A.(Carleton) Instructor, Carleton College, Northfield, Minn.

DOROTHY S. CURRY, M.S.(Chicago) Asso. Professor, Wilberforce University, Ohio.

T. S. DEAN, M.S.(North Texas) President, Design Associates, Inc., Sherman, Texas.

V. E. DIETRICH, M.Sc.(Notre Dame) Grad. Asst., Purdue University, Lafayette, Ind.

D. W. DUBOIS, B.S.(Oklahoma) Grad. Asst., University of Oklahoma, Norman, Okla.

ARTHUR ERDELYI, D.Sc.(Edinburgh) Professor, California Institute of Technology, Pasadena, Calif.

LORRAINE W. FARBER, M.A.(Buffalo) Instructor, University of Buffalo, N. Y.

JEAN B. FEIDNER, B.A.(William Smith) Instructor, University of Buffalo, N. Y.

RUDOLF FEIGE, M.A.(Berlin) Instructor, University of Cincinnati, Ohio.

J. R. FLEMING, Captain, U.S.A.F., Air Weather Service, Warner Robins, Ga.

F. A. FOGG, B.S.(Chile) Engr., Collins Radio Co., Cedar Rapids, Iowa.

P. L. FORD, M.S.(Louisiana State) Instructor, Kemper Military School, Boonville, Mo.

J. H. FOUNTAIN, B.A.(Buffalo) Grad. Student, University of Buffalo, N. Y.

G. C. FRANCIS, M.S.(Minnesota) Instructor, Carleton College, Northfield, Minn.

W. M. FRANK, Student, Yeshiva University, Brooklyn, N. Y.

F. C. GERMAN, M.A. (Illinois) Asso. Professor, Kansas State Teachers College, Pittsburg, Kansas.

GLADYS E. GILLMAN, B.A. (Skidmore) Teacher, Winsor School, Boston, Mass.

R. D. GLAUZ, M.S. (Michigan) Instructor, Montana State College, Bozeman, Mont.

H. E. GOHEEN, Ph.D. (Stanford) Asst. Professor, Syracuse University, N. Y.

B. T. GOLDBECK, M.A. (Texas Christian) Student, Texas Christian University, Fort Worth, Texas.

BEULAH GRAHAM, M.A. (Kentucky) Teacher, Campbellsville College, Ky.

U. S. GREENE, C.P.A., Marine Midland Bldg., Binghamton, N. Y.

VIRGINIA M. HALL, A.M. (Boston Univ.) Instructor, Simmons College, Boston, Mass.

PARKER HAMILTON, B.S. (Harvard) Asst. Professor, Antioch College, Yellow Springs, Ohio.

C. L. HARBISON, M.A. (Indiana) Asst. Professor, South Dakota School of Mines and Technology, Rapid City, S. D.

GERALD HARRISON, Ph.D. (Calif. Inst.) Asst. Professor, Wayne University, Detroit, Mich.

M. J. HELLMAN, M.S. (CCNY) Instructor, Rutgers University, Newark, N.J.

B. O. HOYLE, M.S. (Southern Illinois) Instructor, Southern Illinois University, Carbondale, Ill.

W. R. HYDEMAN, A.M. (Syracuse) Mathematician, Navy Department, Washington, D. C.

EUGENE ISAACSON, Ph.D. (New York) Instructor, New York University, N. Y.

P. W. M. JOHN, M.A. (Oxon) Instructor, University of Oklahoma, Norman, Okla.

G. P. JOHNSON, M.A. (Minnesota) Grad. Student, University of Minnesota, St. Paul, Minn.

R. P. KANIA, M.A. (New Zealand) Lecturer, Teachers Training College, Auckland, N. Z.

C. E. KELLEY, B.S. (Central Missouri) Asst. Instructor, University of Missouri, Columbia, Mo.

D. E. KIBBEY, Ph.D (Illinois) Asso. Professor, Syracuse University, N. Y.

JOSEPH KLEIN, B.S. (Rutgers) Student, Rutgers University, New Brunswick, N. J.

R. J. KOHLMEYER, M.A. (Washington) Instructor, Pratt Institute, Brooklyn, N. Y.

G. M. KUZNETS, Ph.D. (California) Asso. Professor, University of California, Berkeley, Calif.

VIVIAN E. LARSON, M.A. (Wisconsin) Instructor, University of Wisconsin, Milwaukee, Wis.

F. W. LOTT, JR., M.A. (Michigan) Asst. Professor, Iowa State Teachers College, Cedar Falls, Iowa.

R. E. LOWNEY, M.A. (Michigan State) Asst. Professor, Montana State College, Bozeman, Mont.

J. A. MARTIN, SR. Partner, National Oil Co., Lisbon, Ohio.

H. C. MCKENZIE, M.A. (Wisconsin) Instructor, University of Colorado, Boulder, Colo.

GENEVIEVE T. MEYER, M.A. (Marquette) Instructor, University of Wisconsin, Milwaukee, Wis.

MICHAEL MORELLI, S.B. (M.I.T.) Air Material Command, Cambridge Field Station, Mass.

F. J. MURRAY, Ph.D. (Columbia) Professor, Columbia University, New York, N. Y.

R. E. NELSON, M.A. (Dartmouth) Asso. Professor, Dickinson College, Carlisle, Pa.

E. N. NILSON, Ph.D. (Harvard) Asst. Professor, Trinity College, Hartford, Conn.

CHESTER PACHUCKI, M.S. (De Paul) Instructor, De Paul University, Chicago, Ill.

L. J. PAIGE, Ph.D. (Wisconsin) Instructor, University of California, Los Angeles, Calif.

A. J. PEJSA, M.S. (Marquette) Asst. Professor, U. S. Naval Academy, Annapolis, Md.

C. L. PERRY, JR., Ph.D. (Michigan) Asst. Professor, University of Arkansas, Fayetteville, Ark.

M. B. PORTER, M.A. (Texas) Asst. Professor, Southwest Texas State Teachers College, San Marcos, Texas.

R. E. PRIEST, Student, University of Illinois, Urbana, Ill.

N. J. ROSE, M.E. (Stevens) Instructor, Stevens Institute of Technology, Hoboken, N. J.

R. M. ROSS, M.A. (Indiana) Asst. Professor, Rose Polytechnic Institute, Terre Haute, Ind.

J. P. ROTH, M.S.(Michigan) Student, University of Michigan, Ann Arbor, Mich.

BARNEY SANDLER, B.A.(Brooklyn) Instructor, Institute of Applied Arts and Science, Brooklyn, N. Y.

W. L. SAUTER, Student, University of Louisville, Ky.

R. W. SCHMIED, B.A.(Texas) Instructor, Tulane University, New Orleans, La.

A. SCHUMAKER, M.A.(Illinois) Asst. Professor, MacMurray College for Women, Jacksonville, Ill.

SISTER MARY PETRONIA, Ph.D.(Notre Dame) Teacher, Mount Mary College, Milwaukee, Wis.

SISTER MIRIAM FRANCIS, Ph.D.(Catholic) Instructor, Xavier University, New Orleans, La.

R. D. STALLEY, M.A.(Oregon) Instructor, University of Arizona, Tucson, Ariz.

E. A. STAVINOHA, B.S.(Baylor) Grad. Fellow, Oklahoma A & M College, Stillwater, Okla.

ROSCOE STINETORF, Ph.D.(Pennsylvania) Professor of Physics, Wagner College, Staten Island, N. Y.

IRWIN STONER, M.A.(North Carolina) Instructor, University of Minnesota, Minneapolis, Minn.

J. R. SWAFFIELD, Student, Boston University, Boston, Mass.

N. Y. TANG, M.S.(Michigan) Lecturer, University of Washington, Seattle, Wash.

J. R. THOMPSON, M.A.(Minnesota) Instructor, University of Wisconsin, Milwaukee, Wis.

M. E. TITTLE, M.A.(Texas) Instructor, A & M College of Texas, College Station, Tex.

WILMONT TOALSON, M.A.(Kansas) Asst. Professor, Fort Hays Kansas State College, Kan.

C. M. TYLER, JR., Ph.D.(Pittsburgh) Asst. Professor, Carnegie Institute of Technology, Pittsburgh, Pa.

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D. D. WALL, Ph.D.(California) Instructor, University of California, Santa Barbara College, Calif.

FRANCES WEISBECKER, M.A.(North Dakota) Asst. Professor, Milwaukee-Downer College, Wis.

C. S. WILLIAMS, JR., M.A.(Oklahoma) Instructor, University of Oklahoma, Norman, Okla.

H. B. YOUNG, M.C.S.(Boston University) Instructor, Arkansas A. M. & N. College, Pine Bluff, Ark.

THE APRIL MEETING OF THE MISSOURI SECTION

The Missouri Section of the Mathematical Association of America met in Mumford Hall on the campus of the University of Missouri in Columbia, on Saturday, April 9, 1949. Professor P. R. Rider, Chairman of the Section, presided at the morning session, and Professor P. B. Burcham presided at the afternoon session.

Seventy-four persons were in attendance including the following twenty-seven members of the Association: L. M. Blumenthal, C. H. Brown, P. B. Burcham, Mary L. Cummings, W. C. Doyle, R. E. Ekstrom, G. M. Ewing, C. V. Fronabarger, W. L. Graves, Nola L. Haynes, F. F. Helton, R. E. Hogan, L. O. Jones, C. W. Mathews, J. M. Marr, Karl Menger, E. F. Moore, A. D. Pierson, P. R. Rider, J. S. Rosen, J. W. Sawyer, C. E. Shotwell, R. L. Snider, W. L. Stamey, A. D. Talkington, Margaret F. Willerding, and W. D. Williams.

The following officers were elected for the coming year: Chairman, C. W. Mathews, Washington University; Vice-Chairman, G. H. Jamison, Northeast Missouri State Teachers College; Secretary-Treasurer, Margaret F. Willerding, Harris Teachers College. At the conclusion of the meeting tea was served in the lounge of Gwynn Hall.

The Section had Professor Karl Menger of the Illinois Institute of Technology as a guest speaker. Professor Menger spoke on *Some Aspects of the Concept of Convexity*. The following papers were also presented:

1. *An approximate solution of Stieltjes integral by use of Cantelli's inequality*, by Dr. Maria Castellani, University of Kansas City, introduced by the Secretary.

2. *The metrization of torsion*, by Dr. L. M. Blumenthal, University of Missouri.

The paper presents a metric definition of torsion of a metric space at an accumulation point p based merely upon the consideration of the mutual distances of points of quadruples. In distinction to the definition given by Alexits (*Compositio Mathematicae*, vol. 6 (1938–39), pp. 471–477) the new definition is (1) symmetric in all distances, and (2) avoids iterated limits. It is shown that the metric torsion so defined yields the classical torsion for curves of the kind usually dealt with in classical differential geometry.

3. *A reorganization of general mathematics of colleges*, by Dr. Margaret F. Willerding, Harris Teachers College.

Mathematical education is in a sad state of confusion. General mathematics, which is often all the mathematics a student gets in high school, is poorly conceived and inadequately presented. There is great need in this field for better organization and for better teaching. The great organizations of mathematicians and teachers of mathematics should present a united front for improvement to state officers of education and to others who plan the educational offering in the high school.

4. *The trends in mathematical education in high schools*, by Prof. G. H. Jamison, Northeast Missouri State Teachers College.

Every year more and more students who have had only one year of general secondary mathematics are entering colleges. These students are totally unprepared to enter the college freshman mathematics courses in general mathematics as they are now organized. A college course in general mathematics must be organized for these students. The course must supplement the general mathematics courses given in the secondary schools. It must be primarily designed for students who signify that they have no intention of taking more than one year of mathematics. The objectives of such a basic first year course must center around the needs—both present and future—of the college student. It must give the student a better grasp of mathematics as a whole and of the interdependence of its various parts, and accustom him to use, in later applications, the method best adapted to the problems at hand. In order to meet these objectives the content of the course must be studied on an elaborate scale, and the subject matter picked with care. The problem of the reorganization of general mathematics is a fertile field for research and experimentation, and is a challenge to every mathematician and educator.

5. *Use of mnemonic devices in mathematics*, by C. W. Mathews, Washington University.

In teaching mathematics, mnemonic devices are used at times when it is easier to learn the facts from the fundamentals. An example served as an illustration of the truth of this statement. An effective mnemonic device must be one which is based upon well known facts, and should be simple. It should not admit exceptions, or hamper generalizations of the concepts or procedures. Students can learn better with a personalized memory aid than with a poor "mail order" variety. Consider a mnemonic device carefully before you use it.

CHAS. W. MATHEWS, *Secretary-Treasurer*

THE APRIL MEETING OF THE UPPER NEW YORK STATE SECTION

The annual meeting of the Upper New York State Section of the Mathematical Association of America was held at the University of Buffalo, Buffalo, New York, on Saturday, April 30, 1949. The Chairman of the Section, Professor E. B. Allen of Rensselaer Polytechnic Institute, presided at the morning session; the Vice-Chairman, Professor W. H. Durfee of Hobart College, presided at the afternoon session. At the conclusion of the afternoon session tea was served.

One hundred twenty-three persons attended the meeting, including the following seventy members of the Association: Janet E. Abbey, E. B. Allen, H. P. Atkins, H. T. R. Aude, H. W. Baeumler, G. B. Banks, V. N. Behrns, R. L. Beinert, W. W. Bessell, I. S. Boak, A. L. Buchman, C. G. Burger, F. J. H. Burkett, I. S. Carroll, H. S. M. Coxeter, F. F. Decker, E. J. Downie, Walter H. Durfee, F. P. Egan, J. E. Freund, H. M. Gehman, B. H. Gere, R. D. Gordon, Lillian Gough, N. G. Gunderson, May N. Harwood, M. B. Haslam, Margaret W. Hickman, R. G. Hill, H. E. Hoffman, A. W. Jones, C. E. Lemke, Caroline A. Lester, J. V. Limpert, R. K. Longley, R. R. R. Luckey, Dis Maly, E. W. Marchand, June M. McArtney, Rudolph Meyer, F. E. Milliman, Harriet F. Montague, Mabel D. Montgomery, D. S. Morse, Abigail M. Mosey, C. W. Munshower, C. R. Newell, Ruth B. Noller, B. C. Patterson, Theresa L. Podmele, J. F. Randolph, P. C. Rapp, C. E. Rhodes, P. J. Schillo, Edith R. Schneckenburger, L. F. Scholl, B. B. Sharpe, William A. Smith, Ruth W. Stokes, D. D. Strebe, Mary C. Suffa, R. T. Tear, G. W. Walker, J. F. Wardwell, F. C. Warner, E. T. Welmers, Ina W. Welmers, Mary E. Williams, Frances M. Wright, R. C. Yates.

The following officers were elected: Chairman, Walter H. Durfee, Hobart College; Vice-Chairman, B. C. Patterson, Hamilton College; Secretary, C. W. Munshower, Colgate University. The Chairman was authorized to appoint a committee to study the relation between secondary school and college mathematics in New York State. It was announced that E. B. Allen had been elected Sectional Governor. The 1950 meeting will be held at Syracuse University in the spring of 1950.

On Friday evening, April 29, 1949, Professor H. S. M. Coxeter of the University of Toronto delivered a lecture on the Fenton Foundation of the University of Buffalo on the subject *Arithmetical Games and Phyllotaxis*; many members of the Association attended.

The following papers were presented:

1. *The excenters of a hyperbolic triangle*, by Professor H. S. M. Coxeter, University of Toronto.

Since the angle bisectors of a triangle ABC are the loci of points equidistant from two sides, any intersection of two bisectors lies also on a third; therefore, the six bisectors are the sides of a complete quadrangle. This simple argument is valid in elliptic geometry as well as in euclidean. But in hyperbolic geometry it fails to cover the case when the external bisectors of two of the angles are parallel or ultraparallel. With the aid of coordinates it is readily proved that the internal bisector of the third angle still belongs to the same pencil (of parallels or ultraparallels), so that

there is an escribed horocycle or hypercycle. A synthetic proof is obtained by exhibiting these three bisectors as the diagonals of a Brianchon hexagon whose sides are the tangents to the absolute conic at the six "ends" of the lines BC, CA, AB, taken in a suitable order (different from the natural order used for the concurrence of internal bisectors in Coxeter's *Non-Euclidean Geometry*).

2. *The problem of eight points*, by Professor W. B. Carver, Cornell University.

Because of illness Professor Carver was unable to be present; his paper will be published in this MONTHLY.

3. *The addition formulas of trigonometry*, by Professor J. F. Randolph, University of Rochester.

A student will grant that two points on the rim of a wheel remain the same distance apart regardless of how the wheel is turned, and must therefore grant, for A and B arbitrary angles, that

$$\cos (A - B) = \cos A \cos B + \sin A \sin B. \tag{1}$$

It is only necessary to consider the points represented by (1, 0), (cos $[A-B]$, sin $[A-B]$), (cos B, sin B) and (cos A, sin A), note that the distance between the first pair (seen by rotating the first pair along the unit circle through the angle B), express the equality of these distances by using the distance formula, and simplify. Since (1) holds without restrictions on the angles A and B, and the values of the trigonometric functions for quadrantal angles are known, we set $A=0$ in (1) to obtain $\cos(-B)=\cos B$; set $A=90°$ in (1) to obtain $\cos(90°-B)=\sin B$ and hence also $\cos B=\sin(90°-B)$; and from periodicity note $\sin(-A)=\sin(360°-A)=\sin(90°-(A-270°))=\cos(A-270°)=-\sin A$ (the last equality following from (1) with $B=270°$). Thus the sum and difference formulas and all consequences of them (including reduction formulas) are easily obtained without apology.

4. *An algebraic algorithm*, by Dr. N. G. Gunderson, University of Rochester.

The speaker described Rosser's algorithm for solving the linear Diophantine equation, and discussed several applications.

5. *Mathematics in New York State Institutes*, by Professor I. S. Boak, New York State Agriculture and Technical Institute, Canton, New York.

The purpose and scope of mathematics courses was discussed in relation to the technological courses offered at the Institutes. The influence of special requirements of technical courses on methods of teaching and content, and the provisions for variations in ability and mathematical preparation during the early weeks of the course was indicated. The work of the New York State curriculum workshop in relation to mathematics was described, and trends in this field were discussed. The paper concluded with a brief discussion of trends toward standardization of courses and testing programs.

6. *Flying inertias*, by Dr. E. T. Welmers, Bell Aircraft Corporation.

The determination of the inertia properties of aircraft or aircraft components furnishes an interesting application of the formulas developed in elementary calculus. The moments of inertia of the aircraft as a whole are essential in determining the flight path. In the design of control surfaces to prevent flutter, it is necessary to determine the static balance (dependent on the simple moment), moment of inertia about a hinge line, and the dynamic balance (dependent on products of inertia). Illustrations of these requirements were given for unconventional control surfaces involving, particularly, sweep and dihedral. More advanced mathematics is required to evaluate the virtual inertia of the air which surrounds an oscillating wing or the moment of inertia of liquid fuel. Finally, mention was made of the mathematical bases for certain experimental methods of determining inertias.

7. *Extra-sensory perception, a problem in the theory of confirmation*, by Professor J. E. Freund, Alfred University.

Professor Freund discussed the application of the frequency theory of probability to the concept of the confirmation of scientific hypotheses. The probability of a scientific theory was defined in terms of its associated class of observable phenomena, and a formula was derived to show how the number and variety of instances of confirmation effect the degree to which a theory may be considered to be confirmed. These results were applied to the problem of confirming an "unusual" theory, e.g. the theory of extra-sensory perception, and it was demonstrated why theories are not accepted at times in spite of the fact that the positive evidence seems overwhelming. The danger of the statement "It couldn't have happened by chance" was discussed with reference to this problem.

C. W. MUNSHOWER *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The thirty-third annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio on Saturday, April 2, 1949. Professor R. H. Marquis, Chairman of the Section, presided.

Ninety-two persons registered attendance, including the following sixty-five members of the Association: W. E. Anderson, Max Astrachan, Grace M. Bareis, I. A. Barnett, H. M. Beatty, Henry Blumberg, J. B. Brandeberry, Foster Brooks, C. D. Calhoon, V. B. Caris, E. H. Clarke, W. F. Cornell, Rufus Crane, Grace Cutler, Wayne Dancer, Violet Davis, R. H. Downing, B. B. Dressler, P. L. Evans, H. E. Fettis, B. C. Glover, L. J. Green, Marshall Hall, Jr., Frances Harshbarger, C. H. Heinke, P. S. Herwitz, Carl Holtom, S. J. Jasper, E. D. Jenkins, Margaret E. Jones, Chosaburo Kato, L. C. Knight, L. L. Lowenstein, W. C. Lowry, R. H. Marquis, H. R. Mathias, E. J. Mickle, L. H. Miller, Max Morris, H. C. Parrish, H. S. Pollard, Irene Price, Tibor Rado, S. E. Rasor, O. W. Rechard, F. W. Reed, P. V. Reichelderfer, S. A. Rowland, K. C. Schraut, C. E. Sealander, J. L. Solomon, V. C. Stechschulte, R. L. Swain, Hellen Tappan, H. E. Tinnappel, H. S. Toney, E. P. Vance, R. W. Wagner, D. R. Whitney, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, E. H. Wohler, Alberta Wolfe, G. F. Woodson, Jr.

The following officers were elected for the coming year: Chairman, E. P. Vance, Oberlin College; Secretary-Treasurer, Foster Brooks, Kent State University; Member of Executive Committee, R. L. Swain, Ohio State University; Program Committee, I. A. Barnett, University of Cincinnati (Chairman), L. C. Knight, Jr., Muskingum College, H. R. Mathias, Bowling Green State University.

The following papers were presented:

1. *Some current questions in administration of college mathematics*, by Professor R. H. Marquis, Ohio University.

The speaker made the following points: Textbooks written for high school students of elementary algebra and plane geometry are not suited to the more mature students required to take these subjects in college, hence there is a need for new texts for them. There is a great deal of

duplication in courses in elementary statistics given in various departments. The requirement of a course in the fundamentals given by the mathematics department would permit better courses to be built by the specialists in the other fields. The problems of securing a well-trained staff are still acute but becoming easier. The high schools will soon resume the trend toward the masters degree for the better jobs. The field of education is the easiest field for the schoolman to follow for this degree. A better approach to the needs of these men is needed on the part of mathematicians if they are to share in the advanced preparation of secondary teachers.

2. *Dynamics of variable mass systems*, by Professor L. F. Doty, University of Cincinnati (introduced by Professor I. A. Barnett).

It is known that the classical laws of motion of Newtonian mechanics require special interpretation when applied to a particle of varying mass. This concept is reviewed and extended to a system of particles of variable mass. By adopting a postulate concerning the continuous distribution of matter, vector equations of motion are derived for a finite body whose mass is a function of the time. As a special case, the scalar equations of plane motion are exhibited and compared with the usual rigid body equations.

3. *Some extensions of a geometrical construction problem*, by Mr. L. L. Ross, Ohio State University (introduced by the Secretary).

This paper presents several distinct solutions of the problem: Given four points, to construct a line through each such that they intersect to form a square. The number of solutions, and some special cases, are also discussed.

One of the methods used in solving the square problem is extended to solve these related problems: (1) Construct a quadrilateral similar to a given one so that each side contains one of four given points; (2) Construct lines tangent to each of four given circles so that the lines form a quadrilateral similar to a given one; (3) In a fixed quadrilateral, inscribe a quadrilateral similar to a given one.

By considering a triangle as a special case of a quadrilateral, one obtains solutions to problems such as that of constructing a triangle, congruent to a given one, having each side tangent to one of three given circles.

4. *Linear periodic functions*, by Professor F. W. Reed, Ohio University.

The linear periodic functions ($\sin t$ and $\cos t$ and others derived from these by iteration) are used for writing the equations of figures made up of polygons and their degenerate forms. Segmental curves of cursive type are analysed into periodic and secular parts. Examples are: field of hexagons, packed circles, family of all regular polygons, checker-board. The methods yield equations of surfaces having plane and curved facets, including the polyhedrons, the crystals, toroids, etc. Other examples are: packed spheres, stacked cubes, and honey-comb surfaces.

5. *A machine for solving simultaneous linear equations*, by Mr. Edmund Churchill and Mr. Herman Berman, Antioch College (introduced by Professor Max Astrachan).

A machine for the solution of simultaneous linear equations by the transformation of the mathematical problem into an electrical problem was demonstrated. Multiplications and additions of currents needed for solution is accomplished by simple resistances and potentiometers. Circuit diagrams together with directions for the construction of a model (cost about $35.00) for the solution of two equations (can be extended to more variables) are available from the authors.

6. *History of mathematics for secondary school teachers*, by Professor O. L. Dustheimer, University of Toledo.

After a short review of the general plan of a course for young teachers of high school mathe-

matics, several allied topics were discussed. The real history of elementary mathematics centers around fifty-seven mathematicians. The special topics mentioned were: old, new, and world calendars, square and cube roots by arithmetic, $\sqrt{2}$, $\sqrt{3}$, π, insurance and annuities, and determination of the date of Easter.

7. *Discussion of classroom notes.*

This discussion was led by Professor Wayne Dancer, University of Toledo, with contributions by R. L. Swain, I. A. Barnett, F. B. Wiley, H. S. Toney and Tibor Rado.

8. *Some questions in metric geometry*, By Professor E. J. Mickle, Ohio State University.

Professor Mickle's paper was a one-hour address delivered by invitation of the Program Committee. Let T and T^* be two continuous mappings from a unit square Q in a plane into euclidean three space E_3. Assume that for every pair of points in Q the images under T^* are never farther apart than the corresponding images under T. Then T and T^* are representations of surfaces whose Lebesgue areas $A(T)$ and $A(T^*)$ satisfy the inequality $A(T^*) \leqq A(T)$. The proof of this inequality utilizes the fact that a Lipschitzian transformation with constant k from a set in a euclidean space into a euclidean space can be extended to a Lipschitzian transformation with the same constant k defined on the whole space. Recent work in surface area utilizes a similar result on the extension of a Lipschitzian transformation in Banach spaces.

FOSTER BROOKS, *Secretary*

CALENDAR OF FUTURE MEETINGS

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Southern Illinois University, Carbondale, May 12–13, 1950.

INDIANA, Wabash College, Crawfordsville, April 29, 1950.

IOWA, State University of Iowa, Iowa City, April 21–22, 1950.

KANSAS, Spring, 1950.

KENTUCKY, University of Kentucky, Lexington, April 29, 1950.

LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, February 24–25, 1950.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.

METROPOLITAN NEW YORK, Spring, 1950.

MICHIGAN, March, 1950.

MINNESOTA, Macalester College, St. Paul, May 6, 1950.

MISSOURI, Spring, 1950.

NEBRASKA, Nebraska Wesleyan University, Lincoln, May 6, 1950.

NORTHERN CALIFORNIA, Berkeley, January 28, 1950.

OHIO, Denison University, Granville, April 22, 1950.

OKLAHOMA

PACIFIC NORTHWEST, University of Washington, Seattle, June, 1950.

PHILADELPHIA

ROCKY MOUNTAIN, University of Denver, April, 1950.

SOUTHEASTERN, University of Florida, Gainesville, April 7–8, 1950.

SOUTHERN CALIFORNIA, Immaculate Heart College, Hollywood, March 11, 1950.

SOUTHWESTERN, Spring, 1950.

TEXAS, Abilene, Spring, 1950.

UPPER NEW YORK, Syracuse University, Spring, 1950.

WISCONSIN, Marquette University, Milwaukee, May, 1950.

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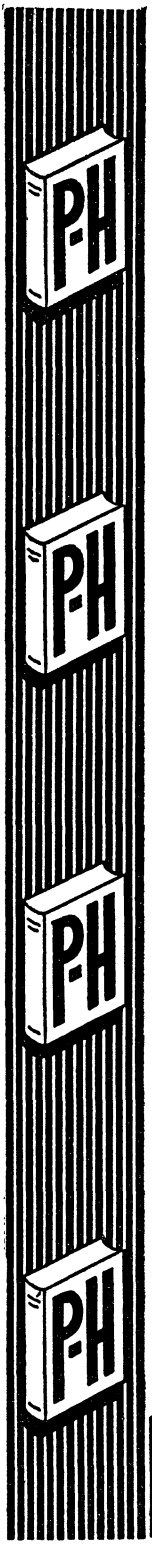
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OBJECTIVES IN THE TEACHING OF COLLEGE MATHEMATICS*

F. S. NOWLAN, University of Illinois

1. Introduction. Any constructive study of objectives in the teaching of mathematics must presuppose an agreement as to what is meant by mathematics. In this paper, we concern ourselves with mathematics in the sense of mathematical thinking and not as an inclusive term applying to the vast body of mathematical truths; also we are not thinking primarily of the mastery of rules and devices for the manipulation of numbers and algebraic expressions. What, then, is mathematics in this sense and what is its value?

It is a truism to state that mathematical thinking is deductive reasoning, but we can profitably examine this statement. Deductive reasoning is based upon assumptions, and these apply to what, for lack of a better word, we will designate elements. The elements may, in part, be accepted without definition, for example, our concept of point in geometry. Any mathematical development begins by assuming that the elements under consideration satisfy certain conditions, that is, are subject to certain rules of behavior. Mathematics consists in the logical deduction of the necessary consequences of the initial assumptions.

It may happen that the assumptions clash, or lead to conclusions which contradict established or accepted theory. One may then modify the assumptions, or start with a new set, or, in the latter contingency, one may be radical and hold to the initial assumptions and ignore the accepted theory. This radical action has occurred frequently in the development of science and has made scientific progress possible.

The procedure that we have outlined is basic in mathematical thinking, and, although not always followed, is useful in every field of human endeavor.

Unfortunately, mathematics, in this sense, will fail to bring about a Utopia. The initial assumptions may include the doctrine that the greatest good results from self-advancement, or that one's chief mission in life is the dissemination of certain creeds or ideologies, or the retention of certain creeds or ideologies, and that the end justifies the means. The individual may then reason logically and honestly, that is, in agreement with his assumptions, but the result may be a Hitler, or worse, whose success or failure, proves or disproves (according to accepted standards of proof) his theories.

All this goes to show that mathematical thinking, in itself, is no cure-all for human ills, but it is essential if these ills are to be alleviated. To repeat, mathematical thinking merely means the study of the necessary consequences of initial assumptions which motivate courses of action. As such it applies even to the social and economic sciences. It is non-moral in character and has no concern with variable standards of rightness or wrongness in behavior.

* Presented to the Illinois Section of the Association at Chicago, Illinois, May 14, 1948. Also presented at the meeting of the National Council, Columbus, Ohio, December 29, 1948.

2. Objectives in teaching mathematics. By now you are wondering what this has to do with objectives in the teaching of college mathematics. Let me explain, and in doing so you will understand that I am merely presenting my personal views and reactions to conditions which are fairly common in our colleges and universities. Also, I would emphasize at this point that I do not underestimate the value of practical applications in the teaching of mathematics, nor the necessity for the mastery of mechanical skills in numerical and algebraic operations. I assume that these are taken care of, as a matter of course, in our instruction.

There are two main aims in the teaching of college mathematics, whether to liberal arts students or to students of engineering. These, in order of importance, are: (1) Training in precise thinking and a grasp of basic principles. (2) The acquisition of information and a mastery of certain technical skills.

It appears, however, that, in general, a disproportionate stress has been placed upon the second objective to the almost total neglect of the first. The excuse has been that the time at the instructor's disposal is limited and that present day students are poorly prepared and intellectually immature.

In answer to these objections, we point out that mechanical work that is not based upon an understanding of principles has little value, practical or otherwise, and that a student can progress rapidly, and without mental strain, only when he works understandingly.

It is true that in every student body there are those who are incapable of logical thinking but we should hardly permit them to determine our standards. Also, there are those who are so gifted intellectually that they will progress under any system and in spite of handicaps of instruction and textbooks. It happens, however, that the majority of students are in a middle class; they need guidance and can profit from guidance. They are capable of logical thinking but, in many cases, have never undergone that experience. These students constitute the main reason for instructors in our colleges.

One of the first duties of the instructor should be to purge the student's mind of the belief that mathematics is a purely mechanical art and that it consists in the formal application of rules that must be memorized. This concept of mathematics, which is fairly common among both students and instructors, stultifies the intellect and the imagination.

To digress, I recall the case of an ex-marine, in a navy V-12 course, a young man of more than average perseverance and intelligence. This student was asked to show that three points A, B, and C were collinear. He found that the lengths of AB, BC, and AC were $\sqrt{13}$, $\sqrt{13}$, and $\sqrt{52}$, respectively, and remarked, "We must prove $\sqrt{52}=2\sqrt{13}$." Then follows the deduction:

$$\sqrt{13} = \sqrt{4+9} = 2+3 = 5,$$

$$\sqrt{52} = \sqrt{16+36} = 4+6 = 10,$$

but

$$10 = 2 \times 5,$$

therefore

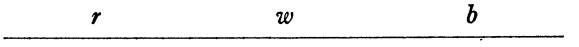
$$\sqrt{52} = 2\sqrt{13}.$$

In doing this, the young man showed imagination and some power of analysis and, in the best tradition of his service, he attained his objective, but he had never learned to associate mathematics and thinking.

3. Some suggestions. How then can we remedy matters? Let me presume to suggest a means. Because I believe that it is only by virtue of an intelligent understanding and grasp of principles that a student can obtain lasting profit from his mathematical studies, and also make progress and acquire speed in his later work, and because I believe that algebra is the study that can best give the student the requisite viewpoint and training, I suggest that more time be devoted to that subject. Not only are we obliged to give the student a new point of view but, frequently, we must eradicate the result of earlier training. This takes time and cannot be accomplished overnight; certainly we can not build a lasting structure upon a superficial foundation. I therefore suggest that the instruction in algebra courses, for the first few weeks, be deliberate, and that the stress, in the first few lessons, be upon the nature of deductive reasoning.

In this connection, the instructor might find profit in reading a work such as the recently published book by B. W. Jones, entitled *Elementary Concepts of Mathematics*. Although serving as an introduction to abstract thinking, the discussion and examples which appear in this book are such as to arouse the interest of the student and stimulate his imagination. Unfortunately, on the other hand, we find influential writers on the teaching of mathematics who seem obsessed with the notion that student interest and mental stimulus can be obtained only by a study of problems which deal with dollars and cents or which concern some form of physical activity. A learning process that is so motivated can scarcely broaden the student's intellectual outlook, and would tend to accentuate his adherence to preconceived notions and personal prejudices. It would be of doubtful cultural value.

The instructor might next take up with his class the growth of the number system and indicate the part that postulates play in its development. The treatment would be abstract since it deals with numbers which are abstractions, but it could be presented so as to capture the students' interest. One of the difficulties at this stage of the work is the necessity of convincing the student that rules of operation must be reexamined with every extension of the number system. This can be done, and the student's interest captured, by an illustration somewhat similar to the following: Consider three colored beads, red, white, and blue, upon a wire in the order



An interchange of the beads to produce a new order, say,

$$w \qquad b \qquad r$$

is called a substitution and may be designated by the symbol

$$s_1 = \begin{pmatrix} r & w & b \\ w & b & r \end{pmatrix},$$

which indicates the replacement of r by w, w by b, and b by r. In case we are dissatisfied with the new color scheme, we may continue with another substitution, say:

$$s_2 = \begin{pmatrix} w & b & r \\ w & r & b \end{pmatrix}$$

to produce the order

$$w \qquad r \qquad b$$

We then observe that the single substitution,

$$\begin{pmatrix} r & w & b \\ w & r & b \end{pmatrix},$$

is the resultant, or equivalent, of the two substitutions s_1 and s_2, in the indicated order. For lack of a better name, we call the resultant substitution the product of s_1 and s_2, in that order, and we agree to represent it by the notation s_1s_2. We also call the process of forming the product multiplication. In doing all this we have used the words product and multiplication in a new sense; that is, we have extended the meaning of these words. It is then fairly obvious that rules for multiplication which were established in our earlier work, say for positive numbers, will not carry over automatically to the multiplication of substitutions. In fact, upon trial, we obtain for the substitution s_2 followed by s_1, the product

$$s_2s_1 = \begin{pmatrix} r & w & b \\ r & b & w \end{pmatrix}.$$

Thus it follows for substitutions that

$$s_1s_2 \neq s_2s_1,$$

whereas, in our earlier work with positive numbers,

$$s_1s_2 = s_2s_1.$$

An illustration of this sort should prepare for a discussion of the problems that attend any extension of the number system. It should also suggest that if,

by definition, we introduce a new type of number, we should never assume that rules, which were not included in our definitions, apply merely because they were established for more elementary types of numbers. We must inquire into the rules of operation every time we extend the number system.

All this means that we propose beginning the study of college algebra with an attempt to indoctrinate the student in the rudiments of abstract thinking. This has the advantage that the student becomes conscious of the postulational character of mathematics and learns at an early stage to refer back to his assumptions before arriving at conclusions. In other words, he gets training in arriving at conclusions which are not based on prejudice, or preconceived notions, or on the word of some authority. He must think for himself, and work from first principles.

At a later time, the introduction of complex numbers into the number system furnishes another excellent opportunity for training in deductive reasoning.

We have now completed the introductory stage in the study of college algebra. From here on, our remarks apply equally well to any course in college mathematics and merely suggest how the instructor may attain the objective that the student shall understand his mathematics and shall be capable of independent thinking. Of course, as mentioned at the outset, we assume throughout that the student shall receive the usual instruction and drill in manipulative processes, except that this shall not dominate the picture. With all this in mind, it is most important that the student shall be required to express himself precisely, and without ambiguity, in speech, in symbolism, and in writing, especially in the latter since while writing he has more opportunity for deliberation. Information which one cannot impart to another has little intrinsic value. Furthermore, a person's inability to express his ideas clearly is indicative of confusion in thinking, and the converse is equally true.

The student must be aware of the exact meaning of the notation which he employs, and this, of course, involves an awareness of any restriction upon the interpretation of notation. I have in mind, for instance, the symbol $\sqrt{a}$, in which a is positive, which by universal agreement represents the positive square root of a. How many students realize that this agreement will compel him to write

$$\sqrt{x^2} = -x,$$

when x represents the abscissa of a point in the second quadrant? In any case, an example of this sort will prove useful. Also, in this connection, the student should be able to locate the fallacy in the statement

$$1 = \sqrt{1} = \sqrt{i^4} = i^2 = -1.$$

The same necessity for proper interpretation arises in the evaluation of the length of arc in the case of the astroid

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

The usual formula gives the integral

$$\int \sqrt{\frac{a^{2/3}}{x^{2/3}}}\, dx.$$

The student should be able to explain why

$$\int_{-a}^{a} \frac{a^{1/3}}{x^{1/3}}\, dx$$

gives zero and not the length of the portion of the astroid above the x-axis. The difficulty is not in the improper integral but in failing to observe that $-a^{1/3}/x^{1/3}$ is the positive square root for (x, y) a point in the second quadrant.

Since writing the above, I have seen a reference to trigonometric identities in a recent number of a leading mathematical journal. The statement was made: "As a matter of fact, both members of the identity will tolerate any fundamental process, except, and here the caution must be flashed, the extraction of roots. For example,

$$\sin^2 (-x) = 1 - \cos^2 x, \qquad \text{for any } x.$$

Extracting roots

$$\sin (-x) = \sqrt{1 - \cos^2 x},$$

$$-\sin x = \sqrt{1 - \cos^2 x}."$$

The author seems to labor under the confusion that we have just pointed out. He should have written

$$\pm \sin x = \sqrt{1 - \cos^2 x},$$

with the proviso that the sign plus or minus must be chosen so as to make the expression on the left side positive. For example, if

$$x = 240°,$$

he would have the true relation

$$\sin (-240°) = -\sin 240° = \sqrt{1 - \cos^2 240°}.$$

Another example in interpretation, one which involves a restriction in a differentiation, is the substitution

$$x = a \sin \theta$$

in the integral

$$\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx.$$

Why must the limits in the resulting integral read from $-\pi/2$ to $\pi/2$, rather than from $3\pi/2$ to $\pi/2$? Attention to such details can mean the difference between the student working mechanically or intelligently.

The word "equals" is one of the important technical terms in elementary mathematics. The student must employ this word in its proper sense and also be aware of the fact that an equality sign is not a punctuation mark. The instructor should not accept the statement: "The equation equals," or, "The formula equals." Quite regardless of the numerical result that he obtains, a student should not receive a passing grade on the solution of a problem that concerns the speeds of two airplanes if he starts out: "Let x equal the first airplane."

I have seen these statements in recent student work, and also the following which illustrate additional misuse of the equality sign:

(1)
$$2x - 3y = 7$$
$$= 2x - 3y - 7 = 0.$$

(2)
$$x/2 + y/3 = 1$$
$$= 3x + 2y = 6.$$

(3) In finding the seventh term in the expansion of $(3x-2y)^{10}$:

$$(3x - 2y)^{10} = 7\text{th term} = \textit{etc.}$$

(4) In finding the foci of the ellipse

$$16x^2 + 25y^2 = 400:$$

$$F = ae = \sqrt{a^2 - b^2} = \sqrt{25 - 16} = 3 = F_1(3, 0) \quad \text{and} \quad F_2(-3, 0).$$

(5) Also,

$$\frac{27.8 \times .0364}{.456} = \frac{\log 27.8 + \log .0364}{\log .456}$$
$$= \frac{1.4440 + 8.5611 - 10}{9.6590 - 10} = \frac{10.0051 - 10}{9.6590 - 10}$$
$$= .3461 = 2.218 \text{ (the correct answer to the problem).}$$

I suggest that to help the student appreciate the fault in work such as the last, the instructor might assign a simple problem of the type: Determine a number x which is less by 3 than the product of 6 and 7. He could then submit to the class, for their approval or disapproval, the solution.

$$x = 6 + 7 \div 3 = 42 \div 3 = 39.$$

If they disapprove, he could exhibit the logarithmic computation and make comparisons. If this makes no impression, the instructor may conclude that the student is not a prospective mathematical genius.

One final example: The student of college algebra is likely to believe that

polynomial equations in x and y can be combined freely by addition, subtraction, and by the substitution of the one into the other and that, provided there have been no numerical errors, the values that are obtained for x and y constitute solutions. The student might, with profit, consider this procedure as applied to the equations

$$(a)\quad (x^2 + y^2 + x)^2 = 9(x^2 + y^2) \qquad \text{and} \qquad (b)\quad x^2 + y^2 = 1.$$

The substitution of (b) into (a) gives

$$(x + 1)^2 = 9, \qquad x + 1 = \pm 3, \qquad x = 2, \qquad -4.$$

If he substitutes 2 for x in (a), he gets the equation

$$(y^2 + 6)^2 = 9(y^2 + 4),$$

or,

$$y^2(y^2 + 3) = 0,$$

which has the real solution

$$y = 0.$$

The student would likely conclude that (2, 0) is a common solution of the given equations and so a common point of their graphs. It happens, however, that the graphs have no point in common and the equations no common solution.

It is desirable that the student should realize that the propriety of combining the equations as we have done is based on an assumption that there is a common pair of values for x and y which satisfy the two equations and that, as a consequence of such an assumption, even when the numerical work is known to be correct, the values that have been obtained for x and y must be checked in both equations.

The student should learn to employ words in their correct sense and so as to convey his meaning without ambiguity, and he must realize that symbolism constitutes a shorthand notation and must be employed correctly if it is to convey information. He should be expected, in his various written tests, to give precise definitions of terms that are employed in his studies and in doing this and in proving principles he should be encouraged to employ, so far as possible, his own phrasing rather than that of the instructor, or the text, verbatim. It would be interesting, and perhaps surprising, to learn the small percentage of those in classes in college mathematics who have any clear conception of the meaning of the terms which they employ in their daily work, words and phrases such as rational number, polynomial in x, polynomial in x of the nth degree, sum of an infinite series, convergent series, *etc.* Of course, in fairness, we should not place the entire blame for the student's inability to express himself upon his early training in mathematics. His instruction, or lack of instruction, in English must, in part, be held responsible.

We find that in many cases the student is adverse to reading his textbooks and is inclined to depend too much upon the instructor. It would help matters if he were not only required to read the text but were held responsible for interpreting assignments from it to the class.

From all that has been said, one might infer, and correctly so, that the speaker has little sympathy with many of the so-called objective tests and multiple choice questions in examinations. These appear, to him, to defeat the very aim of education, since they require little more of the student than the ability to mark the spot with an x, and surely there is more to a measure of educational attainment than the winning of credits, especially when we simplify the process to this extent.

I realize that these remarks are out of step with some present-day educational theory. However, when, in educational journals, I find the emphasis on mathematical textbooks (and I quote) "being easy," "meaningful to the student," "not a fusion of abstract mathematics," and find references to "hatred towards mathematics" engendered by its difficulty, and this in the very elementary books, then I admit that I am not greatly concerned if I seem out of step with certain theorists. It happens that we are not traveling in the same direction. Also, unlike these people, I am not prepared to grant that American youth is less intelligent than the youth of other countries. It seems desirable to me that a textbook should be sufficiently difficult to require some mental effort on the part of the student. Otherwise, it can scarcely serve as a means of mental stimulus and intellectual development. Furthermore, I feel that we should not be disturbed by adverse criticisms of mathematics which may come from the man on the street, or from non-mathematicians. Their views on mathematics should carry the same weight as their views on relativity or on the technique of atomic research. In general, the so-called popular treatments of mathematics and science are a doubtful blessing and are likely to give rise to misleading concepts.

4. Summary. In summary: We have indicated that mathematics should be taught with the emphasis upon the thinking process. However, we have warned that this statement must not be interpreted as discounting the value of applied mathematics. Applied mathematicians must think. The statement merely implies that there are social values in the study of mathematics, and that these are as important as its applications in industry and in science.

We have shown that proper conclusions are inevitable in a process of reasoning and are a necessary consequence of the initial assumptions, but we have given no recipe for the selection of these assumptions. It is unfortunate, but true, that the initial assumptions may generate intolerance and bigotry, and the danger to our social and economic life is the fact that indoctrination into these premises takes place at an intellectually immature age. In this way they become a part of one's being, and constitute preconceived notions, or prejudices, which limit tolerance. They are accepted as realities and are not looked upon as assumptions.

A social value of the postulational approach to the study of mathematics is that it presupposes an examination and study of the premises. The student comes to realize that in a given study there are alternative sets of premises, all equally possible and proper. Thus, for example, in building an algebra (in the sense of linear algebras) he may start with any one of various sets of postulates, all equally possible and proper, yet one generates an algebra which is rich, and another barren, in its possibilities. The abstract treatment of algebra assures that the student will develop the habit of considering and questioning sets of postulates and this without bias in favor of one set or another. It is then inevitable that he shall acquire habits of reflection and judgment, and that these will affect his reactions to the problems of everyday life. They will make him more tolerant of opinions with which he disagrees.

We are living in a remarkable but dangerous age, one in which new and powerful forces, physical and spiritual, have been let loose. The individual is faced with a greater challenge, and a greater opportunity, than perhaps at any other time in world history. The forces, some good and some evil, must be met with understanding. This calls for independent thinking, sound judgment, and intelligent leadership. It follows that teachers of college mathematics hold a key position. It is their obligation to train our future leaders in habits of sound thinking, in clearness of vision and expression, and in tolerance, which is a by-product of mathematical thinking. However, to do this, teachers must shift the emphasis from the mechanics of mathematics to its understanding. In doing all this, teachers of mathematics can play a part in moulding world destiny.

ON SUMS INVOLVING BINOMIAL COEFFICIENTS

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1. Preliminary concepts. Let n, a, b, and c be integers. Let the binomial coefficients be defined as

$$\binom{n}{b} = \begin{cases} \dfrac{n!}{(n-b)!b!} & \text{for } n \geqq b \geqq 0; \\ 0 & \text{otherwise.} \end{cases}$$

We will evaluate the finite sum

$$\sum_k \binom{n+ak}{b+ck}, \tag{1}$$

where a summation over k will always signify a summation over all integral k.

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By the determinant of (1), we shall mean $nc-ab$.

Let the Pascal triangle be mapped onto the Cartesian plane by identification of the binomial coefficient $\binom{x}{y}$ with the point (x, y). All of the summands of (1) will lie on a line,

$$y = \frac{c}{a}x - \frac{(nc - ab)}{a}, \tag{2}$$

provided a is not zero. The case $a=0$ will be considered separately.

From the identity,

$$\binom{n}{b} = \binom{n}{n-b},$$

the sum (1) may also be written

$$\sum_k \binom{n + ak}{(n - b) + (a - c)k}. \tag{3}$$

The line passing through the summands of (3) has the slope $(a-c)/a$. Of the two slopes, c/a and $(a-c)/a$, one is greater than unity and the other is negative. (Otherwise, the sum would be infinite.) Therefore, the finite sum may be written in the form (1) with positive slope c/a.

Changing the signs of both a and c in (1) is equivalent to a substitution of $-k$ for k. But since the summation is over all k, positive and negative, such a substitution does not change the value of the sum. Hence we may assume that in (1), $c>a>0$.

2. The case $(a, c)=1$. The restriction on (1) that $(a, c)=1$ implies that all the integral points of (2) correspond to summands of (1), and hence that such a sum is uniquely specified by the equation of its corresponding line.

Consider the set of sums that can be written in the form (1) with the same a and c. Their corresponding lines, from (2), are parallel. Since a and c are relatively prime, the determinants of these sums form precisely the set of all integers. From (2), the y-intercept of a line is the quotient of the determinant of the corresponding sum and $-a$. Hence these parallels are equally spaced.

Let S_m denote the sum on the line whose determinant is m. Consider the sequence,

$$\cdots S_{-1}, S_0, S_1, \cdots S_m, \cdots . \tag{4}$$

We have $S_m=0$ for $m<0$ since such lines, having slopes greater than one and positive y-intercepts, do not meet the non-zero portion of the Pascal triangle.

It follows from the fundamental identity of the Pascal triangle, namely,

$$\sum_k \binom{n + ak}{b + ck} = \sum_k \binom{n - 1 + ak}{b - 1 + ck} + \sum_k \binom{n - 1 + ak}{b + ck}, \tag{5}$$

that the sequence (4) has the recurrence relation

$$S_m = S_{m-c+a} + S_{m-c}, \qquad (m \neq 0). \tag{6}$$

Let α_r $(r=0, 1, \cdots, (c-1))$ be the roots of

$$1 = x^{-c+a} + x^{-c}, \tag{7}$$

the characteristic equation of (6), or of the polynomial equation with the same roots,

$$x^c - x^a - 1 = 0. \tag{8}$$

Then

$$S_m = \sum_{r=0}^{c-1} d_r \alpha_r^m, \tag{9}$$

where the d_r $(r=0, 1, \cdots, (c-1))$ are arbitrary constants, satisfies (6), as can be verified by direct substitution in (6) and comparison with (7). In order that (9) should give the terms of (4), the d_r must satisfy the following set of equations:

$$\begin{aligned}
S_0 &= 1 = d_0 + d_1 + \cdots + d_{c-1} \\
S_{-1} &= 0 = d_0\alpha_0^{-1} + d_1\alpha_1^{-1} + \cdots + d_{c-1}\alpha_{c-1}^{-1} \\
S_{-2} &= 0 = d_0\alpha_0^{-2} + d_1\alpha_1^{-2} + \cdots + d_{c-1}\alpha_{c-1}^{-2} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
S_{-(c-1)} &= 0 = d_0\alpha_0^{-(c-1)} + d_1\alpha_1^{-(c-1)} + \cdots + d_{c-1}\alpha_{c-1}^{-(c-1)}.
\end{aligned}$$

The Vandermonde determinant,

$$|\alpha_i^{-j}| = \prod_{i>j} (\alpha_i^{-1} - \alpha_j^{-1}), \quad i, j = 0, 1, \cdots, (c-1),$$

is not zero since, as will be shown later, (8) has no multiple roots. The determinant formed by replacing the column of $|\alpha_i^{-j}|$ involving α_r by the S_{-j}'s reduces immediately to a product involving another Vandermonde determinant:

$$(-1)^r \alpha_0^{-1} \cdots \alpha_{r-1}^{-1} \cdot \alpha_{r+1}^{-1} \cdots \alpha_{c-1}^{-1} \prod_{i>j} (\alpha_i^{-1} - \alpha_j^{-1}),$$
$$i, j = 0, 1, \cdots, (r-1), (r+1), \cdots, (c-1).$$

By Cramer's rule, we have

$$d_r = \frac{(-1)^r \alpha_0^{-1} \cdots \alpha_{r-1}^{-1} \cdot \alpha_{r+1}^{-1} \cdots \alpha_{c-1}^{-1}}{(-1)^{c-r-1} \prod (\alpha_r^{-1} - \alpha_j^{-1})} = \frac{\alpha_r^{c-1}}{\prod (\alpha_r - \alpha_j)}$$

where $j=0, 1, \cdots, (r-1), (r+1), \cdots, (c-1)$. But the denominator of the last expression is the derivative of the left member of (8) evaluated at $x=\alpha_r$. Hence

$$d_r = \frac{\alpha_r^{c-1}}{c\alpha_r^{c-1} - a\alpha_r^{a-1}},$$

and, therefore,

$$S_m = \sum_{r=0}^{c-1} \frac{\alpha_r^m}{c - a\alpha_r^{a-c}}.$$

3. The general case. In order to sum (1) when a and c are not relatively prime, we must first generalize the above result in a slightly different direction.

Consider the sum

$$S_m(t) = \sum_k \binom{n + ak}{b + ck} t^{b+ck}, \tag{10}$$

where $m = nc - ab$, $|t| \leqq 1$, and a and c have the same restrictions as before. After multiplying each element $\binom{x}{y}$ of the Pascal triangle by t^y, we have the relation, derived from (5),

$$\sum_k \binom{n + ak}{b + ck} t^{b+ck} = t\sum_k \binom{n - 1 + ak}{b - 1 + ck} t^{b-1+ck} + \sum_k \binom{n - 1 + ak}{b + ck} t^{b+ck},$$

which may be written

$$S_m(t) = tS_{m-c+a}(t) + S_{m-c}(t), \qquad (m \neq 0). \tag{11}$$

We must show that the characteristic equation of (11),

$$1 = tx^{-c+a} + x^{-c},$$

or the equivalent equation,

$$x^c - tx^a - 1 = 0, \tag{12}$$

has no multiple roots. Taking the derivative of the left member of (12), removing the roots $x = 0$ (since they obviously are not roots of (12)), and eliminating x between the result and (12), we obtain

$$\left(\frac{at}{c}\right)^{c/(c-a)} - t\left(\frac{at}{c}\right)^{a/(c-a)} = 1$$

or

$$\left(\frac{at}{c}\right)^{c/(c-a)} = \frac{a}{a - c}.$$

But this conflicts with our original assumption that $c > a > 0$. Hence (12) and (8) have no multiple roots.

The sum (10) is then given by

$$S_m(t) = \sum_{r=0}^{c-1} \frac{\alpha_r^m}{c - at\alpha_r^{a-c}},$$

where the α_r $(r=0, 1, \cdots, (c-1))$ are the roots of (12).

Consider the expression

$$\frac{1}{d} \sum_{s=1}^{d} e^{(2\pi iks)/d},$$

where d is an integer greater than one. If k has d as a factor, the sum is one; otherwise the sum is zero. Hence

$$\frac{1}{d} \sum_k \left\{ \binom{n+ak}{b+ck} \sum_{s=1}^{d} e^{(2\pi iks)/d} \right\} = \sum_k \binom{n+adk}{b+cdk}. \tag{13}$$

But the left member of (13) also reduces to

$$\frac{1}{d} \sum_{s=1}^{d} \left\{ e^{(-2\pi ibs)/dc} \sum_k \binom{n+ak}{b+ck} e^{(2\pi i)(b+ck)s/dc} \right\}. \tag{14}$$

Hence,

$$\sum_k \binom{n+adk}{b+cdk} = \frac{1}{d} \sum_{s=1}^{d} \omega^{-bs} S_m(\omega^s), \qquad \omega = e^{(2\pi i)/dc}. \tag{15}$$

We can now also consider the case where a is zero, that is,

$$\sum_k \binom{n}{b+dk}. \tag{16}$$

Suppose $d=1$, then (16) is merely 2^n. Assume $d>1$. Then, setting $c=1$ and $k'=b+k$, we have, from (14) and (15),

$$\sum_k \binom{n}{b+dk} = \frac{1}{d} \sum_{s=1}^{d} \left\{ e^{(-2\pi ibs)/d} \sum_{k'=-\infty}^{+\infty} \binom{n}{k'} e^{(2\pi i)k's/d} \right\}$$

or, the known result,

$$\sum_k \binom{n}{b+dk} = \frac{1}{d} \sum_{s=1}^{d} \omega^{-bs}(1+\omega^s)^n, \qquad \omega = e^{(2\pi i)/d}. \tag{17}$$

It is interesting to note that since the sums

$$\sum_k \binom{n+k}{b+2k}$$

form the Fibonacci series, all of our series of sums may be considered as generalizations of the Fibonacci series.

AN EXTENSION OF FERMAT'S THEOREM

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In Fermat's Theorem it is stated that if p is a positive, rational, odd prime integer and if a is a rational integer such that

$$a \not\equiv 0 \pmod{p},$$

then

$$a^{p-1} - 1 \equiv 0 \pmod{p}. \tag{1}$$

From this theorem, it follows immediately that

$$\begin{aligned} a^{p(p-1)} - 1 &\equiv 0 \pmod{p^2}, \\ a^{p^2(p-1)} - 1 &\equiv 0 \pmod{p^3}, \\ a^{p^3(p-1)} - 1 &\equiv 0 \pmod{p^4}, \\ &\cdots\cdots\cdots\cdots, \\ a^{p^{n-1}(p-1)} - 1 &\equiv 0 \pmod{p^n}, \end{aligned} \tag{2}$$

where n may possess any positive, rational, integral value.

Let this sequence of congruences be considered in an opposite sense to that in which it has been written above; that is, let us write

$$\begin{aligned} a^{p^{n-1}(p-1)} - 1 &\equiv 0 \pmod{p^n}, \\ a^{p^{n-2}(p-1)} - 1 &\equiv 0 \pmod{p^{n-1}}, \\ a^{p^{n-3}(p-1)} - 1 &\equiv 0 \pmod{p^{n-2}}, \\ &\cdots\cdots\cdots\cdots, \\ a^{p^2(p-1)} - 1 &\equiv 0 \pmod{p^3}, \\ a^{p(p-1)} - 1 &\equiv 0 \pmod{p^2}, \\ a^{p-1} - 1 &\equiv 0 \pmod{p}. \end{aligned}$$

It will be shown that within the domain of algebraic integers this "downward sequence" can be extended as follows:

$$\begin{aligned} a^{(p-1)/p} - 1 &\equiv 0 \pmod{p^{1/p}}, \\ a^{(p-1)/p^2} - 1 &\equiv 0 \pmod{p^{1/p^2}}, \\ a^{(p-1)/p^3} - 1 &\equiv 0 \pmod{p^{1/p^3}}, \\ &\cdots\cdots\cdots\cdots \\ a^{(p-1)/p^n} - 1 &\equiv 0 \pmod{p^{1/p^n}}. \end{aligned} \tag{3}$$

Let $K(Z)$ be the cyclotomic field generated by a p^{n+1}th root of unity

$$Z = e^{2i\pi/p^{n+1}}; \tag{4}$$

then if E is the algebraic number defined by

$$p = E(1 - Z)^{p^n(p-1)} \tag{5}$$

it may be shown that E is a unit. For on setting up the irreducible equation satisfied by $1-Z$ it is found that $N(1-Z)=p$, while

$$(1 - Z)^{p^n(p-1)} = p\overline{E} \tag{6}$$

where $\overline{E}$ is an integer of $K(Z)$. Hence, taking norms in $K(Z)$, $p^{p^n(p-1)} = p^{p^n(p-1)}N(\overline{E})$, and $N(\overline{E})=1$. Therefore $\overline{E}$, and likewise its reciprocal E, is a unit.

Let

$$x = \theta$$

be one of the roots of the irreducible equation

$$x^{p^n} - a = 0; \tag{7}$$

then the p^n conjugate roots of (7) are

$$\theta_j = Z^{p(j-1)}\theta, \qquad j = 1, 2, 3, \cdots, p^n. \tag{8}$$

Thus

$$x^{p^n} - a = \prod_{j=1}^{j=p^n} (x - \theta_j). \tag{9}$$

From the binomial expansion

$$(\theta^{p-1} - 1)^{p^n} = \theta^{p^n(p-1)} - \binom{p^n}{1}\theta^{(p^n-1)(p-1)} + \binom{p^n}{2}\theta^{(p^n-2)(p-1)} - \cdots$$

$$- \binom{p^n}{p^n - 2}\theta^{2(p-1)} + \binom{p^n}{p^n - 1}\theta^{p-1} - 1,$$

it follows that

$$(\theta^{p-1} - 1)^{p^n} \equiv 0 \pmod{p}, \tag{10}$$

because

$$\binom{p^n}{1} \equiv \binom{p^n}{2} \equiv \binom{p^n}{3} \equiv \cdots \equiv \binom{p^n}{p^n - 1} \equiv 0 \pmod{p}$$

and

$$\theta^{p^n(p-1)} - 1 = a^{p-1} - 1 \equiv 0 \pmod{p}.$$

Let

$$(\theta^{p-1} - 1)^{p^n} = B(1 - Z)^{p^n(p-1)}, \tag{11}$$

and let $x=H$ be one of the p^n roots of the equation $x^{p^n}=B$.

From (11) it is evident that a suitable value of H will satisfy the equation

$$\theta^{p-1} - 1 = H(1 - Z)^{p-1}. \tag{12}$$

Thus

$$a^{p-1} - 1 = \prod_{j=1}^{j=p^n} (\theta_j^{p-1} - 1) = \left[\prod_{j=1}^{j=p^n} H_j\right](1 - Z)^{p^n(p-1)} \tag{13}$$

and, from (6),

$$a^{p-1} - 1 = \overline{E}\left[\prod_{j=1}^{j=p^n} H_j\right] p. \tag{14}$$

From (6), it follows that a suitable value

$$E_Z = \sqrt[p^n]{\overline{E}}$$

satisfies the equation

$$(1 - Z)^{p-1} = E_Z p^{1/p^n},$$

where the value of E_2 depends on the value assigned to $p^{1/p}$.

From (12), it follows that

$$a^{(p-1)p^n} - 1 \equiv 0 \pmod{[1 - Z]^{p-1}}; \tag{15}$$

hence

$$a^{(p-1)/p^n} - 1 \equiv 0 \pmod{p^{1/p^n}}. \tag{16}$$

As an example, let $p=3$ and $a=2$. According to (16), for $n=1$, it should be that

$$(\sqrt[3]{2})^2 - 1 \equiv 0 \pmod{\sqrt[3]{3}}.$$

This is so, because

$$(\sqrt[3]{2})^2 - 1 = [\sqrt[3]{(\sqrt[3]{2} - 1)^2}] \times \sqrt[3]{3}$$

if real roots are employed.

A SYMMETRICAL NOTATION FOR NUMBERS

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The possibility of representing real numbers in various scales of notation is well known. Thus, in the scale r an arbitrary positive number b may be expanded in the form,

$$b = \sum_{-\infty}^{N} a_n r^n, \qquad 0 \leqq a_n \leqq r - 1,$$

and represented in the "decimal" notation as $a_N a_{N-1} \cdots a_0 \cdot a_{-1} a_{-2} \cdots$. Negative numbers are represented by prefixing a minus sign to the representation of the corresponding positive numbers. Although it seems unlikely that the scale ten will ever be changed for ordinary work, the use of other scales and systems of notation is still of practical as well as mathematical interest. In some types of computing machines, for example, scales other than ten lend themselves more readily to mechanization.

A slight modification of the ordinary expansion gives a representation for numbers with certain computational advantages. Assuming r to be odd, it is seen easily that any positive or negative number b can be represented as

$$b = \sum_{-\infty}^{N} a_n r^n, \qquad -\frac{r-1}{2} \leqq a_n \leqq \frac{r-1}{2},$$

and we may denote b as usual by the sequence of its digits

$$b = a_N \cdots a_0 \cdot a_{-1} \cdots .$$

Both positive and negative numbers are thus represented by a standard notation without a prefixed sign, the sign being implied by the digits themselves; the number is positive or negative according as the first (nonvanishing) digit is greater or less than zero. Every real number has a unique representation apart from those whose expansion ends in an infinite sequence of the digits $(r-1)/2$ or $-(r-1)/2$, each of which has two representations. If this notation were to be used, a simple notation should be invented for the negative digits which suggested their close relation to the corresponding positive digits. For typographical simplicity we shall here denote the negative digits by placing primes on the corresponding positive digits. The notation for the first nine positive and negative integers with $r = 3, 5, 7, 9$ is as follows:

r	−9	−8	−7	−6	−5	−4	−3	−2	−1
3	1′00	1′01	1′11′	1′10	1′11	1′1′	1′0	1′1	1′
5	2′1	2′2	1′2′	1′1′	1′0	1′1	1′2	2′	1′
7	1′2′	1′1′	1′0	1′1	1′2	1′3	3′	2′	1′
9	1′0	1′1	1′2	1′3	1′4	4′	3′	2′	1′

r	0	1	2	3	4	5	6	7	8	9
3	0	1	11′	10	11	11′1′	11′0	11′1	101′	100
5	0	1	2	12′	11′	10	11	12	22′	21′
7	0	1	2	3	13′	12′	11′	10	11	12
9	0	1	2	3	4	14′	13′	12′	11′	10

In general the negative of any number is found by placing a prime on each unprimed digit and taking it off each primed digit. Arithmetic operations with this system are considerably simplified. In the first place the symmetries introduced by this notation make the addition and multiplication tables much easier to learn. For the scale $r=9$ these tables are, respectively, as follows:

+	4′	3′	2′	1′	0	1	2	3	4
4′	1′1	1′2	1′3	1′4	4′	3′	2′	1′	0
3′	1′2	1′3	1′4	4′	3′	2′	1′	0	1
2′	1′3	1′4	4′	3′	2′	1′	0	1	2
1′	1′4	4′	3′	2′	1′	0	1	2	3
0	4′	3′	2′	1′	0	1	2	3	4
1	3′	2′	1′	0	1	2	3	4	14′
2	2′	1′	0	1	2	3	4	14′	13′
3	1′	0	1	2	3	4	14′	13′	12′
4	0	1	2	3	4	14′	13′	12′	11′

·	4′	3′	2′	1′	0	1	2	3	4
4′	22′	13	11′	4	0	4′	1′1	1′3′	2′2
3′	13	10	13′	3	0	3′	1′3	1′0	1′3′
2′	11′	13′	4	2	0	2′	4′	1′3	1′1
1′	4	3	2	1	0	1′	2′	3′	4′
0	0	0	0	0	0	0	0	0	0
1	4′	3′	2′	1′	0	1	2	3	4
2	1′1	1′3	4′	2′	0	2	4	13′	11′
3	1′3′	1′0	1′3	3′	0	3	13′	10	13
4	2′2	1′3′	1′1	4′	0	4	11′	13	22′

The labor in learning the tables would appear to be reduced by a factor of at least two from the corresponding $r=9$ case in ordinary notation. There is no need to learn a "subtraction table"; to subtract, one primes all digits of the subtrahend and adds. The sign of the difference automatically comes out correct, and the clumsy device of "borrowing" is unnecessary. More generally, to add a set of numbers, some positive and some negative, all are placed in a column without regard to sign and added, *e.g.* ($r=9$):

(1′)		(1)		carried numbers.
1	3′	1′	2	
2′	3	1	4	
4′	1′	2	3	
3	2′	3′	4	
3	0	0	1′	
1′	2′	1	3′	
1′	4	1	0	

This process may be contrasted with the usual method where the positive and negative numbers must be added separately, the smaller sum subtracted from the larger and the difference given the sign of the larger, that is, three addition or subtraction processes and a sign rule, while with the symmetrical system one standard addition process covers all cases. Furthermore, in such a sum cancellation is very common and reduces considerably the size of numbers to be carried in memory in adding a column; this follows from the fact that any digit cancels its negative and these may be struck out from a column without affecting the sum. If all digits are equally likely and independent, the sum in a column will have a mean value zero, standard deviation $\sqrt{p(r^2-1)/12}$ where p is the number of numbers being added, while in the usual notation the mean value is $p(r/2)$ with the same standard deviation.

Multiplication and division may be carried out also by the usual processes, and here again signs take care of themselves, although in these cases, of course, the advantage of this is not so great.

We may note also that in the usual system of notation, when we wish to "round off" a number by replacing all digits after a certain point by zeros, the digits after this point must be inspected to see whether they are greater or less than 5 in the first place following the point. In the former case the preceding digit is increased by one. With the symmetrical system one always obtains the closest approximation merely by replacing the following digits by zeros. Numbers such as $1.444\cdots = 2.4'4'4'\cdots$ with two representations are exactly half way between the two nearest rounded off approximations, and in this case we obtain the upper or lower approximation depending on which representation is rounded off. If we were using this notation, department stores would find it much more difficult to camouflage the price of goods with \$.98 labels.

We have assumed until now that the scale r is odd. If r is even, say 10, a slightly unbalanced system of digits can be used; for example, 4′, 3′, 2′, 1′, 0, 1, 2, 3, 4, 5. The dissymmetry introduced unfortunately loses several of the advantages described above, *e.g.*, the ease of negation and hence of subtraction, and also the round off property.

A more interesting possibility is that of retaining symmetry by choosing for "digits" numbers halfway between the integers. In the case $r=10$ the possible digits would be

$$a_n = \frac{9'}{2}, \frac{7'}{2}, \frac{5'}{2}, \frac{3'}{2}, \frac{1'}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2},$$

and any number b can be expressed as

$$b = \sum_{-\infty}^{N} a_n r^n.$$

In this system the properties of positive-negative symmetry, automatic handling of signs, and simple round off are retained. One curious and disadvantageous feature is that the integers can only be represented as infinite decimals, and this is possible in an infinite number of different ways. For example.

$$0 = \cdot\frac{1}{2}\frac{9'}{2}\frac{9'}{2}\cdots = \frac{1}{2}\cdot\frac{9'}{2}\frac{9'}{2}\frac{9'}{2}\cdots = \frac{1'}{2}\cdot\frac{9}{2}\frac{9}{2}\frac{9}{2}\cdots \text{ etc.}$$

Symmetrical notation offers attractive possibilities for general purpose computing machines of the electronic or relay types. In these machines it is possible to perform the calculations in any desired scale and only translate to the scale ten at input and output. The use of a symmetrical notation simplifies many of the circuits required to take care of signs in addition and subtraction, and to properly round off numbers.

MATHEMATICAL NOTES

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A NOTE ON AN INEQUALITY FOR INTEGRALS

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This note is concerned with the question as to when the inequality

$$\left| \int_a^b f(x)dx \right| \leqq \int_a^b |f(x)|\,dx \tag{1}$$

becomes an equality. We shall consider continuous functions of the real variable x; the function values may be real or complex, or more generally, may lie in a Banach space B. In the latter event the absolute value signs are to be read as norms in the Banach space. It is immaterial whether the interval of integration is finite or infinite, provided the integrals in question exist.

For real continuous functions the equality in (1) holds if and only if $f(x)$ is

either always nonnegative or always nonpositive on the interval (a, b). This familiar theorem may easily be reduced to the theorem that the integral of a continuous nonnegative function is zero only if the integrand is identically zero.

For complex continuous functions the equality in (1) holds if and only if the function values $f(x)$ all lie on a fixed ray emanating from the origin in the complex plane. An equivalent statement of the condition is that all the function values must be nonnegative multiples of a fixed nonzero complex number. This theorem seems intuitively evident when we think of the integrals as sums; it is not as familiar as the corresponding theorem for real functions, however, and the proof is not as simple. We shall give a proof whose form is at once applicable to the case in which the function values lie in a Banach space. The theorem stated at the beginning of the present paragraph is a special case of Theorem 2. For an instance where it is essential to know that the inequality (1) is strict, see Landau [1].

THEOREM 1. *Let B be a Banach space, and let $f(x)$ be continuous at all finite points of the interval $a \leqq x \leqq b$, with values in B. In case the interval is infinite we assume that the integral on the right in* (2) *exists. Then the equality*

$$\left\| \int_a^b f(x)dx \right\| = \int_a^b \|f(x)\|dx \tag{2}$$

holds if and only if for every finite set of distinct points on (a, b) we have

$$\left\| \sum_{i=1}^{n} f(x_i) \right\| = \sum_{i=1}^{n} \|f(x_i)\|. \tag{3}$$

Proof: The sufficiency of condition (3) is easily demonstrated, and we omit the argument. In case the interval (a, b) is infinite we first prove that (3) implies (2) for each finite subinterval of (a, b).

In proving the necessity of (3) we first observe that if (2) holds, then

$$\left\| \int_E f(x)dx \right\| = \int_E \|f(x)\|dx, \tag{4}$$

where E is any subset of (a, b) for which the integrals have a meaning. For, denoting the complement of E in (a, b) by CE, we have

$$\left\| \int_a^b f dx \right\| \leqq \left\| \int_E f dx \right\| + \int_{CE} \|f\|dx,$$

$$\int_a^b \|f\|dx = \int_E \|f\|dx + \int_{CE} \|f\|dx.$$

Thus (2) implies that

$$\int_E \|f\|dx \leqq \left\| \int_E f dx \right\|.$$

Since the reverse inequality also holds, (4) is true.

It suffices to prove (3) for points $x_1, \cdots, x_n$ interior to (a, b), since f is continuous. Suppose then that we are given such points, and let $y_i = f(x_i)$. If $\epsilon > 0$, let E_i be the interval $|x - x_i| \leqq \delta$, where δ is a positive number chosen in such a way that the sets $E_1, \cdots, E_n$ lie in (a, b) and do not overlap, while $\|f(x) - f(x_i)\| < \epsilon$ in E_i. Let $E = E_1 + \cdots + E_n$. Then

$$\int_E f dx = \sum_{i=1}^n \int_{E_i} (f(x) - y_i) dx + \sum_{i=1}^n \int_{E_i} y_i dx,$$

$$\left\| \int_E f dx \right\| \leqq 2n\delta\epsilon + 2\delta \left\| \sum_{i=1}^n y_i \right\|;$$

$$\int_E \|f\| dx = \sum_{i=1}^n \int_{E_i} \|f(x) - y_i + y_i\| dx$$

$$\geqq \sum_{i=1}^n \int_{E_i} (\|y_i\| - \|f(x) - y_i\|) dx$$

$$\geqq 2\delta \sum_{i=1}^n \|y_i\| - 2n\delta\epsilon.$$

From (4) and the foregoing considerations we conclude that

$$\sum_{i=1}^n \|y_i\| \leqq 2n\epsilon + \left\| \sum_{i=1}^n y_i \right\|.$$

It follows that

$$\sum_{i=1}^n \|y_i\| \leqq \left\| \sum_{i=1}^n y_i \right\|;$$

the reverse inequality holds as well, and therefore (3) is true.

THEOREM 2. *Let the hypotheses on $f(x)$ be as in Theorem* 1, *and suppose that the surface of the unit sphere in B is strictly convex (i.e. that given a chord joining two points of the surface, all interior points of the chord are inside the sphere). Then* (2) *holds if and only if all the values of the function $f(x)$ are nonnegative multiples of a fixed nonzero vector in B.*

Proof: The sufficiency proof is so simple that we omit it.

Suppose that (2) holds; then (3) holds also, by Theorem 1. If $f(x)$ is identically zero there is nothing to prove. Hence suppose $y_0 = f(x_0) \neq 0$ for some x_0. Let $y = f(x)$. We have to express y as a nonnegative multiple of y_0. Dismissing the trivial case $y = 0$, let us write

$$u_0 = \frac{y_0}{\|y_0\|}, \qquad u = \frac{y}{\|y\|}.$$

Then $y_0+y=\|y_0\|u_0+\|y\|u$, and by (3)

$$1 = \frac{\|y_0 + y\|}{\|y_0\| + \|y\|} = \left\| \frac{\|y_0\|u_0 + \|y\|u}{\|y_0\| + \|y\|} \right\|.$$

But then u_0,u, and

$$v = \frac{\|y_0\|u_0 + \|y\|u}{\|y_0\| + \|y\|}$$

are all points on the surface of the unit sphere in B. However, if the points u_0, u are distinct, v is an interior point of the chord joining them. By the strict convexity of the surface of the unit sphere in B we conclude that u_0 and u coincide. In other words,

$$y = \frac{\|y\|}{\|y_0\|} y_0.$$

This completes the proof of the theorem.

The conclusion of Theorem 2 may be false if the surface of the unit sphere in B is not strictly convex. An example is afforded by taking B as the vector space of the complex numbers, with the norm defined by

$$\|\xi + i\eta\| = \max \{ |\xi|, |\eta| \}.$$

Suppose $f(x)=\xi(x)+i\eta(x)$, where $\xi(x)\geqq 0$ and $|\eta(x)| \leqq \xi(x)$. Then it is easily seen that (2) holds; but the function values $f(x)$ need not be confined to a single ray through the origin.

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A DIFFERENTIATION FORMULA

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1. Introduction. The question of changing the order of two infinite processes has in many cases been both interesting and fruitful. In tensor analysis, the form of important developments occasionally turns on the pattern of a commutation formula. In combinatorial analysis, commutation formulas have been used to discover relations among binomial coefficients.

This paper presents a generalization of a relationship which might be described as a commutation formula for the partial derivative of a total derivative of a point function. As an example of its application, the generalized relationship is used to prove a sum rule for binomial coefficients. Specifically, the theorem to be generalized is the following.

If $F(x^1, \cdots, x^N)$ is either a scalar or a vector point function of class M, then along a parameterized arc C: $x^a = x^a(t)$ of class M and for $0 \leqq P \leqq M$, $F^{(P)} = d^P F/dt^P$ is of functional order P,

$$F^{(P)} = F^{(P)}(x^1, \cdots, x^N, x^{(1)1}, \cdots, x^{(1)N}, \cdots, x^{(P)1}, \cdots, x^{(P)N}),$$

where $x^{(A)a} = d^A x^a/dt^A$. The following theorem,[1] in which

$$F^{(P)}_{;(A)a} = \partial F^{(P)}/\partial x^{(A)a},$$

$$\binom{P}{A} = \frac{P!}{A!(P-A)!},$$

and

$$F^{(P-A)}_a = d^{(P-A)}(\partial F/\partial x^a)/dt^{(P-A)},$$

is fundamental for the transformation theory associated with the extended coordinate transformation.

THEOREM: *If $F(x^1, \cdots, x^N)$ is either a scalar or a vector point function of class M, then along a parameterized arc C: $x^a = x^a(t)$ of class M and for $0 \leqq (P, A) \leqq M$*

$$F^{(P)}_{;(A)a} = \binom{P}{A} F^{(P-A)}_a.$$

Here, as in what follows, it is understood that

$$\binom{P}{A} = 0$$

when $A > P$ or $0 > A$, and that other "out of range" symbols are also zero. The purpose of the present note is to generalize this theorem to the case in which F is of functional order J.

2. The Formula. We shall establish the following result.

THEOREM: *If $F(x^1, \cdots, x^N, \cdots, x^{(J)1}, \cdots, x^{(J)N})$ is either a scalar or a vector point function of class M, then along a parameterized arc C: $x^a = x^a(t)$ of class $M+J$ and for $0 \leqq P \leqq M$, $0 \leqq A \leqq M+J$*

$$F^{(P)}_{;(A)a} = \sum_{i=0}^{J} \binom{P}{A-i} F^{(P+i-A)}_{;(i)a}.$$

Proof: For any J, the case $P = 0$, A arbitrary, and the cases $A \geqq P+J$, P arbitrary, are obvious. To complete the proof, it is sufficient to show that if the theorem is true for J, P, A then it is true for J, $P+1$, A. The proof follows.

[1] H. V. Craig, Vector and Tensor Analysis, p. 215, McGraw-Hill, 1943.

Differentiate $F^{(P)}$ with respect to t by the chain rule:

$$F^{(P+1)} = \sum_{B=0}^{P+J} F^{(P)}_{;(B)b} x^{(B+1)b},$$

and differentiate this with respect to $x^{(A)a}$:

$$F^{(P+1)}_{;(A)a} = \sum_{B=0}^{P+J} F^{(P)}_{;(B)b;(A)a} x^{(B+1)b} + \sum_{B=0}^{P+J} F^{(P)}_{;(B)b} \frac{\partial x^{(B+1)b}}{\partial x^{(A)a}}.$$

Now, because of the independence of the various quantities $x^{(A)a}$, the derivative $\partial x^{(B+1)b}/\partial x^{(A)a}$ which appears at the last of the foregoing equation may be replaced by the product of Kronecker deltas $\delta^{B+1}_{A}\delta^{b}_{a}$, and thus we may write

$$F^{(P+1)}_{;(A)a} = (F^{(P)}_{;(A)a})' + F^{(P)}_{;(A-1)a}.$$

If the formula is true for J, P, A then

$$F^{(P+1)}_{;(A)a} = \sum_{i=0}^{J}\left[\binom{P}{A-i} F^{(P+i-A+1)}_{;(i)a} + \binom{P}{A-i-1} F^{(P+i-A+1)}_{;(i)a}\right];$$

and since

$$\binom{P}{A-i} + \binom{P}{A-i-1} = \binom{P+1}{A-i},$$

we have

$$F^{(P+1)}_{;(A)a} = \sum_{i=0}^{J}\binom{P+1}{A-i} F^{(P+i-A+1)}_{(i)a}.$$

This is what was to have been shown.

As an example of the way in which the formula may be applied to discover relations among binomial coefficients, it will be shown that

$$\sum_{i=k}^{P}\binom{M}{B-i}\binom{P}{i-k} = \binom{P+M}{B-k},$$

where C is the smaller of B and $P+k$.

Proof: Now

$$F^{(P+M)}_{;(B)b} = [F^{(P)}]^{(M)}_{;(B)b}.$$

An immediate application of the generalized relationship gives

$$[F^{(P)}]^{(M)}_{;(B)b} = \sum_{i=0}^{P+J}\binom{M}{B-i} [F^{(P)}_{;(i)b}]^{(M+i-B)}$$

and applying it again to the quantity in the brackets we obtain

$$F_{;(B)b}^{(P+M)} = \sum_{i=0}^{P+J} \binom{M}{B-i} \sum_{k=0}^{J} \binom{P}{i-k} F_{(k)b}^{(M+i-B+P+k-i)}.$$

But applying the generalized relationship directly to the left-hand side we get

$$F_{;(B)b}^{(P+M)} = \sum_{k=0}^{J} \binom{P+M}{B-k} F_{(k)b}^{(P+M+k-B)}.$$

A comparison of the two results shows that

$$\sum_{i=0}^{P+J} \binom{M}{B-i}\binom{P}{i-k} = \binom{P+M}{B-k}.$$

But because of the second binomial coefficient on the left-hand side, i must be greater than k and less than $P+k$, which is less than $P+J$. Thus the lower limit for i can be replaced by k and the upper limit by $P+k$. Because of the first binomial coefficient on the left-hand side, the upper limit for i can be put equal to B if B is less than $P+k$. This completes the proof.

A PROOF OF INDEPENDENCE

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Hilbert and Bernays give on p. 66 of their *Grundlagen der Mathematik*, vol. I (Berlin, 1934), a system of axioms for the propositional calculus. The system may be described as follows:

(1) *Any statement of one of the following form is a theorem*:

I.1. $A \to (B \to A)$. I.2. $(A \to (A \to B)) \to (A \to B)$. I.3. $(A \to B) \to ((B \to C) \to (A \to C))$.

II.1. $A \& B \to A$. II.2. $A \& B \to B$. II.3. $(A \to B) \to ((A \to C) \to (A \to B \& C))$.

III.1. $A \to A \vee B$. III.2. $B \to A \vee B$. III.3. $(A \to C) \to ((B \to C) \to (A \vee B \to C))$.

IV.1. $(A \sim B) \to (A \to B)$. IV.2. $(A \sim B) \to (B \to A)$. IV.3. $(A \to B) \to ((B \to A) \to (A \sim B))$.

V.1. $(A \to B) \to (\overline{B} \to \overline{A})$. V.2. $A \to \overline{\overline{A}}$. V.3. $\overline{\overline{A}} \to A$.

(2) *If A and $A \to B$ are theorems, then B is a theorem.*

They prove the independence of each of I.1—V.3 from the rest of the system on pp. 72–79, *ibid.*, and prove the independence of all of these except I.1 and I.3 with tables involving two or three truth values. The proofs of I.1 and I.3 make use of tables involving four truth values.

Now I propose to contribute an alternative proof for the independence of I.1 by a table which involves only three truth values. The table I use is such that when A and $A \to B$ both take the value 1, then B also takes the value 1. It is as follows.

A	B	$\overline{A}$	$A \vee B$	$A \& B$	$A \leftharpoondown B$	$A \rightarrow B$
1	1	3	1	1	1	1
1	2	3	1	2	3	3
1	3	3	1	3	3	3
2	1	2	1	2	3	1
2	2	2	2	2	1	1
2	3	2	2	3	3	3
3	1	1	1	3	3	1
3	2	1	2	3	3	1
3	3	1	3	3	1	1

According to the table, all the axioms I.2–V.3 always take the value 1, while I.1, $(A \rightarrow (B \rightarrow A))$, takes the value 3 when A and B take the values 2 and 1 respectively.

It can be shown that no proof of independence involving only three truth values, whether with one or two designated truth values, can be constructed for I.3, and from this it follows that no proof can be constructed involving only two truth values. The proof consists essentially in constructing all possible truth tables for "$\rightarrow$", and observing that any truth table which preserves designated values in accordance with (2), and always assigns designated values to I.1 and I.2, also always assigns designated values to I.3. The number of tables which need be checked can be materially reduced by proper use of I.1 and I.2.

HYPERBOLIC FUNCTION SERIES ARISING FROM A SIMPLE POTENTIAL PROBLEM

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1. Introduction. If Laplace's equation in two dimensions,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

is solved by separating the variables, we obtain solutions of the type $\phi = \cos kx \cosh ky$, $\phi = \sinh kx \cos ky$, and so on. If the function, harmonic in a square region and constant on the boundaries, is expanded by means of these solutions, the identity (4) of this note is suggested, from which can be deduced many simple and interesting series involving the hyperbolic functions, which are not readily obtained by other methods.

2. The fundamental identity. Consider the function ϕ_1, harmonic inside the square $x = \pm \frac{1}{2}$, $y = \pm \frac{1}{2}$, with the boundary values

$$\phi_1 = 0 \quad \text{on} \quad x = \pm \tfrac{1}{2},$$
$$\phi_1 = 1 \quad \text{on} \quad y = \pm \tfrac{1}{2},$$

the values at the vertices being indeterminate. If such a function exists and can be expressed in terms of products of circular and hyperbolic functions, then

$$\phi_1 = \sum_{n=0}^{\infty} a_n \cos (2n + 1)\pi x \cosh (2n + 1)\pi y. \tag{1}$$

Sine and hyperbolic sine functions are ruled out by the fact that ϕ_1 must be an even function of x and y, while its vanishing at $x = \pm \frac{1}{2}$ fixes the factors involving x as $\cos (2n+1)\pi x$. Taking into account the second boundary condition, we must have, for $|x| < \frac{1}{2}$,

$$1 = \sum_{n=0}^{\infty} a_n \cos (2n + 1)\pi x \cosh (2n + 1) \frac{\pi}{2} \cdot$$

Using the well-known Fourier expansion for the "square-wave" function,

$$1 = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \cos (2n + 1)\pi x}{2n + 1}, \qquad |x| < \tfrac{1}{2}, \tag{2}$$

we must have, in (1),

$$a_n = \frac{4}{\pi} \frac{(-1)^n}{(2n + 1) \cosh (2n + 1) \frac{\pi}{2}} \cdot \tag{3}$$

With this value of a_n in (1), we have a formal solution of our boundary-value problem.

We now consider the symmetrical problem with the boundary conditions

$$\phi_2 = 0 \quad \text{on} \quad y = \pm \tfrac{1}{2},$$
$$\phi_2 = 1 \quad \text{on} \quad x = \pm \tfrac{1}{2},$$

with a similar solution

$$\phi_2 = \sum_{n=0}^{\infty} a_n \cosh (2n + 1)\pi x \cos (2n + 1)\pi y,$$

where the value of a_n as given in (3) is unchanged.

Since the problem is linear, the two solutions may be superposed, giving the solution of the potential problem in which $\phi = 1$ on the sides of the square, except perhaps at the vertices. Since the unique solution of the case where $\phi = 1$ everywhere on the boundary is $\phi \equiv 1$, we are led to suspect strongly that $\phi_1 + \phi_2$ is identically unity inside the square; that is, that

$$\frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1) \cosh (2n + 1) \frac{\pi}{2}} [\cos (2n + 1)\pi x \cosh (2n + 1)\pi y + \cosh (2n + 1)\pi x \cos (2n + 1)\pi y] = 1; \tag{4}$$

and this can in fact be shown to hold for all points of the square except the vertices.

3. Proof. The result (4) may be proved by the method above, but it would be necessary to invoke rather strong theorems of potential theory, and probably the simplest method is to use instead of (4) the series obtained by integrating it twice with respect to each variable, when difficulties due to lack of uniform convergence disappear. We shall need some further Fourier series. Series (2) is uniformly convergent in any range inside $-\frac{1}{2}<x<\frac{1}{2}$ excluding an arbitrarily small region about the end-points, and hence it can be integrated termwise in $-\frac{1}{2}\leqq x\leqq\frac{1}{2}$. Integrating (2) twice, first from 0 to x and then from $-\frac{1}{2}$ to x, we obtain

$$\frac{x^2}{2}-\frac{1}{8}=-\frac{4}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\pi x}{(2n+1)^3}. \tag{5}$$

Repeating the double integration, we have

$$\frac{x^4}{24}-\frac{x^2}{16}+\frac{5}{384}=\frac{4}{\pi^5}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\pi x}{(2n+1)^5}. \tag{6}$$

Both series converge uniformly for $-\frac{1}{2}\leqq x\leqq\frac{1}{2}$.

Returning now to the conjectured identity (4), we integrate it twice with respect to each variable, the integration constants being supplied with the help of identities (5) and (6). The resulting equation is

$$\begin{aligned}-\frac{4}{\pi^5}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^5\cosh(2n+1)\frac{\pi}{2}}&[\cosh(2n+1)\pi x\cos(2n+1)\pi y\\&+\cos(2n+1)\pi x\cosh(2n+1)\pi y]\\=-\frac{1}{24}&\left(x^4-6x^2y^2+y^4+\frac{1}{4}\right).\end{aligned} \tag{7}$$

This result can be readily proved, since the series on the left-hand side, together with its second derivatives, is uniformly convergent in the square, and satisfies Laplace's equation, as does the polynomial on the right-hand side. From (6), both sides have the same values on the boundaries $x=\pm\frac{1}{2}$, $y=\pm\frac{1}{2}$. But for a closed region, the harmonic function with given boundary-values is unique, and thus (7) is an identity throughout the square. Equation (4) may now be proved by differentiating (7) twice with respect to each variable. All the series concerned are uniformly convergent in any closed region inside the square, and hence (4) is verified for all interior points of the square. From (2) it holds also on the boundaries, except at the vertices, where it is false.

4. Derived series. Identity (4) may be made the starting point for obtaining a large number of series involving $\cosh(2n+1)\pi/2$. Putting $x=y=0$, for example, we get

$$\frac{\pi}{8} = \frac{1}{\cosh\frac{\pi}{2}} - \frac{1}{3\cosh\frac{3\pi}{2}} + \frac{1}{5\cosh\frac{5\pi}{2}} - \cdots .$$

Other results are obtained by differentiating or integrating (4). Differentiating once with respect to x, and putting $y=0$, $\pi x=z$, we have

$$\frac{\sin z}{\cosh\frac{\pi}{2}} - \frac{\sin 3z}{\cosh\frac{3\pi}{2}} + \frac{\sin 5z}{\cosh\frac{5\pi}{2}} - \cdots$$
$$= \frac{\sinh z}{\cosh\frac{\pi}{2}} - \frac{\sinh 3z}{\cosh\frac{3\pi}{2}} + \frac{\sinh 5z}{\cosh\frac{5\pi}{2}} - \cdots ,$$

for $|z| < (\pi/2)$. Differentiating (4) twice with respect to each variable, obtaining as usual a series unformly convergent in any region inside the square, and putting $x=y=0$, we get

$$\frac{1^3}{\cosh\frac{\pi}{2}} - \frac{3^3}{\cosh\frac{3\pi}{2}} + \frac{5^3}{\cosh\frac{5\pi}{2}} - \cdots = 0.$$

Integrating (4) once with respect to each variable, and putting $x=y=\frac{1}{2}$, we have

$$\frac{\pi^3}{32} = \tanh\frac{\pi}{2} + \frac{1}{3^3}\tanh\frac{3\pi}{2} + \frac{1}{5^3}\tanh\frac{5\pi}{2} + \cdots .$$

The method here adopted may be extended to three or more dimensions. In three dimensions we arrive at such series as

$$\frac{\pi^2}{48} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{(2m+1)(2n+1)\cosh\frac{\pi}{2}\sqrt{[(2m+1)^2+(2n+1)^2]}} .$$

These series can, of course, be obtained by other methods such as a contour integration or the transformation of the complete elliptic integral [1], but that given seems the simplest.

The fact that rectangular boundary problems could yield such identities was apparently first noted by Thomson (Lord Kelvin) and Tait [2] who saw that the expression they obtained for the torsional rigidity of a rectangular bar was unsymmetrical in the directions of the sides of the rectangle, and produced a non-trivial identity by equating the original expression to that obtained by symmetrical interchange.

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1. Purser, F., Messenger of Math., vol. 11, 1882, p. 26.
2. Thomson and Tait, Treatise on Natural Philosophy, Camb. Univ. Press, 1903, vol. 2, p. 249.

CLASSROOM NOTES

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ON THE SYSTEM OF NATURAL NUMBERS

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1. Introduction. It seems highly desirable to be able to develop a simple and logical basis for the system of natural numbers, abstracted from one's own experiences, that would be suitable for presentation to students in College Algebra. This would have the effect of placing early emphasis on the logical structure of elementary algebra, similar to that in demonstrative geometry in secondary schools.

In this paper we shall establish explicitly and concisely the undefined elements, the defined elements, and the postulates which serve as a logical basis for the arithmetic of natural numbers. These fundamental concepts lead easily and naturally to the concept of natural number and the principle of mathematical induction, as well as the other well known properties of the system of natural numbers.

2. Undefined elements. We shall take the following concepts as undefined elements: *unity, set, belonging to a set, correspondence, and order.* The investigation of the origin of these concepts is a physiological and philosophical problem but not a mathematical one.

The notions of unity and plurality or set are acquired early in life, when one of the principal faculties of our mind, ability to discriminate, comes into action. We then begin to perceive distinctly the objects of the external world and we begin also to have some knowledge of our own individuality, thus originating those primary concepts of unity and plurality.

The entities or objects belonging to a set are called *elements* of the set. We shall denote the elements of the set by capital letters and the set itself by boldface capital letters. We shall express the fact that the set **M** has the elements A, B, C, D, $\cdots$, or that these elements belong to **M** by writing

$$\mathbf{M} = \text{A, B, C, D, } \cdots .$$

For convenience, we shall introduce the concept of null-set (empty set), which has no elements. Since a set is here regarded as a collection or aggregate of things called elements, the null-set is, in general, excluded from consideration. Non-empty sets may be called *proper sets*. We shall consider only proper sets unless otherwise stated.

When we associate, by means of some well defined rule, each element of a set with one and only one element of another set, we say that a one to one correspondence has been established between the sets.

By order we mean, roughly speaking, some distribution of things in time, space, or according to a certain rule or criterion, and an ordered set according to one rule may not be ordered according to some other rule. We shall adopt the terms *preceding* and *succeeding* for describing rules of order. That is, for an ordered set these terms have precise meanings. Conversely, a set may be regarded as *ordered* when precise meaning has been attached to these terms, that is, when some criterion is stated to decide for two elements of the set which is the preceding one.

3. Assumptions. The descriptions of the elements unity, set, belonging to a set, correspondence, and order discussed in the preceding section are not to be considered as definitions of these concepts, but as descriptions of their general characteristics. In this section we shall state four postulates which these undefined elements are to satisfy. We shall see that these four assumptions are sufficient as fundamental postulates of the Arithmetic.

POSTULATE 1. *Given a set* **X** *there exists another set* **Y** *having no elements in common with* **X.**

This postulate establishes the existence of elements which do not belong to any set, transcending our faculty to find in nature or create in the imagination. It serves as the antecedent of Postulate 2 and jointly with it the base of the operation called addition of numbers to be defined later.

POSTULATE 2. *Given a set* **X,** *it is possible to form another set either by removing an element from* **X** *or by adjoining another element to those of* **X.**

POSTULATE 3. *Given the sets* **X** *and* **Y,** *it is possible to choose an element in each as corresponding elements.*

This postulate is known as the principle of free choice or axiom of Zermelo.

POSTULATE 4. *For any elements* A, B, *and* C *of an ordered set we have*

(1) A *does not precede* A,

(2) *If* A *precedes* B, *then* B *does not precede* A,

(3) *If* A *precedes* B *and* B *precedes* C, *then* A *precedes* C.

These conditions characterize what is more specifically known as *linear order.*

4. Fundamental definitions. We are now ready to define certain concepts explicitly.

DEFINITION 1. *A set is said to be determined or defined when there exists some rule to decide whether a given object belongs or does not belong to the set.*

DEFINITION 2. *The sets* **M** *and* **N** *are said to be equal whenever every element of* **M** *belongs to* **N** *and conversely.*

The equality of the sets **M** and **N** is denoted by the symbolism

$$\mathbf{M} = \mathbf{N}.$$

DEFINITION 3. *The set* **H** *is said to be a proper subset of the set* **K** *if the elements of* **H** *belong to* **K** *but there are elements in* **K** *which do not belong to* **H.**

We shall indicate that **H** is a subset of **K** by the notation

$$\mathbf{H} < \mathbf{K} \quad \text{or} \quad \mathbf{K} > \mathbf{H}.$$

DEFINITION 4. *If in an ordered set there exists an element preceding all the remaining elements, this element is called the first element of the set. If there exists an element which does not precede any element it is called the last element of the set.*

DEFINITION 5. *Elements* A *and* B *of an ordered set are said to be consecutive when there does not exist an element* C *such that* A *precedes* C *and* C *precedes* B. *In this case the element* A *is said also to be the immediate predecessor of* B.

Sets in mathematics may be finite, as the set composed of the sides of a polygon, or infinite, as the set of points in a circle. The usual procedure in the past has been to define infinite sets first and then define a finite set as one which is not infinite. It seems far more natural and, therefore, desirable to define finite sets first. That it is possible to define finite sets in terms of the notion of order is shown by the next definition.

DEFINITION 6. *A set* **M** *is said to be finite if it can be ordered in such a way as to make every subset of* **M** *have a first and a last element.*

The following properties are immediate consequences of this definition:

(1) *All finite sets have a first and a last element.*

(2) *All subsets of a finite set are finite.*

(3) *The elements of a finite set are consecutive.*

It is always assumed that the same ordering is retained in the subsets of an ordered set.

DEFINITION 7. *A set that is not finite is said to be infinite.*

DEFINITION 8. *A simply infinite set* **S** *is an infinite set such that*

(1) *Every subset of* **S** *has a first element,*

(2) *Every element of* **S** *except the first has an immediate predecessor.*

It follows from the first condition in this definition that every element of **S** has an immediate successor. The elements of a simply infinite set are, therefore, consecutive. The simply infinite sets differ from the finite sets in the fact that they do not have a last element.

As an example, let us consider the set of points A, A′, A″, A‴, $\cdots$, B, where A′ is the mid-point of the segment AB; A″ is the mid-point of A′B; A‴ is the mid-point of A″B; and so forth.

This set may be ordered by letting the word *precede* mean *to the left of*, and then the first element is A and the last element is B. The set is infinite, although it has a first and a last element. This shows why we stated the condition for a

finite set in terms of its subsets, requiring for each one the property of having a first and a last element. The converse of this is not true, that is, if the set has a first and a last element it does not follow that all subsets have a first and a last element. Thus, in the above example, the subset A, A′, A″, · · · does not have a last element.

The total set in the above example is not simply infinite, since the second condition of Definition 8 is not satisfied because the element B does not have an immediate predecessor. The set obtained by removing the element B is simply infinite, since it has the properties required in Definition 8.

DEFINITION 9. *Two sets are said to be equivalent when there exists a one to one correspondence between their elements.*

It follows easily that the relation of equivalence is *reflexive*, *symmetric*, and *transitive*.

We shall indicate that two sets **X** and **Y** are equivalent by the notation

$$\mathbf{X} \cdot\cdot \mathbf{Y}.$$

DEFINITION 10. *When two sets are equivalent we say they have the same number of elements.*

Thus the *concept of number* results from considering equivalent sets when abstraction is made of all other characteristics of their elements such as kind, color, size, etc. A similar mental process leads to the formation of all our general or abstract concepts. Thus, for example, by looking at different flowers and neglecting the particular characteristics of each one, but retaining their essential common properties (those concerning the stamen and pistil), we arrive at the general concept of flower.

In analogous fashion the number concept is an abstraction elaborated spontaneously in our mind for the purpose of representing equivalent sets.

Definitions of this type are called *definitions by abstraction.* It is the most suitable manner of definition for those concepts which are formed by considering in some sense the equality or equivalence of objects. Two objects are said to be equal or equivalent when they have in common a certain specified property, being capable of mutual substitution with respect to this property. From each kind of equivalence a new concept is created.

5. Principle of Complete Induction. *If the first element of a finite or simply infinite set* **X** *has a certain property, and if from the assumption that some element has the property it follows that the next element also has that property, then all the elements of the set* **X** *have said property.*

Suppose on the contrary that some elements of **X** do not have said property and let **Z** denote the set of such elements. The set **Z,** being a subset of **X,** has a first element K. That is, K is the first element not having the said property. Let J denote the immediate predecessor of K. Since J has the said property, it follows

that K has the property, which contradicts the definition of K. Therefore, the principle is proved.

THEOREM. *A finite set can be made empty by removing its elements one at a time.*

Let the given finite set be

$$\mathbf{X} = \text{A, B, C, } \cdots \text{, H, I, } \cdots \text{, L.}$$

The theorem is true for the set

$$\mathbf{X}' = \text{A}$$

since the single element A can be removed by Postulate 2. If the theorem is true for the set

$$\mathbf{X}'' = \text{A, B, C, } \cdots \text{, H,}$$

it is true for the set

$$\mathbf{X}''' = \text{A, B, C, } \cdots \text{, H, I}$$

by removing the element I and thus obtaining the set $\mathbf{X}''$ having the property. By the principle of induction the property is certain for the entire set.

6. Properties of Equivalent Sets. We shall now state certain useful properties of equivalent sets.

(1) *If the sets* **X** *and* **Y** *are equivalent, then the sets resulting from removing an element from each set are equivalent; and the sets resulting from adjoining an element to each set are equivalent.*

(2) *Finite sets* **X** *and* **Y** *are either equivalent or one of them is equivalent to a part of the other.*

This follows easily from the theorem on the exhaustion of finite sets.

(3) THEOREM OF DIRICHLET. *No finite set is equivalent to a part of itself.*

An indirect proof can be easily made, based on the theorem on exhaustion of finite sets.

This property is not true of infinite sets. In fact, Dedekind and others have used this property to distinguish finite and infinite sets. That is, they defined an infinite set as one equivalent to a part of itself and then defined finite set as one that is not infinite. The approach used here, that of defining finite sets first, seems far more natural and it constitutes a direct approach to the most important fundamental concepts of arithmetic.

(4) *If the finite sets* **X** *and* **Y** *are equivalent by a certain mode of correspondence, then they are equivalent under any mode of correspondence.*

Let us suppose that

$$\mathbf{X} \cdot\cdot \mathbf{Y}. \tag{1}$$

If the correspondence is established in some other manner and the results are not equivalent, then we have by Property 2

(2) $$\mathbf{X}' \cdot\cdot \mathbf{Y} \quad \text{or} \quad \mathbf{X} \cdot\cdot \mathbf{Y}'$$

where $\mathbf{X}'$ is a part of $\mathbf{X}$ and $\mathbf{Y}'$ a part of $\mathbf{Y}$. But it follows from (1) and (2), and the transitive property of equivalence that

$$\mathbf{X}' \cdot\cdot \mathbf{X} \quad \text{or} \quad \mathbf{Y}' \cdot\cdot \mathbf{Y}$$

which contradicts the Theorem of Dirichlet.

COROLLARY. *If the finite sets* $\mathbf{X}$ *and* $\mathbf{Y}$ *are not equivalent under a certain mode of correspondence, then they remain non-equivalent under any mode of correspondence.*

(5) *Every set equivalent to a finite set is also finite.*

A finite set has the property, by definition, that every subset has a first and a last element. It is obvious that any set equivalent to a finite set would have the same property in view of the correspondence between the sets.

(6) *All orderings of a finite set are equivalent.*

For, any finite set is equivalent to itself and, therefore, equivalent to any ordering of it by Property 4.

COROLLARY. *If a set is infinite with a certain ordering, then it remains infinite under any ordering.*

7. Fundamental Succession. The above properties of equivalent sets, which are intuitively known and applied by everyone, lead to the construction of a sequence of sets such that no two of them are equivalent and such that one of them is equivalent to an arbitrary given finite set. For, let A be an object; by adjoining an object B, we obtain a set A, B. By adjoining another object C, we obtain a set A, B, C. Proceeding always in the same manner (Postulate 2) we form the so-called fundamental sequence of sets:

$$\text{A; A, B; A, B, C; A, B, C, D; } \cdots .$$

In this sequence each set is a subset of the succeeding sets, whence no two of them are equivalent. Also, it is easily seen that any finite set $\mathbf{W}$ is equivalent to one of these sets in the fundamental sequence. For if $\mathbf{W}$ is equivalent to the sets $\mathbf{X}$ and $\mathbf{Y}$ of the fundamental sequence, then $\mathbf{X}$ and $\mathbf{Y}$ are equivalent which we have seen is not true.

Since every finite set is equivalent to one and only one set of the fundamental sequence, the human mind has created number symbols to represent the sets which are equivalent to each one in the sequence. These number symbols have names. Thus, the number corresponding to the sets equivalent to (A) is denoted by the symbol 1 and is called *one;* the number corresponding to the sets which are equivalent to (A, B) is denoted by 2 and is called *two;* the number corresponding to the sets which are equivalent to (A, B, C) is denoted by the symbol 3 and is called *three;* and so forth.

The concept of these particular numbers develops little by little in the child's mind and the general concept of number is formed with them, that is,

the concept of the numbers as the common entity to any equivalent sets (Def. 10). This general concept of number may be thought of as a *genus* for which the particular numbers 1, 2, 3, · · · are the *species.*

Since the fundamental sequence constitutes an infinite set of sets (simply infinite) it is impracticable to have distinct and unrelated symbols for each number. This difficulty has been overcome by establishing a set of rules by means of which all numbers can be represented with a finite number of symbols. This set of rules is called a system of numeration. The system in use, the so-called *decimal system*, makes use of the symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, together with the symbol 0 (zero) which is taken to represent the null or empty set.

The set of numbers

$$1, 2, 3, \cdots, 9, 10, 11, \cdots, 101, 102, 103, \cdots$$

which represents the fundamental sequence of sets in the manner described above is called the *sequence of natural numbers.*

8. Cardinal Numbers. Take a set of objects of any kind, for example

$$\mathbf{X} = \mathbf{M}, \ \mathbf{N}, \mathbf{P}, \mathbf{S}, \mathbf{T}.$$

If we suppose for a moment that this set contains only the element M, it is equivalent to (A) and is represented by the number 1, which we shall indicate by writing M_1. Similarly, we write M_1, N_2 if the set contains only the elements M and N. Continuing this way, we write

$$\mathbf{M}_1, \ \mathbf{N}_2, \mathbf{P}_3, \mathbf{S}_4, \mathbf{T}_5$$

to indicate that the set **X,** considered as having all its elements, is equivalent to (A, B, C, D, E) of the fundamental sequence and represented by the number 5. It is seen that the number of a set is assigned to the last element and for that reason is called *cardinal* (principal) *number* of the set. In practice, this method of designating the elements in an ordering of a set is called counting the elements of the set. The number corresponding to the last element is the cardinal number of the set.

Observing that if we count the elements in distinct orderings we obtain the same cardinal number, because we have already seen that all orderings of a finite set are equivalent, we have

The cardinal number of a set is independent of the ordering of the set.

This is called, according to Schröder, the principle of the invariance of the number.

The sets of the fundamental succession are considered ordered in the following way: A set precedes another if it is a part of the other, as for example, the set (A, B, C) precedes the set (A, B, C, D, E). The set of natural numbers becomes ordered by correspondence with the fundamental succession, that is, for example, 3 precedes 5. Finally, the sets in nature may be distinctly ordered by correspondence, by means of the process called enumeration, considering that an

object precedes another when its attached number precedes that of the other. Numbers used for this purpose are called *ordinal* numbers.

9. Equality and Inequality of Natural Numbers. Let **M** and **N** denote any two finite sets with cardinal numbers m and n respectively. It follows from Property 2 of Section 6 that one and only one of the following cases can arise:

(a) $\mathbf{M} \cdot\cdot \mathbf{N}$, in which case m and n represent the same number and this is denoted by the symbolism $m=n$;

(b) $\mathbf{M} \cdot\cdot \mathbf{N}'$, where $\mathbf{N}'$ is a part of **N**, in which case m is said to be less than n and this is denoted by the symbolism $m<n$;

(c) $\mathbf{M}' \cdot\cdot \mathbf{N}$, where $\mathbf{M}'$ is a part of **M**, in which case m is said to be greater than n and this is denoted by the symbolism $m>n$.

It is easily shown that the relation of equality of numbers is *reflexive*, *symmetric*, and *transitive* and that the inequality relations $>$ and $<$ are not reflexive nor symmetric but are transitive.

10. Sum of Natural Numbers. We shall first consider the sum of finite sets.

EXISTENCE THEOREM. *Given the finite sets* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \cdots, \mathbf{N}$ *which have no elements in common, then there exists a set* **S** *formed exclusively of the elements of the sets. The set* **S** *is called the sum of the sets* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \cdots, \mathbf{N}$.

This follows from the principle of exhaustion of finite sets and Postulate 2.

THEOREM. *The sum of a finite number of finite sets is a finite set.*

For, the two conditions of Definition 6 are satisfied:

(1) The sum may be ordered. It will suffice to order the set of sets and the elements of each set. This is possible since all sets involved are finite. Once this has been done, for any two elements P and Q we have the following: If they are in different sets, we shall consider P as preceding Q when P belongs to a set which precedes the set to which Q belongs; if they belong to the same set, we keep for them the same order as they have in the set.

(2) Every subset of the total set has a first and a last element. For, among the sets there will be a first one and a last one containing elements of the chosen subset. These elements will form within each set partial subsets with a first and a last element.

The following laws of sums of finite sets are easily proved:

(1) UNIFORM: *The sum of equivalent sets results in equivalent sets.*

(2) COMMUTATIVE: *The sum set* **S** *of the sets* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \cdots, \mathbf{N}$ *is independent of the order in which they are added.*

(3) ASSOCIATIVE: *The same sum set* **S** *is obtained by adding any two of the sets* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \cdots, \mathbf{N}$ *and to that result the remaining sets.*

(4) MONOTONIC: *If* $\mathbf{A}>\mathbf{B}$, *then* $(\mathbf{A}, \mathbf{C})>(\mathbf{B}, \mathbf{C})$ *for every finite set* **C**.

To define the *sum of natural numbers*, suppose that the finite sets $\mathbf{A}, \mathbf{B}, \mathbf{C}, \cdots, \mathbf{N}$ have the cardinal numbers $a, b, c, \cdots, n$, respectively. Then the sum set **S** is finite and its cardinal number s is defined to be the sum of the numbers $a, b, c, \cdots, n$, and is written

$$s = a + b + c + \cdots + n.$$

The operation of addition satisfies the following laws which are entirely analogous to the above laws for sum of sets.

(1) UNIFORM: *If* $a=a', b=b', \cdots, n=n'$, *then* $a+b+\cdots+n=a'+b'+\cdots+n'$.

(2) COMMUTATIVE: $a+b=b+a$.

(3) ASSOCIATIVE: $a+b+c=(a+b)+c=a+(b+c)$.

(4) MONOTONIC: *If* $a>b$, *then* $a+c>b+c$ *for every number* c.

These laws follow from the corresponding laws for sums of sets.

A number which if added to any number x results in the same number x is called the *unit of addition*. It is easily seen that the unit of addition is zero, that is,

$$a + 0 = a.$$

Furthermore, from the monotonic law, if $c>0$, then

$$a + c > a.$$

11. Multiplication of Natural Numbers. We call the *product* of a number a and a number b $(b>1)$ the sum of b summands each equal to a, and is written ab or $a\times b$.

The operation of multiplication obeys the following laws:

(1) UNIFORM: *If* $a=a'$ *and* $b=b'$, *then* $ab=a'b'$.

(2) DISTRIBUTIVE: $(a+b+c)n=an+bn+cn$.

(3) COMMUTATIVE: $ab=ba$.

(4) ASSOCIATIVE: $abc=(ab)c=a(bc)$.

(5) MONOTONIC: *If* $a>b$, *then* $ac>bc$, *for* $c>0$.

The products $1\times b$ and $0\times b$ are expressed by

$$1 \times b = 1 + 1 + \cdots + 1 = b,$$

$$0 \times b = 0 + 0 + \cdots + 0 = 0,$$

according to the above definition. This definition has no meaning, however, in the case of the expressions $b\times 1$ and $b\times 0$. With the object of preserving the commutative law for multiplication of any two numbers without exception, we shall agree to attribute to the expressions $b\times 1$ and $b\times 0$ the same values as to $1\times b$ and $0\times b$, respectively. That is, we agree upon the following relations:

$$b \times 1 = 1 \times b = b, \qquad b \times 0 = 0 \times b = 0,$$

and in particular $1\times 1=1$, $1\times 0=0$, and $0\times 0=0$. It is easily verified that the above five laws of multiplication are valid under these interpretations.

The operations of subtraction and division can now be defined and the arithmetic of natural numbers further developed. This development will not be carried further here since our primary object has been to develop a simple and logical basis for the arithmetic of natural numbers, suitable for presentation to students in what might be called College Arithmetic.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 901. *Proposed by Leo Moser, University of Manitoba*

A company offers its 350 employees a bonus of \$10.00 to each male and \$8.25 to each female. All the females accept, but a certain percentage of the males refuse to accept. Knowing what this percentage is, one can deduce the total amount paid out. What is the total amount paid to the women?

E 902. *Proposed by R. V. Andree's class in Solid Analytics, University of Oklahoma*

A similar argument to that used in E 832 [1949, 407] may be used to "prove" the following theorem: "If a sphere has a center with at least one irrational coordinate, then there are at most three points on the sphere with rational coordinates." Give a counter example to show that this theorem is not true and discuss the point at which the "proof" breaks down.

E 903. *Proposed by F. A. E. Pirani, Carnegie Institute of Technology*

Show that every even integer greater than 46 can be expressed as the sum of two abundant numbers. (See E 848, [1949, 478].)

E 904. *Proposed by C. S. Ogilvy, Columbia University*

Find the equation of the locus of centers of curvature of $y=x^n$ at the point (1, 1) as n varies continuously from $-\infty$ to $+\infty$.

E 905. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let A_1 and A_2, B_1 and B_2, C_1 and C_2 be the points of intersection with the sides BC, CA, AB of a triangle ABC of the polars of a point P with respect to the escribed circles (I_b) and (I_c), (I_c) and (I_a), (I_a) and (I_b). For what position of P will the points A_1 and A_2, B_1 and B_2, C_1 and C_2 be, simultaneously, symmetric with respect to the midpoints of BC, CA, AB?

SOLUTIONS

A Hyperboloid of One Sheet

E 868 [1949, 338]. *Proposed by P. D. Thomas, Washington, D. C.*

Let P and Q be, respectively, the feet of the common perpendicular to two

fixed skew lines p and q. A variable line r meets p in R and q in S. Find the locus of r if the volume of the tetrahedron $PQRS$ is constant. Also find the locus of the centroid of $PQRS$.

Solution by N. D. Lane, Acadia University, Nova Scotia. Let P be the origin, p the x-axis, PQ the z-axis, Q the point $(0, 0, k)$, q the line $y=mx$, $z=k$, R the point $(a, 0, 0)$, and S the point (b, mb, k). Then equations for r are

$$(x - a)/(b - a) = y/mb = z/k,$$

from which

$$a = k(y - mx)/m(z - k) \quad \text{and} \quad b = yk/zm.$$

Now the constant volume, V, of the tetrahedron $PQRS$ is

$$V = \pm (1/6)\begin{vmatrix} a & 0 & 0 & 1 \\ b & mb & k & 1 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \pm abmk/6,$$

whence

$$ab = \pm 6V/mk = k^2y(y - mx)/m^2z(z - k),$$

and the equation satisfied by any point on r is

$$k^3y^2 \mp 6Vmz^2 - k^3mxy \pm 6mVkz = 0,$$

which represents a hyperboloid of one sheet.

For the centroid $(\bar{x}, \bar{y}, \bar{z})$ we have

$$\bar{x} = (a + b)/4, \qquad \bar{y} = mb/4, \qquad \bar{z} = 2k/4,$$

whence

$$a = 4\bar{x} - b = 4\bar{x} - 4\bar{y}/m, \qquad b = 4\bar{y}/m,$$

and the relation $ab = \pm 6V/mk$ becomes

$$8k\bar{y}^2 - 8km\bar{x}\bar{y} \pm 3mV = 0.$$

Thus the locus of the centroid is the above hyperbola in the plane $z=k/2$.

Also solved by J. E. Darraugh, Roger Lessard, and the proposer.

A Polynomial the Square of Another Polynomial

E 869 [1949, 338]. *Proposed by P. T. Bateman, Institute for Advanced Study*

If a polynomial $f(x)$ with integral coefficients has the property that $f(n)$ is a perfect square for all integers n, then $f(x)$ is the square of another polynomial with integral coefficients.

Solution by W. H. J. Fuchs, Cornell University. The given problem is the special case $g(x)=x^2$ of the following more general

THEOREM: *If $f(x)$ and $g(x)$ are polynomials and if for every integer $p>p_0$ there is an integer $q=g(p)>0$ such that $f(p)=g(q)$, then $f(x)=g(h(x))$, where h is a polynomial. If f and g have integral coefficients and g has leading coefficient* 1, *then h also has integral coefficients.*

Proof: We note that $g(y)=f(x)$ defines y as an algebraic function of x. If g is of degree n, f of degree m, then for large x

$$y = x^{m/n}\sum_{k=0}^{\infty}\gamma_k x^{-k/n}. \tag{1}$$

If x is a positive integer, then, for a suitable determination of the nth root of x, y is also a positive integer. This determination is the same for all large x, since $y=\gamma_q x^{(m-q)/n}(1+o(1))$, where q is the least integer with $\gamma_q\neq 0$. There is no loss of generality in assuming that $x^{1/n}$ has to be chosen positive.

Let $x=\nu^n p$, $(\nu=1, 2, \cdots, h)$. By (1)

$$y = q(\nu^n p) = \sum_{k=0}^{h-1}\gamma_k p^{(m-k)/n}\nu^{m-k} + O(p^{(m-h)/n}) \tag{2}$$

as $p\to\infty$. Regard (2) as a system of equations for the unknowns $\gamma_k p^{(m-k)/n}$, $(k=0, 1, \cdots, h-1)$. The determinant of this system is

$$\Delta = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 2^m & 2^{m-1} & \cdots & 2^{m-h+1} \\ \cdot & \cdot & \cdots & \cdot \\ h^m & h^{m-1} & \cdots & h^{m-h+1} \end{vmatrix} \neq 0.$$

Solving (2),

$$\Delta\gamma_k p^{(m-k)/n} = \text{integer} + O(p^{(m-h)/n}), \qquad (k = 0, 1, \cdots, h-1). \tag{3}$$

If $h>k>m$ this is impossible unless $\gamma_k=0$. Hence (1) contains no negative powers of $x^{1/n}$ and so the O-term in (2), and therefore in (3), does not occur if $h=m+1$. Then (3) shows that $\gamma_k p^{(m-k)/n}$ is rational for every large p. This is possible only if $\gamma_k=0$ for all $k\not\equiv m \pmod n$. Hence (1) reduces to a polynomial $h(x)$ in x. By (3) $h(x)$ has rational coefficients.

To prove the last part of the theorem observe that if $N>1$ is the least common denominator of the coefficients of $h(x)$, then N^p is the least common denominator of the coefficients of $h^p(x)$, by a well known theorem. Hence $g(h(x))$ cannot have integral coefficients, since $N^{n-1}g(h(x))$ does not have integral coefficients ($n=$degree of g).

Also solved by L. M. Kelly.

Alex Rosenberg said that the problem, with a corresponding generalization to higher powers, seems first to have been proposed by Ch. Brisse as problem

No. 37 on p. 10 of vol. 1 of the *Intermédiaire des Mathématiciens* (1894). Two correct solutions were published in the same journal by I. Ivanoff (p. 74, vol. 1, 1894) and J. Franel (p. 94, vol. 2, 1895). The problem was then again proposed by R. Jentzsch in the *Archiv der Mathematik und Physik* (ser. 3, vol. 19, p. 361, 1912), and a third distinct solution was published in the same journal by W. Grosch (ser. 3, vol. 21, p. 368, 1913). The first two solutions are elementary in character, whereas the third solution is somewhat more sophisticated.

Also see problem 114, p. 135 of Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, II.

Forty-five Degree Parallelograms

E 870 [1949, 338]. *Proposed by Joseph Rosenbaum, Hartford, Conn.*

Characterize quadrilaterals $A_1B_1C_1D_1$ such that if A_2, B_2, C_2, D_2 are the circumcenters of $A_1B_1C_1$, $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, then A_1, B_1, C_1, D_1 are the circumcenters of $A_2B_2C_2$, $B_2C_2D_2$, $C_2D_2A_2$, $D_2A_2B_2$.

Solution by S. T. Thompson, Tacoma, Washington. By hypothesis

$$A_1A_2 = A_2C_1 = C_1C_2 = C_2A_1 = A_1B_2 = A_2B_1 = C_1D_2 = C_2D_1, \tag{1}$$

$$B_1B_2 = B_2D_1 = D_1D_2 = D_2B_1 = B_1C_2 = B_2C_1 = D_1A_2 = D_2A_1. \tag{2}$$

From the first three equalities of (1) and the first three of (2) it follows that $A_1A_2C_1C_2$ and $B_1B_2D_1D_2$ are rhombi. From the last three equalities of (1) and the last three of (2) it follows that the two rhombi are squares with a common center. This, in turn, guarantees that $A_1B_1C_1D_1$ is a parallelogram. Moreover, since $A_2A_1=A_2B_1=A_2C_1$, we have that angle $A_1B_1C_1$ equals 45° or 135°. Thus $A_1B_1C_1D_1$ is a 45° parallelogram. One can also easily show, conversely, that a 45° parallelogram satisfies the required conditions.

If the words "taken in some order" should be tacked onto the statement of the problem, then we would have, in addition to 45° parallelograms, all orthocentric quadruples.

Also solved by R. G. Buschman, L. M. Kelly, and C. S. Ogilvy. The proposer stated that the corresponding problem for five points in space seems to be not easy.

Values of a Parameter in a Trigonometric Identity

E 871 [1949, 403]. *Proposed by W. R. Ransom, Tufts College*

Is there any value of k other than $k=1$ for which $k \tan x \cot kx = 1$?

Solution by N. D. Lane, St. Andrew's College, Ontario. If $k=0$ the left side is indeterminate. Consider $y_1=k \tan x$ and $y_2=\tan kx$. If $y_1=y_2$ in any interval $a \leqq x \leqq b$ for which y_1 is finite, then $dy_1/dx=dy_2/dx$ in that interval. That is, $k \sec^2 x = k \sec^2 kx$, and we must have $k=\pm 1$. Clearly each of these values is acceptable.

Also solved by P. M. Anselone, W. D. Berg, A. B. Boggs, N. G. Gunderson,

Vern Hoggatt, M. S. Klamkin, Roger Lessard, C. S. Ogilvy, L. B. Rall, A. P. Rhodes, L. A. Ringenberg, and J. W. Sawyer.

The Enthusiastic Problemist

E 872 [1949, 403]. *Proposed by Leo Moser, University of Manitoba*

An enthusiastic problemist proposes at least one problem every day. In order not to overwork problem editors, however, he does not propose more than 730 problems a year. Given any positive integer n, show that he proposes exactly n problems in some set of consecutive whole days. (Dedicated to Victor Thébault.)

Solution by the Proposer. Let S_i denote the number of problems proposed in the first i days, starting today, say. Consider the set of integers $\{S_i\}$, $i=1, 2, \cdots, N$, where $N=2921n$. Notice that $2921n$ days $\leqq 8n$ years, taking leap years into account. Consider also the set of integers $\{n+S_i\}$, $i=1, 2, \cdots, N$. In the two sets together we have $2N=5842n$ integers, the largest of which is $n+S_N \leqq n+8n(730)=5841n$. Hence there is at least one integer common to both sets, say $S_j=S_k+n$, and exactly n problems will be proposed on the days numbered $k+1, k+2, \cdots, j$.

A Generalized Scalar Product

E 873 [1949, 404]. *Proposed by I. M. Hostetter, Oregon State College*

Let a vector be interpreted as a directed line segment and let $A * B$ be a symbolic product of two arbitrary vectors A and B such that (1) $A * B$ is a scalar, (2) $A * B=B * A$, (3) $A * (B+C)=A * B+A * C$. Is there an interpretation of $A * B$ other than the familiar inner, or scalar, product: $|A|\ |B| \cos\theta$, where $|A|$, $|B|$ are the (positive) lengths of A and B, and θ is the angle between A and B?

Solution by C. M. Ablow, Brown University. Under the very weak assumption that $f(x)$ is bounded in the neighborhood of some point, one may prove the

Lemma: If, for all x and y, $f(x+y)=f(x)+f(y)$, then $f(x)=kx$, where k is a constant.

Returning to the problem, let the components in rectangular coordinates of vector A be indicated by (a_1, a_2, a_3). Consider vectors A, B, C with but one non-zero component each, a_i, b_j, c_j, (note that B and C have the same component appearing), and denote these vectors by (a_i), (b_j), (c_j) respectively. Then (3) becomes

$$(a_i) * (b_j + c_j) = (a_i) * (b_j) + (a_i) * (c_j).$$

Thinking of a_i as fixed and b_j and c_j arbitrary this last is of the form of the lemma. Hence $(a_i) * (b_j)=kb_j$, where k is a number independent of b_j. However, (2) shows that we must have $(a_i) * (b_j)=k_{ij}a_ib_j$, where k_{ij} is now a constant, being independent of both variables, for which $k_{ij}=k_{ji}$. Now

$$(a_i) * (b_1, b_2, b_3) = (a_i) * (b_1) + (a_i) * (b_2) + (a_i) * (b_3)$$

$$= k_{i1}a_ib_1 + k_{i2}a_ib_2 + k_{i3}a_ib_3,$$

and applying this last to $(a_1, a_2, a_3) * (b_1, b_2, b_3)$ we see finally that our most general result is

$$A * B = \sum_{i,j=1}^{3} k_{ij}a_ib_j,$$

with $k_{ij}=k_{ji}$ constants.

Also solved by Vern Hoggatt and M. S. Klamkin who gave the rather trivial interpretations $A * B = -A \cdot B$ and $A * B = kA \cdot B$ respectively.

Editorial Note. The proposer is primarily interested in a geometrical or physical interpretation. Can a nice geometrical meaning be assigned to the above general symmetric bilinear form?

The Life-Buoy

E 875 [1949, 404]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A life-buoy is formed by two right circular cones joined by their common base. (1) Show that the centroid of the solid coincides with the centroid of the vertices of the quadrangle obtained by cutting the surface of the buoy by an axial plane. (2) Determine the ratio of the altitudes of the two cones in order that the buoy may be in equilibrium when it rests, on a horizontal plane, on one of the two surfaces, or indifferently on one or the other of the two surfaces, of the two cones.

Solution by Vern Hoggatt, Oregon State College. Take the x-axis coincident with the axis of the buoy, the origin at the center of the common base of the cones. Let the altitudes of the cones be h_1 and h_2, $(h_2 \geqq h_1)$, and the radius of the common base be r. Since the centroid of a right circular cone lies on the axis of the cone at one fourth of the altitude from the base plane we have, where $\bar{x}$ is the x-coördinate of the centroid of the buoy,

$$(1/3)\pi r^2 h_2(h_2/4 - \bar{x}) = (1/3)\pi r^2 h_1(h_1/4 + \bar{x}),$$

or $\bar{x}=(h_2-h_1)/4$. This is also the x-coördinate of the centroid of the vertices of the quadrangle. The buoy will rest indifferently on either conical surface if and only if the perpendicular dropped from the centroid of the buoy upon an element of the smaller cone will hit that element. This happens if and only if $\bar{x}/r \leqq r/h_1$, or if and only if

$$h_2 \leqq (4r^2 + h_1^2)/h_1.$$

If h_2 exceeds this value the buoy will always come to rest on the surface of the larger cone.

Also solved by D. H. Browne, N. D. Lane, W. O. Pennell, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4380. *Proposed by P. A. Piza, San Juan, Puerto Rico*

For arbitrary positive integers n and k let

$$S_1(n, k) = 1^k + 2^k + \cdots + n^k,$$

and put

$$S_{p+1}(n, k) = S_p(1, k) + S_p(2, k) + \cdots + S_p(n, k)$$

with $p = 1, 2, \cdots$. Thus $S_p(n, k)$ is the pth iterated sum, the sum of the sum of . . . the sum of the first n perfect kth powers.

Show that $S_p(n, k)$ is a polynomial in n of degree $p+k$ and determine the form of the polynomial for $k = 3, 4, 5$. If possible, determine the form for general k. (The case $k = 2$ is the subject of problem 22, *Mathematics Magazine*, vol. XXII, no. 1, p. 51.)

4381. *Proposed by R. J. Walker, Cornell University*

The inverse square law has the property that the attraction, at an external point, due to a sphere of uniform density is the same as if the sphere were concentrated at its center. Are there any other laws of attraction which have this property?

4382. *Proposed by E. P. Starke, Rutgers University*

Let $f_1(x)$ be Riemann integrable in the interval $0 \leqq x \leqq M$ and let

$$f_{n+1}(x) = \int_0^x f_n(x)dx, \qquad n = 1, 2, \cdots.$$

Show that

$$\phi(x) = \sum_{n=1}^{\infty} f_n(x)$$

is defined and continuous in the interval except perhaps at discontinuities of $f_1(x)$, and find a simple expression for $\phi(x)$.

4383. *Proposed by Robert Steinberg, University of California, Los Angeles*

Let n symbols be ordered in two different ways, and let a_{ij} denote the number of symbols common to the first i in the first ordering and the first j in the second ordering. Prove that the determinant $|a_{ij}|$ is 1 if the transition from the first ordering to the second is effected by an even permutation and -1 if it is effected by an odd permutation.

4384. *Proposed by H. F. Sandham, Trinity College, Ireland*

In the following modification of the harmonic series there is a change of sign after the reciprocal of each square. Prove that

$$\frac{1}{1}-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{1}{10}-\frac{1}{11}-\frac{1}{12}-\cdots$$

$$=\sum_{n=1}^{\infty}\frac{\pi}{\sqrt{n}\sinh \pi\sqrt{n}}.$$

SOLUTIONS

Twelve Point Spheres

4265 [1947, 479]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If two tetrahedra are homothetic with respect to their common centroid, the twelve point sphere of one of these tetrahedra is tangent to the twelve point spheres of the four tetrahedra which the planes of its faces cut off from the trihedral angles of the other tetrahedron.

*Solution by the Proposer.** Let $T\equiv ABCD$ and $T'\equiv A'B'C'D'$ be the two given tetrahedra, homothetic with respect to their common centroid G. Since the trihedral angles (A) and (A') are homothetic with respect to G, the line GAA' contains the centroids G_a and G_a' of the homologous faces BCD and $B'C'D'$. Also the centroid of the triangle $B_1C_1D_1$ cut from the plane BCD by the trihedral angle (A') coincides with G_a. Hence the triangles BCD and $B_1C_1D_1$ and the tetrahedra T and $A'B_1C_1D_1$ are homothetic with respect to G_a (provided that the homothetic ratio for center G is not $-\frac{1}{3}$). The twelve point spheres of T and $A'B_1C_1D_1$ are also homothetic with respect to G_a. The twelve point sphere of T passes through G_a, hence the twelve point sphere of $A'B_1C_1D_1$ is tangent to that of T at G_a.

Note. The second twelve point spheres of two orthocentric tetrahedra homothetic with respect to their common orthocenter have an analogous property.

Nine-point Circle and Artzt Parabolas

4294 [1948, 254]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The lines joining the orthocenter of a triangle to the points of intersection of the medians with the nine-point circle (other than the midpoints of the sides),

* Translated by W. E. Byrne, Virginia Military Institute.

pass through the vertices of parabolas tangent to two sides of the triangle and having the third side for chord of contact.

Solution by L. M. Kelly, Michigan State College. We use the following notation: A_i are the vertices of the triangle, M_i are the midpoints of the sides, H_i the feet of the altitudes, H the orthocenter, P_i the specified intersections of the medians with the nine-point circle. Let the equation of the parabola be $y=kx^2$. Using familiar properties of the parabola we know that (a) the median A_3M_3 is parallel to the axis of the parabola, (b) the line M_1M_2 is tangent to the parabola, and (c) the orthocenter of the triangle $A_3M_1M_2$, circumscribed to the parabola, is on the directrix. Since the triangle $A_3M_1M_2$ is homothetic to triangle $A_1A_2A_3$ in the ratio 1/2, the orthocenter K_3 of $A_3M_1M_2$ is at the midpoint of the segment A_3H. This point is also on the nine-point circle, as are P_3, M_3, and H_3. Thus $K_3P_3M_3H_3$ is a cyclic quadrilateral and K_3P_3 is therefore perpendicular to the axis, that is, P_3K_3 is the directrix. The coördinates of P_3 are thus easily seen to be $(x_1+x_2)/2$, $-1/4k$.

The coördinates of the orthocenter are now computed in a straightforward fashion and are seen to be $(x_1+x_2)(2k^2x_1x_2+1)$, $-(2k^2x_1x_2+1)/2k$. Thus the slopes of the lines P_3O and HO are both equal to $-1/2k(x_1+x_2)$, so that P_3, O, and H are collinear as required.

Also solved by J. A. Bullard, Joseph Langr, and the Proposer.

Editorial Note. Bullard points out how easily the present proposition follows from his Note, *Further properties of parabolas inscribed in a triangle* (1937, 368). The Proposer gives the following further references to the properties of these parabolas, treated for the first time by Artzt, *Programme scolaire du gymnase de Recklinghausen* (1884): E. Lemoine, P. Barbarin, *Mathesis*, 1884, pp. 12, 142; H. Brocard and G. de Longchamps, *Journal de Mathématiques Spéciales*, 1885, pp. 109, 123; A. Bullard, THIS MONTHLY [1935, 606] and [1937, 368]; V. Thébault, *Archives du Grand-Duché* (Luxembourg) 1948. See also problem 4019 [1943, 517].

Automorphism

4295 [1948, 320]. *Proposed by Irving Kaplansky, University of Chicago*

Show that any group with more than two elements admits an automorphism other than the identity.

Solution by R. C. Buck, Brown University. If the group is not Abelian, any element a not in the center gives a non trivial inner automorphism $x\to a^{-1}xa$. If the group is Abelian, and the square of some element is not the identity, then $x\to x^{-1}$ is such an automorphism. If the square of every element is the identity, and the group has more than two elements, then it is the direct sum of cyclic groups of order 2; the mapping induced by interchanging the generators of two of these furnishes the desired automorphism.

Also solved by Roy Dubisch, J. W. Gaddum, Leo Moser, D. J. Newman, C. R. Phelps, Alex Rosenberg, J. A. Zilber, and the Proposer.

Continued Fractions

4296 [1948, 320]. *Proposed by H. S. Wall, University of Texas*

It is known that when the continued fraction

$$x = p - \cfrac{q}{p - \cfrac{q}{p - \cdot}}$$

converges, then its value is the numerically larger root of the equation $x^2 - px + q = 0$. On the other hand, Newton's formula for computing the roots by successive approximation is

$$x_{k+1} = x_k - \frac{x_k^2 - px_k + q}{2x_k - p} = \frac{x_k^2 - q}{2x_k - p}, \qquad k = 0, 1, 2, \cdots.$$

Show that if x_0 is an approximant of the continued fraction, then $x_1, x_2, x_3, \cdots$ are approximants of the continued fraction.

I. *Solution by J. S. Frame, Michigan State College.* The successive convergents P_n/Q_n to the continued fraction

$$x = a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cdot}}$$

are most easily displayed by the use of matrices. We may prove by induction that

$$\begin{pmatrix} P_n & b_nP_{n-1} \\ Q_n & b_nQ_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & b_n \\ 1 & 0 \end{pmatrix}, \qquad n > 0,$$

if we define $P_0 = 1$, $Q_0 = 0$, so that $P_1/Q_1 = a_1$.

In the present example where $a_i = p$, $b_i = -q$, we have

$$\begin{pmatrix} P_n & -qP_{n-1} \\ Q_n & -qQ_{n-1} \end{pmatrix} = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}\begin{pmatrix} P_{n-1} & -qP_{n-2} \\ Q_{n-1} & -qQ_{n-2} \end{pmatrix}.$$

Hence, $P_n = pP_{n-1} - qQ_{n-1}$, $Q_n = P_{n-1}$ and

$$\begin{pmatrix} P_n & -qQ_n \\ Q_n & P_n - pQ_n \end{pmatrix} = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}^n.$$

Squaring this we have

$$\begin{pmatrix} P_{2n} & -qQ_{2n} \\ Q_{2n} & P_{2n} - pQ_{2n} \end{pmatrix} = \begin{pmatrix} P_n & -qQ_n \\ Q_n & P_n - pQ_n \end{pmatrix}^2,$$

and expanding, we find

$$\frac{P_{2n}}{Q_{2n}} = \frac{P_n^2 - qQ_n^2}{Q_n(2P_n - pQ_n)}.$$

Hence if we set $x_k = P_n/Q_n$ in Newton's Method, we find $x_{k+1} = P_{2n}/Q_{2n}$. Thus each successive application of Newton's method is equivalent to doubling the length of the continued fraction.

If, however, we obtain x_{k+1} from x_k by the Variation of Newton's Method described in this MONTHLY [51 (1944), 36–38; 52 (1945), 212–214; 55 (1948), 90–94] then we find $x_{k+1} = P_{3n}/Q_{3n}$, so that each application of this method is equivalent to tripling the length of the continued fraction.

II. *Solution by Aaron Herschfeld, Washington, D. C.* Explicit formulas for $u_n = P_n/Q_n$ are given in various books (e.g. C. Smith, *A Treatise on Algebra*, 1892, p. 474; Barnard and Child, *Higher Algebra*, 1936, p. 393) where it is shown that if a and b are the roots of the equation $x^2 - px + q = 0$, then

$$u_n = (a^{n+1} - b^{n+1})/(a^n - b^n)$$

or $= a(n+1)/n$, according as $a \neq b$ or $a = b$. Using the relations $a + b = p$ and $ab = q$ it is easy to verify that

$$(u_n^2 - q)/(2u_n - p) = u_{2n}.$$

Hence if x_0 is an approximant, say u_n, then $x_1 = u_{2n}$, $x_2 = u_{4n}$, $\cdots$.

Also solved by R. C. Buck, V. F. Ivanoff, Roger Lessard, W. T. Scott, and the Proposer.

Symmetric Functions of 1, 2, $\cdots$, n

4297 [1948, 320]. *Proposed by Paul Erdös, Syracuse University*

Put $\phi_k^{(n)} = \sum r_1 r_2 \cdots r_k$, where the r's run through all integers $\leqq n$, and the r's are all different. $\phi_n^{(n)} = n!$ It is well known that $\phi_{n-1}^{(n)} \not\equiv 0 \pmod{\phi_n^{(n)}}$. Prove that if n is sufficiently large $\phi_k^{(n)} \not\equiv 0 \pmod{\phi_n^{(n)}}$ for any k. (This is not always true; e.g., $n = 3$, $k = 1$).

Editorial Note. As noted by Leo Moser, the present problem is equivalent to Theorem 1 of the paper, *Some Properties of Partial Sums of the Harmonic Series*, by Erdös and Niven (Bulletin of the American Mathematical Society, v. 52, 1946, pp. 248–251.) The editors regret the late date at which the present proposal was printed since its submission by the Proposer actually antedated the Bulletin article by several years.

A detailed solution by Fritz Herzog gives essentially the same solution as that in the cited paper, except for the use of the prime number theorem in a somewhat different form.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Analytic Geometry. By Robin Robinson. New York, McGraw-Hill Book Company, Inc., 1949. 143 pages. $2.25.

This book is one of many that has appeared of recent date to serve in the capacity of a concise text covering the field of analytic geometry. In each case the authors have decided for themselves as to just where the brevity is to be effected in their texts. The author of this book sets forth three objectives, namely:

1. To provide a background for the calculus; but, at the same time, stress the geometric properties of the loci discussed.
2. To present each of the topics with a minimum amount of material so as not to steal the show from the teacher and thus permit him to develop the subject in his own way.
3. To provide an ample supply of well chosen problems so that students of all calibers may be able to improve their backgrounds and efficiency with the subject.

The reviewer is of the opinion that for the topics included in this *Analytic Geometry* the author has partially accomplished his aims, but has failed to do so completely because of the omission of certain vital topics essential for making analytic geometry the necessary background for the calculus which follows. Such topics as transcendental, exponential, higher plane algebraic and parametric equations are omitted from consideration in this text, and as a consequence it would fail in its most important aim of being a background for the calculus.

There are other minor criticisms such as, for example, the confusing use of the term common chord interchangeably with that of radical axis; the omission of the normal form for the equation of a straight line; and, the omission of the study for the angular bisector locus between two lines.

This text at best gives an above average treatment of analytic geometry up to and including the work on the conic section in the plane; and a standard introduction to space analytics up to and including quadric surfaces. The publishers should be commended for the excellent format of the book which has made whatever contents it does contain attractive in appearance.

S. SELBY

First Year College Mathematics with Applications. By P. H. Daus and W. M. Whyburn. New York. The Macmillan Company. 1949. 13+495 pages. $5.00.

The authors say in the preface, "This book is written with the intention of providing a single text for the study of first year college mathematics, especially

in engineering and technical schools. Its purpose is to provide a strong and ample background for the study of the calculus, and to integrate the subjects of college algebra, analytic geometry, and analytic trigonometry. At the same time the text illustrates all principles by applications taken from science and engineering, so that the course is completely independent of its future use. . . . Suggested lesson outlines for a ninety-hour course (three days a week for two semesters) and for a seventy-five-hour course are given. . . . "

Those who prefer to give relatively strong training in analytic geometry should welcome this book with enthusiasm. In sharp contrast with recent works which combine analytics with calculus and thereby effect a de-emphasis of the former subject, this text retains most of the traditional geometrical topics. Moreover, it provides a vigorous and well-motivated presentation which, in the opinion of the reviewer, is superior to that of many established books. The authors have compiled an extensive list of exercises many of which involve applications to engineering and scientific problems. Numerous sections are devoted to specific applications, for example—Electrical Resistance; Beam Equilibrium; Shear and Moment Diagrams; Electrical Networks; Suspension Bridges, Road Cross-sections; Bending Moments; Vibrating Spring; Vector Impedance; Voltage and Current; Series Circuits; Divided Circuits; Damped Harmonic Functions; Lissajous Figures. Of course these may be omitted without destroying continuity. By including a fair number of these sections, the instructor can introduce his students to several topics normally met in more advanced engineering work. In this way, the authors have furnished ample motivation for study of nearly all material included.

The student is assumed to be proficient in the purely manipulative aspects of algebra. Although the quadratic equation is discussed at some length, little is done with other topics (progressions, binomial theorem, logarithmic computation, *etc.*) normally handled in lower courses. Topics of questionable direct value to engineering students, such as permutations and combinations, classical probability, and mathematics of finance are omitted. Otherwise, most of the traditional college algebra content is included, although some topics such as inequalities are treated only as tools in geometric investigations. In connection with irrational roots of equations, linear and parabolic interpolation and Horner's Method are discussed. Solution of systems of linear equations is accomplished by elimination with an account of the Doolittle Method included. This is in line with the opinion of many engineers and mathematicians that, in practice, methods involving determinants are often not the most desirable. There are sections on hyperbolic, logarithmic and exponential functions and on solutions of transcendental equations.

The reviewer is pleased with the care exercised by the authors in connection with curve tracing where the single-valuedness of the radical is emphasized. He is less pleased with the many-valued interpretation of the same symbol in connection with roots of complex numbers. However, the whole treatment of these topics is certainly not inferior to that of most texts at this level.

In analytics, the straight line and the parabola play leading roles. The various forms of the linear equation are derived in a straight-forward way based upon the concept of slope. In the normal form discussion, the angle between the horizontal axis and the normal is always taken in the first or second quadrant. The positive side of the line is then the upper half-plane.

The treatment of the problem of curve fitting is extensive and well-done. There are sections on the method of moments, the logarithmic papers, least squares, and parabola of regression. Parallel scales and N-charts are discussed. The reviewer did not notice the omission of any topic in plane analytics which is generally considered as appropriate for study by the prospective engineer.

Two chapters are devoted to analytic trigonometry. These are rich in applications and the last chapter contains a superior discussion of polar coordinates. Computational trigonometry is assumed to have been previously covered by the student.

Each set of exercises begins with several labelled "oral." These should be of considerable help to the instructor in directing classroom discussion along profitable lines. The sections on special applications mentioned above will aid the inexperienced instructor as well as the student. The book seems to contain ample material and to be teachable subject to the assumption that the student is at least moderately well prepared. Persons in charge of freshman engineering courses certainly should give consideration to this text.

E. M. BEESLEY

Introduction to Complex Variable and Applications. By R. V. Churchill. New York, McGraw-Hill, 1948. 6+216 pages. $3.50.

This is a text in functions of a complex variable that should prove very popular with students of mathematics, physics and engineering at the senior or first year graduate level. While it is not a treatise on the subject, this book does present logically the parts of complex variable theory necessary for the applications made in the text. Other applications, especially of the theory of residues and in the evaluation of integrals, are to be found in the same author's *Modern Operational Mathematics in Engineering*. These two, with an earlier text, *Fourier Series and Boundary Value Problems*, form a set, closely related in purpose, presenting three important classical methods for the solution of boundary value problems: the separation of variables, the operational method, and now conformal mapping.

In the first four chapters of this book, the author introduces complex numbers and analytic functions, and makes a detailed study of the elementary functions. Then come chapters on integration, power series and residues. Following this are three chapters on conformal mapping and its applications, and the last two chapters are devoted to analytic continuation and Riemann surfaces.

Emphasis is placed on teaching the subject rather than on its development, although there is no sacrifice of logic in the development. The author does not use a standardized procedure in introducing and proving theorems, but states

theorems sometimes before, sometimes after the proof has been made, intending thereby to improve the clarity or preserve the continuity of his development. This technique is one that is carried over from his earlier texts, and helps to keep the wording clear, concise and yet unstrained.

Problems are usually chosen to illustrate points made in the text, with very few designed to extend the theory. In some cases this practice leads to many tedious drill problems, as in the exercises relating to the circular and hyperbolic functions, where proofs of numerous formulas are called for. Such specific problems, however, have more meaning for the student than those calling for an extension of work in the text.

Direct applications of the theory are made in the chapters on conformal mapping and the Schwarz-Christoffel transformation, where problems in steady-state temperature, electric potential and incompressible fluid flow are solved. The solution of many of these problems is facilitated by an illustrated table of conformal transformations in the appendix. Some applications of the theory of residues to the evaluation of integrals are also given. More of these are to be found in *Operational Mathematics*.

This book should be valuable as a text for a one semester course in complex variable, where emphasis on applications of conformal mapping and the theory of residues is desired. Answers to many, but not all, of the problems are given.

Very few minor errors were encountered, all of a typographical nature.

R. E. GASKELL

NEW BOOKS RECEIVED

Non-linear Problems in Mechanics of Continua. (Proceedings of Symposia in Applied Mathematics, vol. 1.) New York, American Mathematical Society, 1949. 8+219 pp. $5.25.

Regular Polytopes. By H. S. M. Coxeter. New York, Pitman Publishing Corp., 1947. 19+321 pp.

Mathematics for Finance and Accounting. By Coleman and Rogers. Pitman Publishing Corporation. 1949. $4.00

The Aims of Education and Other Essays. By A. N. Whitehead. New York, New American Library, 1949. 168 pp. $.35.

Analytic Geometry. Third Edition. By W. A. Wilson and J. I. Tracey. New York, Heath, 1949. 10+318 pp. $2.75.

Geometrical Tools. Revised Edition. By R. C. Yates. St. Louis, Educational Publishers, 1949. 194 pp.

University Mathematics. By J. Blakey. London, Blackie and Son, 1949. 8+527 pp. 25 s.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1948–49

Pi Mu Epsilon, Lehigh University

During the past academic year the *Pennsylvania Gamma* chapter of *Pi Mu Epsilon* at Lehigh University has enjoyed five meetings and an initiation banquet.

Papers presented during the year were as follows:

The Laplace transform, by Prof. R. Stoll

The Laplace transform, a second paper, by J. Harry Dowling

Some approximation methods, by Prof. C. Shook

The deviation of the Lorentz equation, by Mr. Alperin

Some boundary value problems. by Prof. Beer.

At the initiation banquet Prof. A. Pitcher presented a very interesting talk on topology entitled, *The ham sandwich problem.*

New officers were elected for the coming year as follows: Director, Prof. R. Stoll; President, Richard Orford; Vice-President, James Fulleylove; Secretary, Albert Christy; Treasurer, Charles Close.

Kappa Mu Epsilon, Mount Mary College

The *Wisconsin Alpha* chapter of *Kappa Mu Epsilon* held monthly meetings, at which the following papers were given by the members:

Mathematics and Aristotle, by Norma Harding

What about Copernicus?, by Betty Prossen and Rosemary White

Descartes, by Dorothy Vande Walle

Galileo, by Kathleen Hanley

Napier, by Mary Suehr

Trigonometry, by Mary Hunt

The Pythagorean theorem, by Wanda Kropp.

Officers for 1948–49 were: President, Norma Harding; Vice-President, Bernadine Spitznogle; Secretary, Mary Alice Gauerke; Treasurer, Marilyn Briggemann; Corresponding Secretary, Sister Mary Felice.

Dorothy Karner was elected President for the year 1949–50.

Mathematics Colloquia, Purdue University

The following lectures were presented to the members during the year:

The modern concept of infinity and *The topology of general function spaces*, by Prof. C. Kuratowski

Non-cut points, by Prof. K. Zarankiewicz

Grassmann algebras and determinants and *Cohomology theory of abelian groups*, by Prof. S. MacLane

On the mean value of an analytic function, by Dr. H. Shniad

Continuous spectra of quadratic forms and integral equations, by Prof. E. Hellinger

The application of differential equations to the derivation of sampling distributions, by Prof. O. Reiersol

Coefficients of Schlicht functions, by Prof. A. Schaiffer

Determinantal congruences in algebraic number fields, by Prof. R. Hull.

Pi Mu Epsilon, Montana State University

The following papers have been presented to the *Montana Alpha* chapter of *Pi Mu Epsilon* since its reorganization in February 1949.

Careers in science, by Prof. G. Shallenberger

Mathematics and chemistry, by Ted Burton

New absorption bands of the hydrogen molecule, by Paul Pfleuger.

Pi Mu Epsilon entrance prizes of twenty-five, fifteen, and ten dollars, given on the basis of an examination offered to freshmen from Montana High Schools, were awarded this year to Mr. Thronson, first prize; Mr. Wardahl, John Pecarich, and Keith Wilson, all tied for second prize.

Officers for the coming year are: Director, John Peterson; Vice-Director, Donald Philips; Secretary, Donald Marshall; Treasurer, Prof. T. Ostrom.

Kappa Mu Epsilon, Upsala College

During the first semester, the members studied *nomography*. By direction of the sponsor they followed a basic text and collateral reading, and at three successive meetings the material was discussed by Frances Rischmuller, Martin Monroe and James Gill.

During the second semester, the theme was *famous problems in geometry and analysis*. Each topic was prepared in the form of a seminar paper, and included the mathematical background and history of the problem, its ancient and more modern solutions, and its practical applications, both ancient and modern. The papers presented were:

Symmetric functions, by Edward Pulchlopek

The quadrature of the circle, by Robert Schenk

The duplication of the cube, by John Armstrong

The trisection of the general angle, by Robert Warnken

Axioms and postulates, by Robert Wallace

Hyperbolic functions, by Martin Netzler

The cycloid, by Jerome Drexler.

Other addresses heard were:

Zeno of Elia, by Dr. C. Vedova, of Newark College of Engineering

What's in a name? a talk on mathematical nomenclature, by Dr. C. Boyer, of Brooklyn College.

The officers for the year 1949–50 are: President, Victor Valentino; Vice-President, Hannah Magill; Secretary, Ethel Larson; Treasurer, Robert Ruppert; Historian, Hannah Magill; Corresponding Secretary, Prof. Martin Nordgaard.

Mathematics Club, Haverford College

The following papers were presented before the regular meetings of the Haverford *Mathematics Club*:

Functional equations, Prof. D. L. Thomsen, Jr.

The mathematics of music, by J. E. Brownlee

Fibonacci and Farey series with applications, by R. T. Ives.

Problems for which the Mathematics department offered prizes, were discussed at each meeting.

Prof. Kasimir Kuratowski, Visiting Lecturer of the American Mathematical Society from the University of Warsaw, Poland, was a guest of the college for a day. He talked informally with the students and delivered an address on *Modern concepts of infinity*.

Prof. C. O. Oakley served as Faculty Advisor of the club.

Kappa Mu Epsilon, Mount St. Scholastica College

Papers presented to the *Kansas Gamma* Chapter of *Kappa Mu Epsilon* for 1948–49 were:

Mathematics related to the modern world, by Gloria Jaskowiak

Empirical and theoretical statistics, by Mary Alice Weir

A survey of the history and development of non-euclidean geometries, by Sister Evangeline Anderson, O.S.B.

Modern men of mathematics, by Jeri Sullivan, Mary J. Martin, Frances Donlon, and Sally Crisanti

Modern women mathematicians, by Noreen Hurter, Gertrude Harrison, and Jeanne Culivan

Nomography for science students, by Mary Alice Weir.

The recipient of the Hypatiean Award, for her outstanding and original research on *Nomography*, was Mary Alice Weir. The award is made each year to the member making an outstanding contribution to the Chapter.

The Chapter, together with the mathematics department, presented an assembly for the school body centering around the application of mathematics in the life of the student. It included *A Mathematical Playlet*, produced by the pledges showing the application of mathematics in the various fields of knowledge, a choral reading *An Ode to the Queen of Sciences* and a discussion on the popular history of mathematics.

Newly elected officers are: President, Noreen Hurter; Vice-President, Mary Alice Weir; Secretary, Jeanne Culivan; Treasurer, Jeri Sullivan; Musician, Evelyn Wagner; Publicity Chairman, Patti Shideler.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

APPOINTMENTS AVAILABLE AT THE INSTITUTE FOR NUMERICAL ANALYSIS

Appointments of the following two types are being offered to properly qualified graduate students in pure and applied mathematics at the Institute for Numerical Analysis of the National Bureau of Standards at the University of California, Los Angeles.

(1) *Summer Studentships*

The purpose of these appointments is to introduce promising graduate students in mathematics and related fields to the work of the Institute for Numerical Analysis through the medium of an intensive program of mathematical seminars and courses in the use of modern computing equipment, and through active participation in the work of the Institute. The program lasts for ten weeks during the summer, from the middle of June to the end of August. Stipends range from $500 to $700.

(2) *Thesis Fellowships*

The purpose of these appointments is to provide the Institute for Numerical Analysis with a limited number of talented doctoral candidates whose research program, in the opinion of the Director of Research of the Institute, is relevant to the work of the Institute. A candidate for such a fellowship must have completed all residence and language requirements for the Ph.D. degree at an accredited university and must obtain the permission of the university to write the thesis in residence at the Institute. Preference is given to a candidate who has selected his thesis topic and has made some progress on the thesis prior to accepting the fellowship. Holders of these fellowships may absent themselves from the Institute for the purpose of taking necessary examinations, but travel money for such purposes will not be authorized. Additional course work may be taken at the University of California, Los Angeles, or elsewhere, but must be kept within a limit of six semester hours in each semester. The holder of a thesis fellowship will ordinarily not be permitted to accept additional employment for remuneration. If the thesis is published, it is expected that a credit line to the Institute and its financial sponsor, the Office of Naval Research, will be incorporated into the article, presumably as a footnote. The stipend for a thesis fellowship is at present fixed at approximately $2,000 for an 11-month working year, provided that the candidate already possesses an M.A. degree or equivalent experience. A thesis fellowship appointment is normally for one year only, but if circumstances warrant, the appointment may be extended for a period not to

exceed one additional year. Such an extension must receive the prior approval of the Civil Service Commission.

Five thesis fellowships are at present assigned by the Bureau to the Institute for Numerical Analysis.

All applicants should clearly understand that the central function of the Institute is to perform mathematical research and exposition pertinent to the effective use of automatic digital computing machinery. As supplementary to this main function, the Institute supports certain research projects in pure and applied mathematics which, while they do not directly bear upon the development of numerical analysis, nevertheless require extensive computations of a type for which automatic machinery is suitable. In other words, the work of the Institute concerns the development and application of the art and science of computing, and no attempt is made to conduct a well-rounded, general program of mathematical research and instruction.

In the case of Summer Studentships and the Thesis Fellowships, a preliminary application must be made to the Director of Research of the Institute for Numerical Analysis before the first of March preceding the summer or academic year for which the fellowship is requested. This preliminary application is to consist of: (a) a letter from the applicant to the Director of Research indicating his interests and experience; (b) a transcript of the applicant's undergraduate and graduate collegiate record to date; and (c) two supporting letters addressed to the Director of Research from scientists of established reputation who are familiar with the work of the applicant.

Appointees will in general be selected on the basis of the preliminary applications. The chosen candidates will be sent Civil Service Forms 57 to complete, and the formalities of appointment will thereafter be those of regular Civil Service recruitment. The awards of these fellowships will be made on or before April 1.

Other appointments at the Institute are made through regular Civil Service channels. There will be available, from time to time, positions on the staff suitable for younger mathematicians and physicists who have already obtained their doctorates, or who are of comparable standing.

CARE'S BOOK PROGRAM

The Book Program of CARE has been developed to aid in the educational and cultural reconstruction of foreign countries. A professional committee, headed by Dr. Luther Evans, Librarian of Congress, has selected 1300 titles in various categories. CARE is equipped to handle the purchasing and shipping of these books to institutions that need them. At present the list of countries to be served includes Austria, Belgium, Czechoslovakia, Finland, France, the American and French Zones of Germany and Berlin, Greece, Italy, Japan, the Netherlands, Norway, and the United Kingdom.

Contributions may be sent to: CARE, Book Program, 20 Broad St., New York 5, N. Y. (in Canada, Canadian CARE, Ottawa). Donors of $10 or more

may specify one or all of the following: country, type of institution or specific institution and categories of books.

PERSONAL ITEMS

Professor G. T. Whyburn of the University of Virginia has received an honorary degree of Doctor of Science from Washington and Lee University.

Brooklyn College reports that Assistant Professor Harriet Griffin has been promoted to an associate professorship and that Dr. M. E. Levenson, formerly of Cooper Union, has been appointed to an instructorship.

Columbia University announces: Professor Claude Chevally of Princeton University has been appointed to a professorship; Professor Edward Kasner has retired with the title of Adrain Professor Emeritus.

At Montana State College: Mr. Adrien Hess has been promoted to an assistant professorship; Mr. Robert Glauz, Mr. William Swartz, Mr. J. E. Whitesitt have been appointed to instructorships.

New York University reports: Instructors J. van Heijenoort and John Schoonmaker have been promoted to assistant professorships; Dr. G. E. Noether has been appointed to an instructorship; Professor H. R. Cooley has been on sabbatical leave for the first semester 1949–50.

Ohio State University announces the following promotions and appointments: Associate Professor R. G. Helsel has been named Acting Chairman of the Department of Mathematics; Dr. O. W. Rechard and Dr. R. L. Swain have been promoted to assistant professorships; Dr. E. D. Cashwell of the University of Wisconsin has been appointed to an instructorship.

Ohio University reports the appointments of Mr. John Ridout of the University of Missouri to an acting assistant professorship and Mr. John Hokanson, formerly graduate student at the University of Michigan, to an acting instructorship.

Oregon State College announces the following promotions: Associate Professors C. L. Clark and A. T. Lonseth to professorships and Assistant Professors J. C. R. Li and A. R. Poole to associate professorships.

The Stout Institute reports the promotion of Assistant Professor F. E. Tustison to a professorship and the appointment of Mr. Mathew Renneson to an instructorship.

State College of Washington announces: Professor E. C. Colpitts has retired; Dr. J. L. Brenner has been appointed to an assistant professorship; Dr. P. A. Clement has been appointed to an instructorship.

At the University of Chicago a Conference on Topology will be held during the Spring of 1950; more details will be announced later. The following appointments and promotions have been made: Assistant Professor P. R. Halmos has been promoted to an associate professorship; Mr. A. F. Strehler, formerly assistant at the University of Wisconsin, has been appointed to an instructorship. Professor M. H. Stone, chairman of the Department of Mathematics lectured in Tokyo, Osaka, Kyoto and Nagoya, Japan, during September, 1949 as well as

at the University of Indonesia and in Singapore; Professor Stone is now in India and is giving a course of lectures at the Tata Institute of Fundamental Research in Bombay where he will spend several months. Professor A. A. Albert will be Acting Chairman during the Autumn and Spring Quarters of 1949–50 and Professor Saunders MacLane during the Winter Quarter, 1950.

The University of Maine reports the appointments of Instructors Nelma K. Ebbeson and R. N. Mathias.

The University of Michigan makes the following announcements: Assistant Professors E. H. Rothe and Hans Samelson have been promoted to associate professorships; Instructor Leonard Tornheim has been promoted to an assistant professorship; Professor Fritz Herzog of Michigan State College has been appointed Visiting Associate Professor; Dr. D. A. Darling of Rutgers University, Dr. W. J. LeVeque of Harvard University, Mr. A. J. Lohwater, Dr. Imanuel Marx, Dr. C. J. Titus of Syracuse, and Dr. J. L. Ullman have been appointed to instructorships; Professor R. L. Wilder is on sabbatical leave for 1949–50 and is located at California Institute of Technology; Professor R. V. Churchill was on leave for the first semester, 1949–50; Professor S. B. Myers is on leave for the second semester 1949–50; Dr. E. E. Moise and Dr. M. A. Woodbury are on leave for the academic year.

At the University of Nebraska: Professor M. G. Gaba has retired with the title of Professor Emeritus; Mr. C. C. Buck, University of Michigan, has been appointed to an instructorship.

The University of North Carolina announces the promotion of Lecturer I. R. Hershner, Jr., to an assistant professorship and the appointment of Mr. W. R. Mann to an instructorship.

The University of Notre Dame reports: Assistant Professors Ky Fan and J. P. LaSalle have been promoted to associate professorships; Reverend F. L. Brown and Reverend H. F. DeBaggis have been promoted to assistant professorships; Mr. J. W. Riner and Mr. James C. Smith, Jr., have been appointed to instructorships; Reverend H. F. DeBaggis is on leave of absence for the academic year 1949–50 and is located at Princeton University.

At the University of Tennessee the following have been appointed to instructorships: Mr. E. L. Eagle, University of Arkansas, Mr. A. A. Ritchie of Central State College and Dr. Annette Sinclair of the University of Illinois.

Professor C. B. Allendoerfer of Haverford College has been appointed Visiting Professor of Mathematics at the Massachusetts Institute of Technology for the second semester, 1949–50.

Mr. D. L. Barrick of the University of Colorado has been promoted to an assistant professorship.

Assistant Professor R. V. Blair of Vanderbilt University has received a promotion to an associate professorship.

Dr. C. E. Bures has been appointed Assistant Professor of Philosophy and Psychology at the California Institute of Technology.

Professor H. C. Christofferson, who has been serving in the educational di-

vision of Military Government, Wiesbaden, Germany, has returned to his position as acting director of teacher training and professor of mathematics at Miami University.

Mrs. Joan R. Clark has accepted a position as research assistant with the Institute for Cancer Research, Philadelphia.

Associate Professor W. A. Gager of the University of Florida has been promoted to a professorship.

Associate Professor H. A. Giddings, University of New Hampshire, has been appointed to a professorship in the College of Engineering, New York University.

Assistant Professor E. L. Godfrey of Defiance College has been promoted to an associate professorship.

Assistant Professor F. M. Goen of Mississippi State College has been promoted to an associate professorship.

Dr. D. B. Goodner of the University of Illinois has been appointed to an assistant professorship at Florida State University.

Dr. Verena Haefeli has been appointed to an instructorship at Goucher College.

Assistant Professor A. E. Hallerberg of Illinois College has been promoted to an associate professorship.

Professor W. K. Hayman, University College of the Southwest of England, Exeter, has been appointed Lecturer in Mathematics for 1949–50 at Brown University.

Associate Professor W. P. Heinzman of New Mexico College of Agricultural and Mechanic Arts has been promoted to a professorship.

Mr. A. J. Hoffman has been appointed to an instructorship at Barnard College, Columbia University.

Professor Eberhard Hopf has been appointed to a professorship at New York University.

Instructor R. J. Howerton is now Acting Head of the Department of Mathematics of Regis College.

Mr. T. C. Hutchison has accepted a position as electrical engineer with Haller, Raymond & Brown, Inc., State College, Pennsylvania.

Associate Professor H. H. Irwin of State College of Washington has been promoted to a professorship.

Professor R. D. James of the University of British Columbia has been promoted to the position of Head of the Department of Mathematics.

Assistant Professor Free Jamison has been promoted to an associate professorship at San Jose State College.

Mr. H. J. Jones has been appointed to an instructorship at El Camino College, California.

Dr. J. J. Kinsella, formerly at Ohio State University, has been appointed Chairman of the Department of Mathematics Education, School of Education, New York University.

Mr. L. M. Klauber is now Chairman of the Board, San Diego Gas and Electric Company.

Mr. S. L. Levy has been appointed Research Assistant at Brown Universty.

Professor Z. L. Loflin, previously Acting Head of the Department of Mathematics of Southwestern Louisiana Institute, has been appointed Head of the Department.

Dr. M. M. Lotkin is now Chief, Machines Section of Ballistics Research Laboratory, Aberdeen Proving Ground.

Dr. C. W. Mathews of Washington University has been promoted to an assistant professorship.

Miss Lida B. May has been promoted to an assistant professorship at Texas Technological College.

Associate Professor M. G. McCuan has been promoted to the position of Chairman, Department of Mathematics and Engineering, Amarillo College.

Reverend J. L. McKenney has been promoted to the position of Head of the Department of Mathematics of Providence College.

Mr. M. M. McLynn, previously a student at George Washington University, has accepted a position as mathematician with Engineering Research Associates, Arlington, Virginia.

Mr. S. S. McNeary has been promoted to an associate professorship at Drexel Institute of Technology.

Professor W. M. Miller, formerly head of the Department of Mathematics of Robert College, Istanbul, Turkey, has been appointed to a professorship at Southwestern Louisiana Institute.

Mr. W. H. Mills has been appointed to an instructorship at Yale University.

Associate Professor W. L. Moore of the University of Louisville has been promoted to a professorship.

Assistant Professor R. H. Moorman, Tennessee Polytechnic Institute, has been promoted to an associate professorship.

Associate Professor H. T. Muhly of the United States Naval Academy has been appointed to an associate professorship at the State University of Iowa.

Reverend J. P. Murray has been promoted to an assistant professorship at Fairfield University.

Mr. D. J. Myatt of Clarkson College of Technology has been appointed to an instructorship at Antioch College.

Mr. F. P. Nash is now Head of the Department of Mathematics of Groton School.

Assistant Professor C. G. Peckham of the University of Dayton has been promoted to an associate professorship.

Mr. R. P. Peterson, Jr., of the University of California at Los Angeles has a position as mathematician at the Institute for Numerical Analysis of the National Bureau of Standards, Los Angeles.

Mr. O. L. Phillips, who has been serving as Acting Head of the Department

of Mathematics of Mississippi Southern College, has been promoted to the position of Head of the Department.

Dr. W. L. Pickard, formerly chairman of the Department of Mathematics, University of Illinois, Galesburg Division, has been appointed to the position of Professor of Mathematics at the State University of New York at New Paltz.

Mr. Frank Pomila has been appointed to an instructorship at St. John's University.

Associate Professor R. M. Rankin has been promoted to the position of Professor of Mathematics and Chairman of the Department of Mathematics at Missouri School of Mines.

Mr. Russell Remage has returned to the University of Delaware as Instructor of Mathematics after a year at the University of Pennsylvania.

Assistant Professor H. L. Rice, University of Omaha, has been promoted to an associate professorship.

Assistant Professor R. A. Rosenbaum of Reed College has been promoted to a professorship.

Professor W. E. Roth, formerly at Sampson College, has been appointed to a professorship at the University of Tulsa.

Mr. R. S. Seamons has been appointed to an instructorship at Yakima Valley Junior College, Washington.

Assistant Professor C. L. Seebeck, Jr., of the University of Alabama has been promoted to an associate professorship.

Mr. R. D. Sheffield has been appointed to an instructorship at the University of Mississippi.

Assistant Professor Jack Silber, Roosevelt College, has been promoted to an associate professorship.

Mr. David Singer has accepted an appointment as structural engineer with M. H. Treadwell Company, New York City.

Sister Jeanette Obrist has been promoted to an associate professorship at Mount St. Scholastica College.

Sister M. Thomas à Kempis, previously head of the Department of Mathematics of the College of St. Teresa, has been appointed Principal of St. Mary High School, Sleepy Eye, Minnesota.

Sister Marie Gertrude is now Chairman of the Department of Mathematics, Seton Hill College.

Mr. Rubin Smulin is an instructor at Frame Refrigeration-Electrical Institute, Miami, Florida.

Mr. B. R. Snyder of the University of New Hampshire has been appointed to an instructorship at Johns Hopkins University.

Professor T. H. Southard of Wayne University has been promoted to an associate professorship.

Dr. Vivian E. Spencer is now Chief, Minerals Section, Bureau of the Census, Washington, D. C.

Instructor F. M. Stein of Iowa Wesleyan College has been promoted to an associate professorship.

Mr. C. S. Stuckey has accepted a position with Standard Register Company, New York.

Mrs. Falka G. Sturges is teaching at Hassler High School, Redwood City, California.

Instructor E. A. Sturley of Allegheny College has been promoted to an assistant professorship.

Miss M. Frances Suter of Roanoke College has been promoted to an assistant professorship.

Mr. A. F. Svoboda has been promoted to an assistant professorship at DePaul University.

Dr. Olga Taussky has accepted a position as mathematician with the National Bureau of Standards.

Instructor Takashi Terami has been promoted to an assistant professorship at the College of St. Thomas.

Mrs. Marian W. Thornton has been appointed to an assistant professorship at the University of Minnesota.

Mr. J. C. Tinner of Florida Agricultural and Mechanical College has been promoted to an associate professorship.

Associate Professor W. R. Transue of Kenyon College has been promoted to a professorship.

Mr. C. W. Trigg of Los Angeles City College has accepted the editorship of the Problems and Questions Department of the Mathematics Magazine.

Dr. G. W. Tyler is now Consultant in Mathematics and Statistics, Naval Electronics Laboratory, San Diego, California.

Mr. J. T. Vallandingham has been promoted to the position of Head of the Department of Mathematics of Cumberland College.

Mr. G. H. Van Arkel has been promoted to the position of Head of the Department of Science and Mathematics, Everett Junior College, Washington.

Dr. D. D. Wall has been appointed to an instructorship at Santa Barbara College, University of California.

Instructor William Wallace of Northeastern University has been promoted to an assistant professorship.

Associate Professor Emily H. Watkins of the Woman's College of the University of North Carolina has been promoted to a professorship.

Dr. Margaret F. Willerding has been promoted to an assistant professorship at Harris Teachers College.

Dr. G. H. Wilson of Drexel Institute has been promoted to an assistant professorship in the Department of Physics.

Assistant Professor Emeritus H. L. Beard of Oregon State College died on May 19, 1949.

Professor Emeritus Talmon Bell of Sterling College died on October 20, 1949. He was a charter member of the Association.

Mr. W. J. Bower of Upsala College died November 14, 1949.

Mr. Herman Karnow, teacher at Brooklyn Technical High School, died December 31, 1948.

Professor Emeritus A. C. Lunn of the University of Chicago died on November 19, 1949. He was a charter member of the Association.

Dr. Lao G. Simons, professor emeritus of Hunter College of the City of New York and a charter member of the Association, died November 25, 1949.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 79 persons have been elected to membership by the Board of Governors on applications duly certified:

C. E. Amos, Ph.D. (Ohio State) Asso. Professor, University of Toledo, Ohio.

M. Q. Barton, Student, Central College, Fayette, Mo.

V. W. Beck, M.S. (Minnesota) Elect. Engr., American Television & Radio, St. Paul, Minn.

H. A. Bernhard, B.S.E.E. (Newark) Grad. Student, Columbia University, New York, N. Y.

W. S. Bicknell, M.S. (Michigan) Instructor, Iowa State College, Ames, Iowa.

R. N. Bradt, M.S. (Colorado State) Asst. Instructor, University of Kansas, Lawrence, Kans.

G. M. Brown, D.Sc. (Michigan) Asso. Professor University of Georgia, Atlanta, Ga.

R. C. Campbell, B.A. (Dartmouth) Grad. Asst., University of Colorado, Boulder, Colo.

Ion Carstoiu, Ph.D. (Paris) Asst. Professor, Johns Hopkins University, Baltimore, Md.

C. C. Chouteau, A.B. (Kansas) Teacher, Bowring High School, Okla.

D. E. Coffey, Ed.B. (Illinois State Normal) Grad. Student, Montana State University, Missoula, Mont.

E. J. Cogan, M.A. (Wisconsin) Instructor, Pennsylvania State College Center, Pottsville, Pa.

H. K. Crowder, M.S. (Case) Instructor, Case Institute of Technology, Cleveland 2, Ohio.

D. L. Daly, M.A. (Michigan) Instructor, University of Kentucky, Lexington, Ky.

H. E. Dickey, M.S. (Iowa State) Asst. Professor, Iowa State College, Ames, Iowa.

M. H. Dipert, M.S. (Indiana) Professor, Westmar College, LeMars, Iowa.

William Douglas, Courtenay, B.C., Canada.

W. E. Dwyer, Jr., Student, Catholic University of America, Washington, D. C.

D. F. Eliezer, B.A. (Lehigh) Grad. Asst., Lehigh University, Bethlehem, Pa.

E. R. Epperson, B.S.Ed. (Miami Univ.) Grad. Asst., Miami University, Oxford, Ohio.

A. G. Fadell, B.A. (Buffalo) Teaching Fellow, University of Buffalo, N. Y.

W. J. Feeney, S.J., M.A. (Boston College) Grad. Student, Catholic University of America, Washington, D. C.

J. F. Foley, B.Sc. (Chicago) Asst. Physicist, Gas and Electric Co., Baltimore, Md.

R. A. Forseth, B.A. (S.T.C., Moorhead, Minn.) Instructor, State Teachers College, Mayville, N. D.

E. J. Germino, M.A. (Columbia) Instructor, St. John's University, Brooklyn, N. Y.

D. B. GOODNER, Ph.D. (Illinois) Asst. Professor, Florida State University, Tallahassee, Fla.

DONALD GREENSPAN, M.S. (Wisconsin) Instructor, University of Maryland, College Park, Md.

H. C. GRIFFITH, A.B. (Missouri) Asst. Instructor, University of Missouri, Columbia, Mo.

B. H. GUNDLACH, Ph.D. (Hamburg) Asst. Professor, University of Arkansas, Fayetteville, Ark.

EDWIN HEWITT, Ph.D. (Harvard) Asst. Professor, University of Washington, Seattle, Wash.

I. I. HIRSCHMAN, JR., Ph.D. (Harvard) Asst. Professor, Washington University, St. Louis, Mo.

A. J. HOFFMAN, A.B. (Columbia College) Instructor, Barnard College, Columbia University, New York, N. Y.

C. W. HUFF, M.S. (South Carolina) Instructor, University of South Carolina, Columbia, S.C.

A. R. HYDE, M.A. (Trinity) Instructor, Trinity College, Hartford, Conn.

W. G. JOHNSON, M.A. (Michigan) Asst. Professor, Phillips University, Enid, Okla.

F. B. JONES, Ph.D. (Texas) Asso. Professor, University of Texas, Austin, Tex.

R. S. KINGSBURY, M.S. (Wisconsin) Staff Member, Operations Evaluation Group, Navy Department, Washington, D. C.

J. J. KINSELLA, Ed.D. (Columbia) Asso. Professor, New York University, Washington Square, New York, N. Y.

A. T. KOVITZ, B.S.E.E. (Purdue) Instructor, Polytechnic Institute of Brooklyn, N. Y.

OLIVE LESKOW, M.A. (Minnesota) Teacher, Tolleston School, Gary, Ind.

T. M. LITTLE, Ph.D. (Maryland) Asst. Professor of Biology, University of Nevada, Reno, Nev.

J. D. MATHESON, B.S. (U. S. Military Acad.) Lt. Col., U. S. Army, Alexandria, Va.

R. L. MAY, B.S. (Webb Inst.) Technical Asst., Socony-Vacuum Oil Co., New York, N. Y.

M. J. F. MCKEON, M.A. (New York S.C.T., Albany) Instructor, Rensselaer Polytechnic Institute, Troy, N. Y.

J. R. MCPHERSON, Student, University of North Dakota, Grand Forks, N. D.

KOVINA MILOSEVICH, Student, University of Skoplje, Yugoslavia.

MARIE A. MOORE, D.Ed. (Colorado) Professor, Harris Teachers College, St. Louis, Mo.

GERTRUDE C. NAUMAN, Student, Bryn Mawr College, Pa.

J. D. NEFF, B.A. (Coe) Grad. Asst., Kansas State College, Manhattan, Kans.

AGNES E. NIBARGER, M.S. (Wichita) Instructor, University of Wichita, Kans.

O. M. NIKODYM, Ph.D. (Warsaw) Professor, Kenyon College, Gambier, Ohio.

V. D. NYHOFF, B.S. (Kansas State) Grad. Asst., Kansas State College, Manhattan, Kans.

D. G. O'CONNOR, B.E.E. (Manhattan) Jr. Engr., Sylvania Electric, Bayside, L. I., N. Y.

LEOLA P. ODLAND, Student, University of Oregon, Eugene, Ore.

ALICE OSTERBERG, A.M. (Columbia) Instructor, Brooklyn College, N. Y.

P. B. PATTERSON, M.A.E. (Florida) Instructor, University of Florida, Gainesville, Fla.

G. W. PAYNE, Student, St. Mary's University of San Antonio, Texas.

W. L. PICKARD, Ph.D. (Peabody) Professor, State University of New York, State Teachers College, Paltz, N. Y.

F. R. POMILLA, M.S. (Fordham) Instructor, St. John's University, Brooklyn, N. Y.

H. L. POND, M.S. (George Washington) Physicist, David Taylor Model Basin, Washington, D. C.

H. L. RANDOLPH, B.A. (California) Grad. Asst., University of California, Los Angeles, Calif.

D. P. RICHARDSON, Ph.D. (Chicago) Professor, University of Arkansas, Fayetteville, Ark.

JACOB SAMOLOFF, B.A. (Toronto) Instructor, Lehigh University, Bethlehem, Pa.

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THE NINTH AND TENTH CARUS MONOGRAPHS

Two more Carus Mathematical Monographs will be available in the spring of 1950. They are Monograph 9: *The Theory of Algebraic Numbers* by Harry Pollard and Monograph 10: *The Arithmetic Theory of Quadratic Forms* by B. W. Jones.

Every member of the Association is entitled to one copy of each Carus Monograph at $1.75. Orders (preferably accompanied by remittance) for Monographs 9, 10 or others should be sent to: Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

Additional copies of Monographs 9 and 10 for members and copies for non-members may be purchased at $3.00 from John Wiley and Sons, 440 Fourth Avenue, New York 16, N. Y. In the case of Monographs 1 to 8, orders should continue to be sent to the Open Court Publishing Company, LaSalle, Ill.

GEOGRAPHICAL BOUNDARIES OF THE SECTIONS OF THE ASSOCIATION

At the Columbus meeting, the Board of Governors voted that Sectional Governors shall hereafter be elected by a mail vote to be conducted by the Secretary-Treasurer of the Association. The Committee on Section Meetings was authorized to determine the geographical boundaries of each Section and to allocate members to Sections for voting purposes.

Accordingly the Committee on Section Meetings (H. M. Gehman, Fred Robertson and W. V. Parker, chairman) has assigned to each Section the areas listed below. The officers of each Section have been consulted and have agreed to their assignment. Individual members who wish exceptions made may appeal to the Committee through the Secretary-Treasurer of the Association.

It is understood that the membership of each Section consists of all members of the Association who express a desire to be enrolled as members of the Section. Hence, a member may regularly attend meetings of more than one Section. But for purposes of eligibility to hold office and to vote for officers of a Section, the membership of each Section of the Association shall be defined as consisting of all members of the Association living in the geographical areas indicated below.

Sections	*Territory Included in the Section*
ALLEGHENY MOUNTAIN	Pennsylvania (west of longitude 77°30'W), West Virginia
ILLINOIS	Illinois
INDIANA	Indiana
IOWA	Iowa
KANSAS	Kansas
KENTUCKY	Kentucky
LOUISIANA-MISSISSIPPI	Louisiana, Mississippi
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA	District of Columbia, Maryland, Virginia
METROPOLITAN NEW YORK	New Jersey (north of latitude 40°30' N), New York (south of latitude 42° N)
MICHIGAN	Michigan
MINNESOTA	Minnesota, North Dakota; Manitoba, Saskatchewan
MISSOURI	Missouri
NEBRASKA	Nebraska, South Dakota
NORTHERN CALIFORNIA	California (except for the 14 counties included under Southern California), Hawaii, Nevada
OHIO	Ohio
OKLAHOMA	Arkansas, Oklahoma
PACIFIC NORTHWEST	Alaska, Idaho, Montana, Oregon, Washington; Alberta, British Columbia
PHILADELPHIA	Delaware, New Jersey (south of latitude 40°30' N), Pennsylvania (east of longitude 77°30' W)
ROCKY MOUNTAIN	Colorado, Utah, Wyoming
SOUTHEASTERN	Alabama, Canal Zone, Florida, Georgia, North Carolina, Puerto Rico, South Carolina, Tennessee
SOUTHERN CALIFORNIA	California (the counties of San Luis Obispo, Kings, Tulare, Inyo, and Mono and the counties to the south of these)
SOUTHWESTERN	Arizona, New Mexico, El Paso County in Texas
TEXAS	Texas (except El Paso County)
UPPER NEW YORK STATE	New York (north of latitude 42° N); Ontario, Quebec
WISCONSIN	Wisconsin
NEW ENGLAND REGION	Connecticut, Maine, Massachusetts, New Hampshire, Rhode Island, Vermont; New Brunswick, Newfoundland, Nova Scotia, Prince Edward Island

THE MAY MEETING OF THE WISCONSIN SECTION

The seventeenth annual spring meeting of the Wisconsin Section of the Mathematical Association of America was held at Lawrence College, Appleton, Wisconsin on Saturday, May 14, 1949. Professor A. C. Berry, Chairman of the Section presided at the morning and afternoon sessions.

About sixty-five persons were present, including the following thirty-three members of the Association: K. J. Arnold, R. H. Bardell, Leon Battig, L. J. Berner, A. C. Berry, W. W. Bigelow, R. H. Bing, H. J. Cohen, B. H. Colvin, Rev. L. A. V. DeCleene, H. P. Evans, Fannie Hopkins, C. C. Hsiung, R. C. Huffer, J. F. Kenney, R. E. Langer, C. C. MacDuffee, Morris Marden, Sister Mary Felice, J. R. Mayor, P. E. Meadows, Karl Menger, O. E. Overn, G. A. Parkinson, H. P. Pettit, Elizabeth Sokolnikoff, R. D. Specht, J. C. Stewart, J. V. Talacko, J. J. Van de Castle, J. I. Vass, R. D. Wagner, Louise A. Wolf.

At the business meeting held after the afternoon program, the following officers were elected for the coming year: Chairman, J. R. Mayor, University of Wisconsin; Secretary, Louise A. Wolf, University of Wisconsin in Milwaukee; Program Committee, E. G. Harrell, Platteville State Teachers College, J. C. Stewart, Lawrence College, R. D. Wagner, Marinette Extension Division of the University of Wisconsin. It was agreed to have the May 1950 meeting at Marquette University in Milwaukee.

The following papers were presented:

1. *The projective space*, by Professor Karl Menger, Illinois Institute of Technology.

2. *An equilateral distance*, by Professor R. H. Bing, University of Wisconsin.

This paper will be published in a later issue of the MONTHLY.

3. *Some applications of Newton's formula in finite differences*, by Professor H. P. Pettit, Marquette University.

It has frequently been observed that Newton's formula is valid for interpolation when all differences beyond the kth vanish. Here it was shown that where the differences do not vanish but follow certain definite laws, Newton's formula still provides valid and useful results. This was done by applying the forms of the binomial expansion, notably forms such as $\sum_n C_i a^i$, $\sum i_n C_i a^i$, etc., both in the case of the differences $\Delta^i u_0$, and further differences found by treating these as a sequence to be analyzed just as the set U_n. Applications were given to distribution of various types.

4. *Report on the investigation of the A.A.A.S. committee on the status of the sciences and mathematics*, by Professor R. C. Huffer, Beloit College.

5. *A freshman course for general students*, by Sister Mary Petronia, Mount Mary College, introduced by Sister Mary Felice, Mount Mary College.

LOUISE A. WOLF, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The twenty-eighth annual meeting of the Illinois Section of the Mathematical Association of America was held at Bradley University, Peoria, Illinois, on Friday afternoon and Saturday forenoon, May 13–14, 1949. Professor S. F. Bibb, Chairman of the Section, presided at all meetings.

There were fifty-six in attendance including the following forty-four members of the Association: H. G. Ayre, R. A. Baumgartner, H. R. Beveridge, S. F. Bibb, A. H. Black, B. K. Brown, F. R. Brown, S. S. Cairns, Joseph Clare, E. G. H. Comfort, J. J. Corliss, Paul Cramer, Marjorie L. Croft, I. K. Feinstein, L. R. Ford, A. E. Gault, G. D. Gore, A. E. Hallerberg, E. W. Hellmich, Rose L. Hornacek, E. C. Kiefer, W. A. Lafferty, C. T. McCormick, W. C. McDaniel, A. W. McGaughey, E. B. Miller, C. N. Mills, G. E. Moore, M. G. Moore, Elsie C. Muller, Grace M. Nolan, F. S. Nowlan, Margaret Owchar, Roberta E. Presnell, Haim Reingold, T. E. Rine, E. W. Schreiber, H. W. Schwartz, M. Anice Seybold, H. A. Simmons, D. S. Snader, R. C. Stephens, A. T. Street, and Alice Wright.

At the business meeting Saturday forenoon the following officers were

elected for the coming year: M. G. Moore, Bradley University, Chairman; W. C. McDaniel, Southern Illinois University, Vice-Chairman; and E. C. Kiefer, Millikin University, Secretary. The 1950 meetings are to be held at Southern Illinois University, Carbondale, Illinois, on Friday afternoon and Saturday forenoon May 12–13.

Professor L. R. Ford made a progress report regarding encouragement of mathematics among high school students in the Chicago area. He reported several meetings of the committee, of which he is chairman. They plan a series of contests and finally a careful examination to determine the outstanding mathematical minds. Lack of adequate funds hinders the promotion of the plan, but sources of aid seem to be in sight. Several of the Chicago Universities and Colleges stand ready to offer scholarships to the winners. A resolution was passed, and a copy sent to Bradley University, thanking the university for the cooperation and hospitality extended to the Section.

A committee composed of L. R. Ford (Chairman), M. Anice Seybold, and C. T. McCormack was appointed to formulate plans to organize the Illinois Section under a constitution and by-laws. This committee is to report at next year's meeting in Carbondale.

The following program was given:

1. *So you are going to select a text-book in college algebra*, by Dr. J. J. Corliss, University of Illinois, Navy Pier Branch.
2. *The theorem of Coriolis*, by Dr. G. D. Gore, Roosevelt College of Chicago.
3. *The new interpretation of the "belonging to exponents,"* by Dr. H. A. Simmons, Northwestern University.
4. *The story of the Illinois Section*, by Dr. C. N. Mills, Illinois State Normal University.
5. *The Gibbs phenomenon in Fourier series*, by Dr. E. G. H. Comfort, Illinois Institute of Technology.
6. *The training of mathematical teachers*, by Dr. H. G. Ayre, Western Illinois State College.
7. *The teaching of the calculus from the teachers college point of view*, by Dr. E. W. Hellmich, Northern Illinois Teachers College.

E. C. KIEFER, *Secretary*

MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the University of Virginia, Charlottesville, Virginia on Saturday, May 14, 1949. Mr. Michael Goldberg, Chairman of the Section, presided.

There were seventy-seven persons attending the meeting, including the following fifty-four members of the Association: M. W. Aylor, N. H. Ball, R. R. Bernard, Ruth E. Biggers, W. E. Bleick, J. W. Blincoe, T. A. Botts, R. S. Burington, H. H. Campaigne, Randolph Church, H. F. DeFrancesco, C. H.

Denbow, F. T. Dresser, D. M. Dribin, B. C. Getchell, Michael Goldberg, H. W. Gould, E. S. Grable, E. C. Gras, D. W. Hall, J. R. Hammond, T. W. Hatcher, B. C. Horne, Jr., J. E. Ikenberry, S. B. Jackson, Walter Jennings, W. H. Johnson, R. H. Kasriel, Sidney Kaplan, L. M. Kells, V. L. Klee, Jr., R. C. Lamb, B. Z. Linfield, B. J. Lockhart, M. H. Martin, Carol V. McCamman, L. M. McFadden, E. J. McShane, Florence M. Mears, Joseph Milkman, W. H. Norris, J. W. Popow, W. G. Rouleau, Shirley A. Rubenstein, E. D. Schell, W. F. Shenton, E. R. Sleight, H. E. Spencer, Mildred E. Taylor, J. A. Tierney, C. C. Torrance, I. F. Wagner, C. H. Wheeler, III, G. T. Whyburn.

The following papers were presented:

1. *Application of matrix algebra to the solution of normal equations*, by Mr. J. L. Stearn, U. S. Coast and Geodetic Survey, introduced by the Secretary.

A non-Cayleyian matrix algebra was defined. A proof was then given of the Doolittle technique for solving normal equations when the coefficient matrix is positive definite. The usefulness of this matrix algebra for determining the inverse of a high order symmetrical matrix was indicated.

2. *Steady rotational plane flow of a gas*, by Professor M. H. Martin, University of Maryland.

By taking the stream lines and isobars for curvilinear coordinate lines, the problem of determining the steady, rotational plane flow of a gas is reduced to the integration of a quasi-linear partial differential equation in which the pressure p and the stream function ψ are the independent variables, and the unknown function is $\theta=\theta(p, \psi)$, the direction of flow. This partial differential equation involves the speed $q=q(p, \psi)$, and plays a dual role in the theory depending on whether q or θ is taken as a known function of p, ψ. In the latter case the partial differential equation becomes a linear partial differential equation of the third order for the unknown q. Flows for which the isobars are isoclines, $\theta=\theta(p)$, are excluded from the general theory and are treated separately. As an example of the inverse approach in which θ is preassigned, a detailed investigation is carried out for the special case $\theta=\theta(\psi)$ in which the stream lines are straight lines.

3. *Some remarks concerning the symbolics in vector analysis*, by Dr. M. M. Munk, Naval Ordnance Laboratory, introduced by Mr. M. A. Hyman.

The controversy about signs in vector analysis was discussed, and a few reasons for using Hamilton's sign rule were presented. Clarity and versatility may be improved by the use of Jaumann's symbol for dyadic multiplication, and by Jaumann's rule for the expansion of vector differentiation, and also by the use of a symbol for the space integral which is the inverse of the differentiation "nabla."

4. *Visual education—the slide rule*, by Professor W. F. Shenton, American University.

A lantern slide with various motions was demonstrated. By means of it settings of a slide rule were projected on a screen in size large enough to make explanation of the operation of a rule of the Mannheim or duplex type possible to a large number of students at one time.

5. *A theorem on local convexity*, by Mr. V. L. Klee, Jr., University of Virginia.

A subset X of a linear space is called "locally convex" if each of its points has a neighborhood whose intersection with X is convex. In 1928 it was proved independently by H. Tiete and S.

Matsumura that a closed, connected, locally convex subset of euclidean n space must actually be convex. In 1942 this result was extended to normed linear spaces by I. J. Schoenberg and M. Nakamura (independently). Mr. Klee presented a new proof of this theorem, simpler than its predecessors and valid in an arbitrary topological linear space (in the sense of Hille).

6. *The Russian peasants' method of multiplication*, by Mr. Sidney Kaplan, Naval Ordnance Laboratory.

The Russian peasants' method of multiplication was presented and compared with the Egyptian method.

7. *Non-digital computing devices in the teaching of mathematics*, by Professor Randolph Church of the U. S. Naval Postgraduate School.

A few devices for adding, multiplying, integrating, producing special functions and combinations of these for solving systems of linear equations and algebraic equations, for obtaining Fourier coefficients and for other special purposes, were very briefly described. These were selected to support the thesis that the mathematical explanation of such devices can provide illuminating and valuable illustrative material at various levels in the teaching of mathematics. Several of the instruments described were exhibited, including a Mader-Ott harmonic analyzer and an instrument for evaluating Stieltjes integrals designed by E. J. Nystrom.

FLORENCE M. MEARS, *Secretary*

CALENDAR OF FUTURE MEETINGS

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December 30, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Southern Illinois University, Carbondale, May 12–13, 1950.

INDIANA, Wabash College, Crawfordsville, April 29, 1950.

IOWA, State University of Iowa, Iowa City, April 21–22, 1950.

KANSAS, State Teachers College, Pittsburg, April 22, 1950.

KENTUCKY, University of Kentucky, Lexington, April 29, 1950.

LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, February 24–25, 1950.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Richmond, Virginia, May, 1950.

METROPOLITAN NEW YORK, City College, April 1, 1950.

MICHIGAN, University of Michigan, March 25, 1950.

MINNESOTA, Macalester College, St. Paul, May 6, 1950.

MISSOURI, Washington University, St. Louis, March 24–25, 1950.

NEBRASKA, Nebraska Wesleyan University, Lincoln, May 6, 1950.

NORTHERN CALIFORNIA

OHIO, Denison University, Granville, April 22, 1950.

OKLAHOMA

PACIFIC NORTHWEST, University of Washington, Seattle, June, 1950.

PHILADELPHIA

ROCKY MOUNTAIN, University of Denver, April, 1950.

SOUTHEASTERN, University of Florida, Gainesville, April 7–8, 1950.

SOUTHERN CALIFORNIA, Immaculate Heart College, Hollywood, March 11, 1950.

SOUTHWESTERN, Spring, 1950.

TEXAS, Abilene, April 14–15, 1950.

UPPER NEW YORK STATE, Syracuse University, April 22, 1950.

WISCONSIN, Marquette University, Milwaukee, May 13, 1950.

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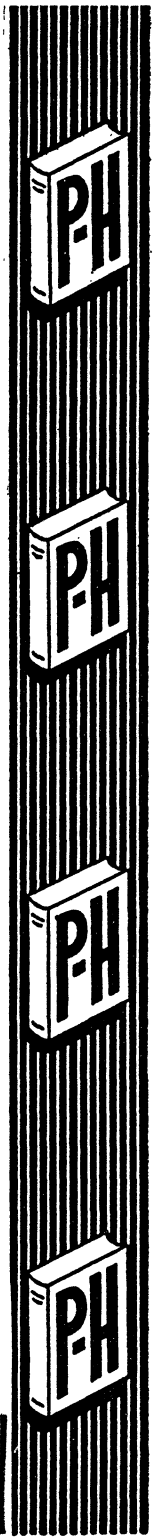
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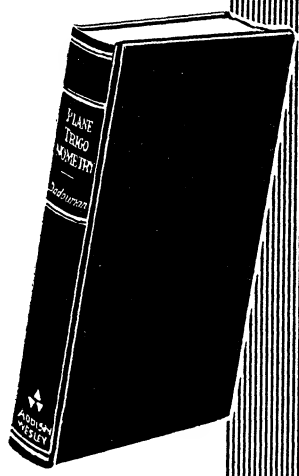
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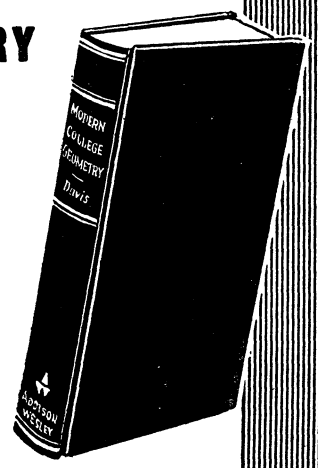


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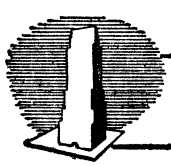
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MARCH 1950

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Carroll V. Newsom, *Editor*

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THE INTERNATIONAL CONGRESS OF MATHEMATICIANS

The next International Congress of Mathematicians will be held in Cambridge, Massachusetts, from August 30 to September 6, 1950, under the auspices of the American Mathematical Society. The Society planned originally to act as host for a Congress in 1940. Plans for the 1940 Congress were practically completed when the outbreak of World War II made it necessary for the Society to postpone the Congress to a more favorable date.

The 1950 Congress will be the third International Congress of Mathematicians to be held on the continent of North America. The first was held at Northwestern University in 1893 and the second at the University of Toronto in 1924. International Congresses were held at intervals of approximately four years, except when war intervened, until 1936. There has been no international gathering of mathematicians since that time, and it is the sincere hope of the Organizing Committee that the gathering in 1950 will be a truly international one, that American mathematicians will attend in large numbers, and that all other countries will be well represented. Mathematicians of all national and geographical groups are invited.

Place of Meeting. Harvard University will be the principal host institution for the Congress. A number of other institutions in metropolitan Boston will join in the entertainment of visitors to the Congress by arranging special features on their campuses. Mathematicians so desiring will be housed at a modest charge in the Harvard University dormitories and meals will be served in the University dining rooms. There will be accommodations for members of families, special provision being made for the care of children. Those who prefer to live in hotels can be comfortably provided for in Cambridge or Boston.

Organization. The Organizing Committee has invited more than a score of outstanding mathematicians to deliver stated addresses. There will be Conferences in four different fields; these are described more fully below. There will be seven Sections for the presentation of short contributed papers, and by invitation of the Section Chairmen there will be a small number of half-hour addresses as part of the Section programs.

In recent years mathematicians have been much impressed by the success of the conference method for presenting research in fields in which vigorous advances have just been made or are in progress. There will accordingly be a coordinated program of formal lectures and informal open discussion, and the stated addresses will be integrated as far as possible with the work of the Conferences. The following list of topics gives an indication of the nature of each of the Conferences:

ALGEBRA (Chairman, A. A. Albert): 1, Groups and universal algebra; 2, Structure theory of rings and algebras; 3, Arithmetic algebra; 4, Algebraic geometry.

ANALYSIS (Chairman, Marston Morse): 1, Algebraic tendencies in analysis; 2, Analysis and geometry in the large; 3, Extremal methods and geometric theory of functions of a complex variable.

APPLIED MATHEMATICS (Chairman, John von Neumann): 1, Partial differential equations; 2, Statistical mechanics; 3, Random processes in physics and communication.

TOPOLOGY, (Chairman, Hassler Whitney): 1, Homology and homotopy theory; 2, Fibre bundles and obstructions; 3, Differentiable manifolds; 4, Topological groups.

The Sections for the presentation of short contributed papers will be as follows: I, Algebra and Theory of Numbers; II, Analysis; III, Geometry and Topology; IV, Probability and Statistics, Actuarial Science, Economics; V, Mathematical Physics and Applied Mathematics; VI, Logic and Philosophy; VII, History and Education. Each member of the Congress may present only one contributed paper and the time allotted for each paper will be ten minutes. Abstracts for such papers should not exceed 400 words in length and must be submitted on blanks which may be secured from Professor J. R. Kline, the Secretary of the Congress, whose address is: University of Pennsylvania, Philadelphia 4, Pa. Abstracts must be in the hands of the Organizing Committee not later than May 15, 1950. The Organizing Committee has decided that contributed papers must be presented in person. The official languages of the Congress are English, French, German, Italian, and Russian.

Entertainment. There will be many interesting entertainment features, including a reception, a symphony concert, and a banquet. There will probably be a number of excursions.

Membership in the Congress. Membership in the Congress will be open to all qualified persons, whether they are able to be present in person or not. For regular members of the Congress the fee is $15.00; these persons will receive the Proceedings of the Congress. Members of families, accompanying Congress members and not participating in the scientific meetings, may become associate members, for whom the fee is $7.50; they will not present papers or receive the Proceedings, but will be entitled to many of the other privileges of membership. Only members and associate members of the Congress will have the privilege of residing in the Harvard dormitories.

Financial Support. Besides the support from Harvard University, generous subventions have been subscribed for the Congress by Bell Telephone Laboratories, Carnegie Corporation, General Electric Company, Institute for Advanced Study, Massachusetts Institute of Technology, National Research Council, Rockefeller Foundation, Standard Oil Development Company, and several private donors.

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The Mathematical Association of America will not hold its usual summer meeting in 1950, for it is believed that many members of the Association will desire to participate in the activities of the Congress. The ready opportunity of meeting a large number of distinguished foreign mathematicians and of hearing discussions on virtually every aspect of modern mathematics represents a climatic event in the history of American mathematics.

ON PAIRS OF TRIANGLES

R. GOORMAGHTIGH, Bruges, Belgium

1. Introduction. In this MONTHLY [1] we have considered, particularly in connection with the orthopole, pairs of triangles inscribed in a circle. We also mentioned [2] that a theorem given by Murnaghan [3], Godeau [4], and Ramler [5] had been published before in a more general form by Kantor [6], and we further generalized this last theorem [2]. We will now consider orthopoles related to *any* pair of triangles placed in the same plane.

2. Angle associated with a pair of triangles. Let $A_1A_2A_3$ and $B_1B_2B_3$ be two triangles and denote by O_a and O_b, R_a and R_b the centers and the radii of their circumcircles.

Using complex coördinates, the base or unit circle being the circle $A_1A_2A_3$, let t_1, t_2, t_3 be the turns corresponding to A_1, A_2, A_3. Denote by

$$s_1 = t_1 + t_2 + t_3, \qquad s_2 = t_2t_3 + t_3t_1 + t_1t_2, \qquad s_3 = t_1t_2t_3$$

the symmetric functions of the t_i.

If c is the coördinate of O_b, the coördinate of B_i is $c+\lambda\tau_i$, λ being real and τ_1, τ_2, τ_3 being turns. We call

$$\sigma_1 = \tau_1 + \tau_2 + \tau_3, \qquad \sigma_2 = \tau_2\tau_3 + \tau_3\tau_1 + \tau_1\tau_2, \qquad \sigma_3 = \tau_1\tau_2\tau_3$$

the symmetric functions of the τ_i.

The conjugate to r being denoted by $\bar{r}$, the Simson line of the point t on the base circle as to $A_1A_2A_3$ has for its equation

$$(1) \qquad tx - s_3\bar{x} = \frac{t^3 + s_1t^2 - s_2t - s_3}{2t},$$

and that of the Simson line of the point $c+\lambda\tau$ as to $B_1B_2B_3$ is

$$(2) \qquad \tau x - \sigma_3\bar{x} = \frac{\lambda\tau^3 + (2c + \lambda\sigma_1)\tau^2 - (2\bar{c}\sigma_3 + \lambda\sigma_2)\tau - \lambda\sigma_3}{2\tau}.$$

It follows that the points t and $c+\lambda\tau$ whose Simson lines are parallel to the diameter of the base circle passing through the unit point are s_3 and $c+\lambda\sigma_3$; if θ is such that

$$e^{i\theta} = \frac{\sigma_3}{s_3},$$

θ is the angle between the circumradii of $A_1A_2A_3$ and $B_1B_2B_3$ passing through the two points, respectively, under consideration.

Hence we derive easily from (1) and (2) the following theorem:

The points A and B of the circumcircles of the triangles $A_1A_2A_3$ and $B_1B_2B_3$, whose respective Simson lines as to those triangles are parallel to any given direction, are such that the circumradii, corresponding to those points, form a constant angle.

This angle plays an interesting part in the properties of a pair of triangles, and will be called the *angle associated with $A_1A_2A_3$ and $B_1B_2B_3$*.

3. Segment associated with a pair of triangles. If from any point as origin we draw two segments equal and parallel to O_aA and O_bB, the segment joining the extremities of these segments will be called the *segment associated with $A_1A_2A_3$ and $B_1B_2B_3$*; it will be denoted by l and

$$l^2 = \left(\frac{s_3}{\sigma_3} - \lambda\right)\left(\frac{\sigma_3}{s_3} - \lambda\right),$$

or, in terms of R_a, R_b and θ,

$$l^2 = R_a^2 + R_b^2 - 2R_aR_b\cos\theta.$$

4. Orthopolar ellipse of a pair of triangles. The equation to B_2B_3 being

$$x + \tau_2\tau_3\bar{x} = c + \tau_2\tau_3\bar{c} + \lambda(\tau_2 + \tau_3),$$

the coördinate of the orthopole of B_2B_3 as to $A_1A_2A_3$ is

$$\frac{\left[s_1 + c + \lambda\sigma_1 + \tau_1\left(\frac{s_3}{\sigma_3} - \lambda\right) + \bar{c}\,\frac{\sigma_3}{\tau_1}\right]}{2}.$$

It follows that the orthopoles of the sides of $B_1B_2B_3$ as to $A_1A_2A_3$ are on an ellipse E having for center the point

$$\frac{s_1 + c + \lambda\sigma_1}{2}$$

and for axes

$$(c\bar{c})^{1/2} \pm \left(\frac{s_3}{\sigma_3} - \lambda\right)^{1/2}\left(\frac{\sigma_3}{s_3} - \lambda\right)^{1/2},$$

these axes being parallel to

$$\frac{x}{\bar{x}} = = \frac{c^{1/2}\left(\frac{s_3}{\sigma_3} - \lambda\right)^{1/2}}{\bar{c}^{1/2}\left(\frac{\sigma_3}{s_3} - \lambda\right)^{1/2}}. \tag{3}$$

Similarly the orthopole of A_2A_3 as to $B_1B_2B_3$ is

$$\frac{s_1 + c + \lambda\sigma_1 + \left(\lambda\frac{\sigma_3}{s_3} - 1\right)t_1 - \bar{c}\frac{s_3}{t_1}}{2}$$

and it is easily found that orthopoles of the sides of $A_1A_2A_3$ as to $B_1B_2B_3$ are also on the ellipse E.

Hence we have the following theorem:

If $A_1A_2A_3$ and $B_1B_2B_3$ are two triangles whose circumcenters are at a distance d and if l is their associated segment, the orthopoles of the sides of each of the triangles as to the other triangle are on an ellipse having for center the midpoint of the segment joining the orthocenters of the triangles and having for axes $d \pm l$.

To find a construction for the direction of the axes, we consider, for a moment, that the unit-point is on O_aO_b, so that $\bar{c}=c$, and from (3) we then derive the following construction:

If A_0 and B_0 are the points on the circumcircles of $A_1A_2A_3$ and $B_1B_2B_3$ such that their respective Simson lines as to those triangles are parallel to O_aO_b, and if O_aB_0' is equal and parallel to O_bB_0, then the axes of E are parallel to the bisectors of the angle (O_aO_b, A_0B_0').

5. Kantor's theorem. When $O_a=O_b$ and $R_a=R_b$, we find that the six considered orthopoles are on a circle having for radius $R_a \cos \theta/2$; this is Kantor's theorem [6].

But the ellipse E becomes a circle when the circumcircles of the given triangles are only concentric.

When two triangles $A_1A_2A_3$ and $B_1B_2B_3$ are inscribed in two concentric circles, the orthopoles of the sides of each of the triangles as to the other are on a circle having for center the midpoint of the segment joining their orthocenters, and the square of the radius of that circle is

$$R_a^2 + R_b^2 - 2R_aR_b \cos \theta,$$

θ being the angle associated with the pair of triangles.

6. Orthopolar ellipse of a point as to a triangle. When $R_b=0$, we find that the orthopoles of any three lines passing through a point O_b as to $A_1A_2A_3$ are on the ellipse having for center the midpoint of the segment joining O_b to the

orthocenter of $A_1A_2A_3$ and for axes $R_a \pm O_aO_b$; these axes are parallel to the bisectors of the angles between O_aO_b and the circumdiameter of $A_1A_2A_3$ passing through the point whose Simson line as to that triangle is parallel to O_aO_b.

This is the well known locus [7], [8] of the orthopole, as to $A_1A_2A_3$, of a variable straight line passing through the fixed point O_b.

7. Collinear orthopoles. In the special case when $d=l$, we have this theorem:

When two triangles $A_1A_2A_3$ and $B_1B_2B_3$ are so placed that the distance of their circumcenters equals their associated segment, the orthopoles of the sides of each of the triangles as to the other triangle are six collinear points.

8. Relation between triangle areas. This last theorem may also be derived as a special case from a property about triangle areas.

Call Δ_a and Δ_b the areas of $A_1A_2A_3$ and $B_1B_2B_3$ and Δ_α and Δ_β the areas of the triangles formed by the orthopoles of the sides of $B_1B_2B_3$ as to $A_1A_2A_3$ and by those of the sides of $A_1A_2A_3$ as to $B_1B_2B_3$. Then

$$16\sqrt{-1}\Delta_\alpha = -\left|\left(\frac{s_3}{\sigma_3}-\lambda\right)\tau_i + c\frac{\sigma_3}{\tau_i},\ \left(\frac{\sigma_3}{s_3}-\lambda\right)\tau_i + c\frac{\tau_i}{\sigma_3},\ 1\right|$$

$$= (d^2-l^2)\left|\tau_i,\ \frac{1}{\tau_i},\ 1\right| = -4\sqrt{-1}(d^2-l^2)\Delta_b.$$

Hence

$$\frac{\Delta_\alpha}{\Delta_b} = -\frac{(d^2-l^2)}{4R_a^2}.$$

As we have a similar expression for Δ_β/Δ_a, we obtain the following theorem:

If Δ_a and Δ_b are the areas of two triangles $A_1A_2A_3$ and $B_1B_2B_3$ inscribed in circles having for radii R_a and R_b and if Δ_α and Δ_β are the areas of the triangles formed by the orthopoles of the sides of $B_1B_2B_3$ as to $A_1A_2A_3$ and by those of the sides of $A_1A_2A_3$ as to $B_1B_2B_3$, then

$$\frac{\Delta_a\Delta_\alpha}{\Delta_b\Delta_\beta} = \frac{R_b^2}{R_a^2}.$$

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SOME APPLICATIONS OF MATRICES IN THE THEORY OF EQUATIONS*

C. C. MacDUFFEE, University of Wisconsin

1. The companion matrix. It is a very easy matter to find a solution of any polynomial equation. Thus I can immediately write out a solution of the cubic equation

$$x^3 - 9x^2 + 24x - 20 = 0,$$

namely, the matrix

$$A = \begin{bmatrix} 0 & 0 & 20 \\ 1 & 0 & -24 \\ 0 & 1 & 9 \end{bmatrix};$$

for it can very readily be shown that

$$A^3 - 9A^2 + 24A - 20I = 0.$$

We call A the companion matrix of the equation.

Of course such a solution is not of direct application in engineering practice, for it is the real or complex solutions that we ordinarily desire. The complex roots in the above case are all real, namely, 2, 2, 5. The question might naturally be asked if there is any relation between this matric solution and the complex solutions.

The answer is in the affirmative, for the companion matrix of every polynomial equation is similar to a triangular matrix having the complex roots in the principal diagonal. Thus for the example above

$$P^{-1}AP = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \qquad P = \begin{bmatrix} 10 & 5 & 4 \\ -7 & -6 & -4 \\ 1 & 1 & 1 \end{bmatrix}.$$

The result is general. If the complex roots are distinct, the companion matrix is similar to a diagonal matrix whose elements are the complex roots. If there are multiple roots, these are linked with 1's. Thus the coefficients of an equation and its complex roots are related by the matric equation

$$P^{-1}AP = C$$

where A involves the coefficients and C the roots.

There is another well-known set of relations between the roots and the coefficients which is embodied in the theory of symmetric functions. This powerful method is sometimes tedious in its applications. It seems worth while to inquire if the companion matrix can be applied in any situations as a substitute for the theory of symmetric functions.

* Presented to the annual meeting of the Association in New York, December 30, 1949.

2. A theorem of Frobenius. A matrix

$$M = \begin{bmatrix} M_1 & O \\ O & M_2 \end{bmatrix}$$

where O denotes a rectangular block of 0's is called the direct sum of M_1 and M_2 and may be written

$$M = M_1 \dotplus M_2.$$

The companion matrix A of a polynomial $f(x)$ is similar to a direct sum of matrices

$$C = J_1 \dotplus J_2 \dotplus \cdots \dotplus J_k$$

where k is the number of distinct complex roots of $f(x) = 0$, and each matrix

$$J = \begin{bmatrix} r & 1 & 0 & 0 \cdots \\ 0 & r & 1 & 0 \cdots \\ 0 & 0 & r & 1 \cdots \\ 0 & 0 & 0 & r \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

has as many rows and columns as the multiplicity of the root r.

Now if $g(x)$ is any polynomial,

$$P^{-1}g(A)P = g(C) = g(J_1) \dotplus g(J_2) \dotplus \cdots \dotplus g(J_k)$$

where

$$g(J) = \begin{bmatrix} g(r) & g'(r) & \frac{1}{2}g''(r) & \frac{1}{3!}g'''(r) & \cdots \\ 0 & g(r) & g'(r) & \frac{1}{2}g''(r) & \cdots \\ 0 & 0 & g(r) & g'(r) & \cdots \\ 0 & 0 & 0 & g(r) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

It is now obvious that if $r_1, r_2, \cdots, r_k$ are the distinct roots of $f(x) = 0$, then $g(r_1), g(r_2), \cdots, g(r_k)$ are the characteristic roots of $g(C)$, which are in turn the characteristic roots of $g(A)$. If r_1 is a root of $f(x) = 0$ of multiplicity h, then $g(r_1)$ occurs at least h times among the roots of $g(A)$. Of course, r_1 and r_2 may be different and such that $g(r_1) = g(r_2)$, in which case $g(r_1)$ is of higher multiplicity than h.

This well-known result, which is due to Frobenius, is a very practicable one for the performance of Tschirnhausen transformations. For instance, let it be required to find the equation $F(x)=0$ whose roots are the squares of the roots of

$$f(x) = x^2 + x + 5 = 0.$$

The companion matrix is

$$A = \begin{bmatrix} 0 & -5 \\ 1 & -1 \end{bmatrix}.$$

In this case $g(x)=x^2$ so that the roots of

$$F(x) = |A^2 - xI| = \begin{vmatrix} -5-x & 5 \\ -1 & -4-x \end{vmatrix} = x^2 + 9x + 25 = 0$$

are the squares of the roots of $f(x)=0$.

In general the equation whose roots are the function $g(x)$ of the roots of $f(x)=0$ is

$$|g(A) - xI| = 0$$

where A is the companion matrix of $f(x)=0$.

3. The resultant. Now let $f(x)=0$ and $g(x)=0$ be two polynomial equations, and let A be the companion matrix and $r_1, r_2, \cdots, r_n$ the complex roots of $f(x)=0$. By the Frobenius theorem, the characteristic roots of $g(A)$ are $g(r_1)$, $g(r_2), \cdots, g(r_n)$. The product of these roots is the constant term of the characteristic equation of $g(A)$; that is,

$$(-1)^n |g(A)| = g(r_1)\cdot g(r_2)\cdot \cdots \cdot g(r_n).$$

Thus $g(A)$ is singular if and only if $f(x)=0$ and $g(x)=0$ have a root in common. In fact, $(-1)^n|g(A)|$ is the resultant of $f(x)=0$ and $g(x)=0$. This fact is well known.

The concepts of determinant and matrix have remained sadly tangled up to recent years, and even yet there is much popular confusion. Some results which originally were stated in the language of determinants have proved to be but special cases of more complete theorems concerning matrices. The above result is a case in point, for we can prove the following useful theorem which somehow seems to have been overlooked.

THEOREM 1. *Let A be the companion matrix of the polynomial $f(x)$. The nullity of the matrix $g(A)$ is equal to the number of complex roots which $f(x)=0$ and $g(x)=0$ have in common, a root being counted as many times as its multiplicity.*

Let $f(x)=0$ have a root r of multiplicity 4, for instance. Then $g(A)$ is similar to a direct sum of blocks, each corresponding to a different root. The block corresponding to the root r will have the form

$$g(J) = \begin{bmatrix} g(r) & g'(r) & \frac{1}{2}g''(r) & \frac{1}{3!}g'''(r) \\ 0 & g(r) & g'(r) & \frac{1}{2}g''(r) \\ 0 & 0 & g(r) & g'(r) \\ 0 & 0 & 0 & g(r) \end{bmatrix}.$$

If r is not a root of $g(x)=0$, then $g(J)$ is of nullity 0. If r is a simple root, $g(r)=0$, $g'(r)\neq 0$, and $g(J)$ is of nullity 1. If r is a double root, $g(r)=g'(r)=0$, $g''(r)\neq 0$, and $g(J)$ is of nullity 2, *etc.* Since the nullity of a direct sum is equal to the sum of the nullities of the component matrices, the theorem is now evident.

Thus, let

$$f(x) = x^3 + 3x^2 - 4, \qquad g(x) = x^3 + 6x^2 + 12x + 8.$$

We find

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix}, \qquad g(A) = \begin{bmatrix} 12 & 12 & 12 \\ 12 & 12 & 12 \\ 3 & 3 & 3 \end{bmatrix}.$$

Since $g(A)$ is of nullity 2, $f(x)=0$ and $g(x)=0$ have two roots in common. Actually in this case it is a common double root.

4. The equation of squared differences. There are many applications of the process described in the last paragraph, one of which is the calculation of the equation whose roots are the squared differences of the roots of the given equation.

Consider the equation $f(x)=0$ with roots $r_1, r_2, \cdots, r_n$ and consider an equation $g(x)=0$ to be found whose roots are the differences $y_k=r_i-r_j$. There are n^2 such differences, including n zeros, the remaining n^2-n differences occurring in opposite pairs. Since $y_k+r_j=r_i$ is a root of $f(x)=0$, it is true that

$$f(x) = 0, \qquad f(x+y) = 0$$

have a common root for every y satisfying $g(y)=0$. Thus $g(y)$ is the eliminant of these equations. If A is the companion matrix of $f(x)=0$, then

$$|f(A + yI)| = g(y) = y^n F(y^2).$$

The roots of $F(z)=0$, then, are the square differences

$$(r_i - r_j)^2, \qquad i > j,$$

and the constant term of $F(z)$ with proper sign is the discriminant of $f(x)=0$.

Thus for

$$f(x) = x^3 - 7x + 6^*$$

$$A = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & 7 \\ 0 & 1 & 0 \end{bmatrix}, \qquad A + yI = \begin{bmatrix} y & 0 & -6 \\ 1 & y & 7 \\ 0 & 1 & y \end{bmatrix},$$

$$f(A + yI) = \begin{bmatrix} y^3 - 7y & -18y & -18y^2 \\ 3y^2 & y^3 + 14y & 21y^2 - 18y \\ 3y & 3y^2 & y^3 + 14y \end{bmatrix},$$

$$|f(A + yI)| = y^3(y^6 - 42y^4 + 441y^2 - 400),$$

so that

$$F(z) = z^3 - 42z^2 + 441z - 400.$$

The discriminant is 400.

5. The discriminant. The above method is not the most direct way of finding the discriminant of an equation, however, since the latter is the eliminant of $f(x) = 0$ and $f'(x) = 0$. That is,

$$\Delta = (-1)^n |f'(A)|$$

where A is the companion matrix of $f(x) = 0$.

But an even more valuable theorem than this can be deduced:

THEOREM 2. *If A is the companion matrix of $f(x)$, then the rank of $f'(A)$ is the number of distinct complex roots of the equation $f(x) = 0$.*

Suppose that $f(x)$ has h_1 simple roots, h_2 double roots, h_3 triple roots, *etc.* Then $f(x) = 0$ and $f'(x) = 0$ have in common h_2 simple roots, h_3 double roots, *etc.*, so that by Theorem 1 the nullity of $f'(A)$ is

$$\nu = h_2 + 2h_3 + 3h_4 + \cdots.$$

But the order of A is

$$n = h_1 + 2h_2 + 3h_3 + 4h_4 + \cdots$$

so that the rank of $f'(A)$ is

$$r = n - \nu = h_1 + h_2 + h_3 + h_4 + \cdots,$$

and the number of distinct roots of $f(x) = 0$.

Let us consider the general cubic equation

$$f(x) = x^3 + bx^2 + cx + d = 0.$$

Then

* Burnside and Panton, The theory of equations, Dublin University Press, 1912, v. 1, p. 83.

$$f'(A) = \begin{bmatrix} c & -3d & bd \\ 2b & -2c & bc - 3d \\ 3 & -b & b^2 - 2c \end{bmatrix}$$

whose determinant multiplied by $(-1)^3$ is

$$b^2c^2 - 4c^3 - 4b^3d - 27d^2 + 18bcd,$$

the discriminant.

If we set

$$f(x) = x^3 - 3rx^2 + 3r^2x - r^3 = (x - r)^3,$$

then

$$f'(A) = \begin{bmatrix} 3r^2 & 3r^3 & 3r^4 \\ -6r & -6r^2 & -6r^3 \\ 3 & 3r & 3r^2 \end{bmatrix}$$

which is of rank 1.

For every $f(x)$ of degree n, $f'(A) \neq 0$ since $f(x)$ is the minimum function of A. This does not constitute a proof of the fundamental theorem of algebra.

6. An illustration. I am tempted to make one further application of this method, even though the problem to which it is applied is rather trivial. Let

$$f(x) = f_1(x) \cdot f_2(x)$$

where $f_1(x)$ and $f_2(x)$ are relatively prime. Then the companion matrix A of $f(x)$ is similar to

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

where A_1 is the companion matrix of $f_1(x)$, and A_2 is the companion matrix of $f_2(x)$. Then

$$f'(x) = f_1(x) \cdot f_2'(x) + f_2(x) \cdot f_1'(x)$$

and, since $f_1(A_1) = f_2(A_2) = 0$,

$$f'(A) \overset{s}{=} f'(A_1 \dotplus A_2) = \begin{bmatrix} f_2(A_1) \cdot f_1'(A_1) & 0 \\ 0 & f_1(A_2) \cdot f_2'(A_2) \end{bmatrix}.$$

Hence the discriminant of $f(x) = 0$ is

$$\pm \left| f'(A) \right| = \pm \left| f_1(A_2) \right| \cdot \left| f_2(A_1) \right| \cdot \left| f_1'(A_1) \right| \cdot \left| f_2'(A_2) \right|$$

which is $\pm$ the square of the resultant of $f_1(x)$ and $f_2(x)$ multiplied by the product of their discriminants.

7. Symmetric functions. Since we have been able to use matric methods in place of symmetric function theory so effectively in special instance, it would be interesting to see if the general theory of symmetric functions can be so developed.

It is well known and easily proved that every polynomial which is symmetric in the roots $r_1, r_2, \cdots, r_n$ of an equation $f(x)=0$ is equal to a sum of symmetric polynomials of the type

$$\sum r_1^{e_1} r_2^{e_2} \cdots r_k^{e_k}, \qquad e_1 \geqq e_2 \geqq \cdots \geqq e_k.$$

This polynomial* is the coefficient of $\lambda_{e_1}\lambda_{e_2} \cdots \lambda_{e_k}$ in the expansion of

$$\prod_{i=1}^{n} (1 + \lambda_1 r_i + \lambda_2 r_i^2 + \cdots + \lambda_{e_1} r_i^{e_1})$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are indeterminates.

Let

$$f(x) = x^n + ax^{n-1} + bx^{n-2} + \cdots = 0$$

have the roots $r_1, r_2, \cdots, r_n$, and let A be its companion matrix. Form the polynomial

$$g(x) = 1 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_{e_1} x^{e_1}.$$

Then $|g(A)|$ is the constant term of the equation whose roots are the functions $g(x)$ of the roots of $f(x)=0$, so that

$$(-1)^n |g(A)| = \prod_{i=1}^{n} (1 + \lambda_1 r_i + \lambda_2 r_i^2 + \cdots + \lambda_{e_1} r_i^{e_1}).$$

Upon expanding both sides and equating coefficients of like powers of the λ's we obtain the symmetric functions of the roots of $f(x)=0$ expressed as polynomials in the coefficients of $f(x)$. Thus we have a short proof of the Fundamental Theorem on Symmetric Functions. Let the leading coefficient of $f(x)$ be 1. Every polynomial in the roots of $f(x)=0$ with coefficients in any field which is symmetric in these roots is expressible as a polynomial in the coefficients of $f(x)$ with coefficients in that field.

Moreover, the method is practicable and can be used to compute one or all of the symmetric polynomials in n roots of degree $\leqq k$ without recursive relations. Thus, suppose that $n=3$ and that

$$f(x) = x^3 + bx^2 + cx + d.$$

Let it be required to find $\sum r_1^3 r_2^2$. We have

* M. Faber, Archiv der Mathematik und Physik, 16 (1910), 144.

$$A=\begin{bmatrix}0 & 0 & -d\\ 1 & 0 & -c\\ 0 & 1 & -b\end{bmatrix}, \qquad g(x)=1+\lambda_1 x+\lambda_2 x^2+\lambda_3 x^3,$$

$$g(A)=\begin{bmatrix}-\lambda_3 d+1 & \lambda_3 bd-\lambda_2 d & \lambda_3(cd-b^2d)+\lambda_2 bd-\lambda_1 d\\ -\lambda_3 c+\lambda_1 & \lambda_3(bc-d)-\lambda_2 c+1 & \lambda_3(c^2-b^2c+bd)+\lambda_2(bc-d)-\lambda_1 c\\ -\lambda_3 b+\lambda_2 & \lambda_3(b^2-c)-\lambda_2 b+\lambda_1 & \lambda_3(-d+2bc-b^3)+\lambda_2(b^2-c)-\lambda_1 b+1\end{bmatrix}.$$

Each element of $g(A)$ is linear in 1, λ_1, λ_2, λ_3, so that this determinant can be written as a sum of 64 determinants each of which is equal to a monomial in the λ's. Thus to calculate $\sum r_1^3 r_2^2$ we need the sum of those determinants which are multiples of $\lambda_3\lambda_2$, namely,

$$\begin{vmatrix}-d & -d & 0\\ -c & -c & 0\\ -b & -b & 1\end{vmatrix}+\begin{vmatrix}-d & 0 & bd\\ -c & 1 & bc-d\\ -b & 0 & b^2-c\end{vmatrix}+\begin{vmatrix}0 & bd & 0\\ 0 & bc-d & 0\\ 1 & b^2-c & 1\end{vmatrix}$$
$$+\begin{vmatrix}0 & 0 & cd-b^2d\\ 0 & 1 & c^2-b^2c+bd\\ 1 & 0 & -d+2bc-b^3\end{vmatrix}+\begin{vmatrix}1 & bd & bd\\ 0 & bc-d & bc-d\\ 0 & b^2-c & b^2-c\end{vmatrix}$$
$$+\begin{vmatrix}1 & -d & cd-b^2d\\ 0 & -c & c^2-b^2c+bd\\ 0 & -b & -d+2bc-b^3\end{vmatrix}=2b^2d+cd-bc^2.$$

Similarly

$$\sum r_1^3 r_2 = b^2c - bd - 2c^2$$

is the coefficient of $\lambda_3\lambda_1$ in the expansion of $|g(A)|$. All other symmetric polynomials may be calculated similarly.

THE VECTOR POTENTIAL OF A SOLENOIDAL VECTOR

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If the vector $\mathbf{f}(\mathbf{r})$ is solenoidal in a region R (div $\mathbf{f}=0$), $\mathbf{f}$ can be expressed as the rotation of a vector $\mathbf{g}$, its vector potential; that is,

$$\mathbf{f} = \text{rot}\,\mathbf{g}.$$

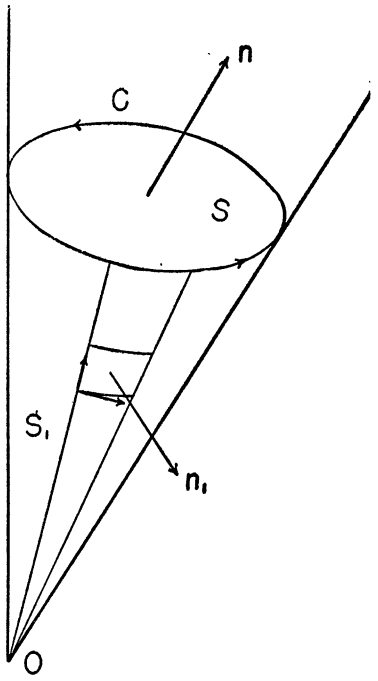
The function $\mathbf{g}$ is arbitrary to an additive gradient of a scalar. The following method of finding $\mathbf{g}$ is independent of the coordinate system and involves only

simple integration.

The integral of $\mathbf{f}$ over a surface S bounded by a closed curve C, $\mathbf{r}=\mathbf{r}(s)$, is

$$\int_S \mathbf{n}\cdot\mathbf{f}\,dS = \int_S \mathbf{n}\cdot\operatorname{rot}\mathbf{g}\,dS = \oint_C \mathbf{\tau}\cdot\mathbf{g}\,ds \tag{1}$$

by Stokes' Theorem. Let S_1 be the surface of a cone with vertex at the origin and having C as directrix and boundary; note the figure. The parametric equa-



tion of S_1 is $\mathbf{r}_1=\lambda\mathbf{r}(s)$, where s is the length of arc on C measured from some fixed point of the curve. Hence the vector element of area on S_1 is

$$\mathbf{n}_1 dS_1 = \frac{\partial \mathbf{r}_1}{\partial s}\times\frac{\partial \mathbf{r}_1}{\partial \lambda}\,ds\,d\lambda = \mathbf{\tau}(s)\times\mathbf{r}(s)\lambda\,ds\,d\lambda,$$

where $\mathbf{\tau}$ is a unit tangent vector along C. Hence the surface integral of $\mathbf{f}$ over S_1 is

$$\int_{S_1}\mathbf{n}_1\cdot\mathbf{f}(\mathbf{r}_1)\,dS_1 = \oint_C \mathbf{\tau}\cdot\left\{\int_0^1 \mathbf{r}(s)\times\mathbf{f}(\lambda\mathbf{r})\lambda\,d\lambda\right\}ds. \tag{2}$$

The unit normals $\mathbf{n}$ and $\mathbf{n}_1$ are both external to the region enclosed by S and S_1. Since div $\mathbf{f}=0$, the sum of the surface integrals (1) and (2) is zero; hence for any closed curve C in R (compatible with the conditions of Stokes' Theorem)

$$\oint_C \mathbf{T}\cdot\left\{\mathbf{g}+\int_0^1 \mathbf{r}(s)\times\mathbf{f}(\lambda\mathbf{r})\lambda d\lambda\right\}ds = 0.$$

The vector in braces is therefore irrotational and may be expressed as the gradient of a scalar. We thus obtain the particular solution

$$\mathbf{g}(\mathbf{r}) = -\mathbf{r}\times\int_0^1 \mathbf{f}(\lambda\mathbf{r})\lambda d\lambda, \tag{3}$$

provided $\mathbf{r}\times\mathbf{f}(\lambda\mathbf{r})\neq 0$ and the integral converges in case it is improper. When $\lambda\mathbf{f}(\lambda\mathbf{r})\to\infty$ as $\lambda\to 0$ we assume that

$$\lim_{\lambda\to 0}\lambda^p\mathbf{f}(\lambda\mathbf{r}) = \mathbf{A}, \qquad 1<p<2, \tag{4}$$

in order to ensure convergence.

The formula (3) is due to H. Liebmann* and has remained unnoticed in the literature. Liebmann apparently regarded his derivation of (3) as heuristic in character, and proceeded to verify that rot $\mathbf{g}=\mathbf{f}$, using rectangular coordinates for the computation. Since his proof is rather long we give a brief vector-analytic verification. We assume that $\nabla\mathbf{f}(\lambda\mathbf{r})$ and $d\mathbf{f}(\lambda\mathbf{r})/d\lambda$ are continuous in the region R and the interval $0<\lambda\leqq 1$; then from (3)

$$\text{rot}\,\mathbf{g} = \int_0^1 \text{rot}\,\{\mathbf{f}(\lambda\mathbf{r})\times\mathbf{r}\}\lambda d\lambda. \tag{5}$$

We now apply the formula

$$\text{rot}\,(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{v} + \mathbf{u}\,\text{div}\,\mathbf{v} - \mathbf{v}\,\text{div}\,\mathbf{u}$$

to the integrand; putting $\mathbf{r}_1=\lambda\mathbf{r}$,

$$\text{rot}\,\{\mathbf{f}(\mathbf{r}_1)\times\mathbf{r}\} = \mathbf{r}\cdot\nabla\mathbf{f}(\mathbf{r}_1) - \mathbf{f}(\mathbf{r}_1)\cdot\nabla\mathbf{r} + \mathbf{f}(\mathbf{r}_1)\,\text{div}\,\mathbf{r} - \mathbf{r}\,\text{div}\,\mathbf{f}(\mathbf{r}_1),$$

where the differential operations ∇, rot, div are all with respect to the vector $\mathbf{r}$. Now if ∇_1 and div_1 refer to the vector $\mathbf{r}_1$, we have

$$\nabla\mathbf{f}(\mathbf{r}_1) = \nabla\mathbf{r}_1\cdot\nabla_1\mathbf{f}(\mathbf{r}_1) = \lambda\mathbf{I}\cdot\nabla_1\mathbf{f}(\mathbf{r}_1) = \lambda\nabla_1\mathbf{f}(\mathbf{r}_1),$$
$$\text{div}\,\mathbf{f}(\mathbf{r}_1) = \lambda\,\text{div}_1\,\mathbf{f}(\mathbf{r}_1) = 0;$$

and since $\nabla\mathbf{r}=\mathbf{I}$, div $\mathbf{r}=3$,

$$\text{rot}\,\{\mathbf{f}(\mathbf{r}_1)\times\mathbf{r}\} = \lambda\mathbf{r}\cdot\nabla_1\mathbf{f}(\mathbf{r}_1) + 2\mathbf{f}(\mathbf{r}_1).$$

Now

$$\frac{d}{d\lambda}\mathbf{f}(\mathbf{r}_1) = \frac{d\mathbf{r}_1}{d\lambda}\cdot\nabla_1\mathbf{f}(\mathbf{r}_1) = \mathbf{r}\cdot\nabla_1\mathbf{f}(\mathbf{r}_1),$$

* Leipziger Berichte, Bd. 60 (1908), pp. 176–189.

$$\lambda \text{ rot } \{\mathbf{f}(\mathbf{r}_1) \times \mathbf{r}\} = \lambda^2 \frac{d}{d\lambda}\mathbf{f}(\mathbf{r}_1) + 2\lambda f(\mathbf{r}_1) = \frac{d}{d\lambda}\{\lambda^2 \mathbf{f}(\mathbf{r}_1)\};$$

and since this expression is the integrand in (5),

$$\text{rot } \mathbf{g} = \lambda^2 \mathbf{f}(\mathbf{r}_1)\Big|_{\lambda=0}^{\lambda=1} = \mathbf{f}(\mathbf{r}).$$

When the integral is improper the lower limit is still zero, for, from (4),

$$\lim_{\lambda\to 0} \lambda^2 \mathbf{f}(\mathbf{r}_1) = \lim_{\lambda\to 0} \mathbf{A}\lambda^{2-p} = 0, \qquad 1 < p < 2.$$

Thus under the conditions stated, rot $\mathbf{g}(\mathbf{r}) = \mathbf{f}(\mathbf{r})$.

When $\mathbf{r} \times \mathbf{f}(\lambda\mathbf{r}) = 0$ or when the integral in (3) diverges Liebmann's formula is inapplicable. We proceed to extend the method above and find $\mathbf{g}$ in these cases of failure.

If $\mathbf{r} \times \mathbf{f}(\lambda\mathbf{r}) \neq 0$, but the integral in (3) is a divergent improper integral, we may replace it by

$$\mathbf{g}(\mathbf{r}) = \mathbf{r} \times \int_1^\infty \mathbf{f}(\lambda\mathbf{r})\lambda d\lambda, \tag{6}$$

provided that this integral converges. The convergence will be assured if we assume that

$$\lim_{\lambda\to\infty} \lambda^p \mathbf{f}(\lambda\mathbf{r}) = \mathbf{A}, \qquad p > 2. \tag{7}$$

For the preceding argument then shows that when $\mathbf{g}$ is given by (6),

$$\text{rot } \mathbf{g} = -\lambda^2 \mathbf{f}(\lambda\mathbf{r})\Big|_{\lambda=1}^{\lambda=\infty} = \mathbf{f}(\mathbf{r})$$

since from (7)

$$\lim_{\lambda\to\infty} \lambda^2 \mathbf{f}(\lambda\mathbf{r}) = \lim_{\lambda\to\infty} \frac{\mathbf{A}}{\lambda^{p-2}} = 0, \qquad p > 2 \tag{8}$$

Formula (7) was obtained by integrating over the cone

$$\mathbf{r}_1 = \lambda\mathbf{r}(s), \qquad 1 \leqq \lambda < \infty,$$

extending from the curve C to infinity. The surface integral over the portion of an infinite sphere about O intercepted by the cone vanishes in view of (8).

If $\mathbf{f}(\mathbf{r})$ is homogeneous of degree n, namely,

$$\mathbf{f}(\lambda\mathbf{r}) = \lambda^n \mathbf{f}(\mathbf{r}), \tag{9}$$

and $n \neq -2$, both (3) and (6) give

$$\mathbf{g}(\mathbf{r}) = \frac{\mathbf{f}(\mathbf{r}) \times \mathbf{r}}{n+2}. \tag{10}$$

When $n \geqq -1$, the integral in (3) is proper; when $-2 < n < -1$, (4) is satisfied with $p = -n$; and when $n < -2$, (7) is satisfied with $p = -n$. Moreover we may verify directly that rot $\mathbf{g} = \mathbf{f}$ if we make use of the extension of Euler's formula

$$\mathbf{r} \cdot \nabla \mathbf{f}(\mathbf{r}) = n\mathbf{f}(\mathbf{r}) \tag{11}$$

obtained by differentiating (9) with respect to λ and then putting $\lambda = 1$. In fact we have from (10)

$$\text{rot } \mathbf{g} = \frac{1}{n+2} \{\mathbf{r} \cdot \nabla \mathbf{f} - \mathbf{f} \cdot \nabla \mathbf{r} + \mathbf{f} \text{ div } \mathbf{r}\} = \mathbf{f}.$$

Neither (3) or (6) apply when $n = -2$. In this case we may use the formula

$$\mathbf{g} = \mathbf{e} \times \int_0^\infty \mathbf{f}(\mathbf{r} + \lambda \mathbf{e}) d\lambda \tag{12}$$

where $\mathbf{e}$ is a constant unit vector of arbitrary direction. Indeed equation (12) applies generally whenever the integral converges, for example, when

$$\lim_{\lambda \to \infty} \lambda^p \mathbf{f}(\mathbf{r} + \lambda \mathbf{e}) = \mathbf{A}, \qquad p > 1. \tag{13}$$

For if we put $\mathbf{r}_1 = \mathbf{r} + \lambda \mathbf{e}$ and use the formula for rot $(\mathbf{u} \times \mathbf{v})$ cited above

$$\text{rot } \mathbf{g} = -\mathbf{e} \cdot \int_0^\infty \nabla \mathbf{f}(\mathbf{r}_1) d\lambda = -\mathbf{e} \cdot \int_0^\infty \nabla_1 \mathbf{f}(\mathbf{r}_1) d\lambda,$$

since

$$\nabla f(\mathbf{r}_1) = \nabla \mathbf{r}_1 \cdot \nabla_1 \mathbf{f}(\mathbf{r}_1) = \nabla \mathbf{r} \cdot \nabla_1 \mathbf{f}(\mathbf{r}_1) = \nabla_1 \mathbf{f}(\mathbf{r}_1),$$
$$\text{div } \mathbf{f}(\mathbf{r}_1) = \text{div}_1 \mathbf{f}(\mathbf{r}_1) = 0.$$

Now

$$\frac{d}{d\lambda} \mathbf{f}(\mathbf{r}_1) = \frac{d\mathbf{r}_1}{d\lambda} \cdot \nabla_1 \mathbf{f}(\mathbf{r}_1) = \mathbf{e} \cdot \nabla_1 \mathbf{f}(\mathbf{r}_1)$$

and hence

$$\text{rot } \mathbf{g} = -\mathbf{f}(\mathbf{r}_1) \Big|_{\lambda=0}^{\lambda=\infty} = \mathbf{f}(\mathbf{r}),$$

for at the upper limit

$$\lim_{\lambda \to \infty} \mathbf{f}(\mathbf{r}_1) = \lim_{\lambda \to \infty} \frac{\mathbf{A}}{\lambda^p} = 0, \qquad p > 1, \tag{14}$$

from (13)

The integral (10) is obtained by integrating over the cylinder

$$\mathbf{r}_1 = \mathbf{r}(s) + \lambda\mathbf{e}. \qquad 0 \leqq \lambda < \infty,$$

extending from the curve C to infinity. Its surface element is

$$\mathbf{n}_1 dS_1 = \frac{\partial \mathbf{r}_1}{\partial s} \times \frac{\partial \mathbf{r}_1}{\partial \lambda} ds d\lambda = \mathbf{t} \times \mathbf{e} ds d\lambda,$$

where $\mathbf{n}_1$ is the external unit normal, and

$$\int_{S_1} \mathbf{n}_1 \cdot \mathbf{f}(\mathbf{r}_1) dS_1 = \oint_C \mathbf{t} \cdot \left\{ \int_0^\infty \mathbf{e} \times \mathbf{f}(\mathbf{r}_1) d\lambda \right\} ds. \tag{15}$$

The cylinder is closed by the cap S (over which $\mathbf{n}$ is now the internal normal) and by a portion of an infinite sphere about O. The surface integral over the latter vanishes in view of (14). Hence the difference of the surface integrals in (1) and (15) vanishes, and we obtain the particular value of $\mathbf{g}$ in (12).

By way of summary we state the

THEOREM. *If* $\operatorname{div} \mathbf{f} = 0$ *the vector potential* $\mathbf{g}$ *of* $\mathbf{f}$ *is given by any one of the vector functions*

$$-\mathbf{r} \times \int_0^1 \mathbf{f}(\lambda\mathbf{r}) \lambda d\lambda, \qquad \mathbf{r} \times \int_1^\infty \mathbf{f}(\lambda\mathbf{r}) \lambda d\lambda, \qquad \mathbf{e} \times \int_0^\infty \mathbf{f}(\mathbf{r} + \lambda\mathbf{e}) d\lambda.$$

for which the integral converges. In particular if $\mathbf{f}$ *is homogeneous of degree* $n = -2$,

$$\mathbf{g} = \frac{\mathbf{f}(\mathbf{r}) \times \mathbf{r}}{n + 2}.$$

Example 1. The vector $\mathbf{f} = \mathbf{a}$ (constant) is solenoidal. Since $\mathbf{f}(\lambda\mathbf{r}) = \mathbf{f}(\mathbf{r})$ we apply (10) with $n = 0$: thus $\mathbf{g} = \mathbf{a} \times \mathbf{r}/2$.

Example 2. The vector $\mathbf{f} = \mathbf{r}/r^3 = -\nabla_{\mathbf{r}} \frac{1}{r}$ is solenoidal and $\mathbf{f}(\lambda\mathbf{r}) = \lambda^{-2}\mathbf{f}(\mathbf{r})$. Hence formula (10) is inapplicable, but we may use (12) since

$$\lim_{\lambda \to \infty} \lambda^2 \frac{\mathbf{r} + \lambda\mathbf{e}}{|\mathbf{r} + \lambda\mathbf{e}|^3} = \mathbf{e}.$$

If we write $\mathbf{r} \cdot \mathbf{e} = r \cos \theta$,

$$(\mathbf{r} + \lambda\mathbf{e}) \cdot (\mathbf{r} + \lambda\mathbf{e}) = r^2 + 2\lambda r \cos \theta + \lambda^2$$

and hence

$$\mathbf{g} = \mathbf{e} \times \mathbf{r} \int_0^\infty \frac{d\lambda}{(r^2 + 2\lambda r \cos \theta + \lambda^2)^{3/2}}.$$

Making the substitution

$$t = \frac{\lambda + r\cos\theta}{r\sin\theta},$$

we find that

$$\mathbf{g} = \frac{\mathbf{e}\times\mathbf{r}}{r^2\sin^2\theta}\int_{\cot\theta}^{\infty}\frac{dt}{(1+t^2)^{3/2}} = \frac{\mathbf{e}\times\mathbf{r}}{r^2\sin^2\theta}(1-\cos\theta) = \frac{\mathbf{e}\times\mathbf{r}}{r^2(1+\cos\theta)}.$$

If we choose $\mathbf{e}=\mathbf{k}$ and write $\mathbf{e}\times\mathbf{r}=r\sin\theta\,\mathbf{c}$, where $\mathbf{c}$ is a unit vector, we have also

$$\mathbf{g} = (1-\cos\theta)\frac{\mathbf{c}}{r\sin\theta} = (1-\cos\theta)\nabla\phi,$$

where r, θ, ϕ are spherical coordinates.

This leads to an expression for the solid angle Ω_0 subtended at O by a closed curve C; for if S is a surface spanned over C

$$\Omega_0 = \int_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}dS = \oint_C \mathbf{T}\cdot\mathbf{g}\,ds = \oint_C (1-\cos\theta)d\phi.$$

Query: We may take $\mathbf{g}=(C-\cos\theta)\nabla\phi$, since $\mathbf{g}$ is arbitrary to an additive gradient. If we now compute the solid angle subtended by the circle $\theta=$const. on a sphere, $\Omega_0=(C-\cos\theta)2\phi$; how can this be true for any value of the constant C? Discuss the formula $\Omega_0=\oint(C-\cos\theta)d\phi$ for a spherical curve that does not enclose either pole.

Example 3. The vector $\mathbf{f}=r^n\mathbf{e}\times\mathbf{r}$ is solenoidal as we see on writing it

$$\mathbf{f} = \frac{1}{n+2}\mathbf{e}\times\nabla r^{n+2}.$$

Moreover, $\mathbf{f}(\lambda\mathbf{r})=\lambda^{n+1}\mathbf{f}(\mathbf{r})$; hence if $n\neq -3$ we may apply (10) and obtain

$$\mathbf{g} = \frac{r^n(\mathbf{e}\times\mathbf{r})\times\mathbf{r}}{n+3}.$$

When $n=-3$ this formula fails. We then use (12) to obtain

$$\mathbf{g} = \mathbf{e}\times(\mathbf{e}\times\mathbf{r})\int_0^{\infty}\frac{d\lambda}{|\mathbf{r}+\lambda\mathbf{e}|^3} = \frac{\mathbf{e}\times(\mathbf{e}\times\mathbf{r})}{r^2(1+\cos\theta)}$$

on using the value of the integral obtained in Example 2.

MATHEMATICAL NOTES

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ENVELOPES OF PLANE CURVES

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As we know, the envelope of a one parameter family of plane curves touches each member at the point they have in common. The usual proof of this result is analytic, that is, by calculating the slopes of the two tangent lines (the writer has seen no other proof); but it is well to give a geometric proof if only for its perspicuity.

As our definition of the envelope we take the usual one, namely that it is the locus of the points of intersection of "consecutive" members of the family.

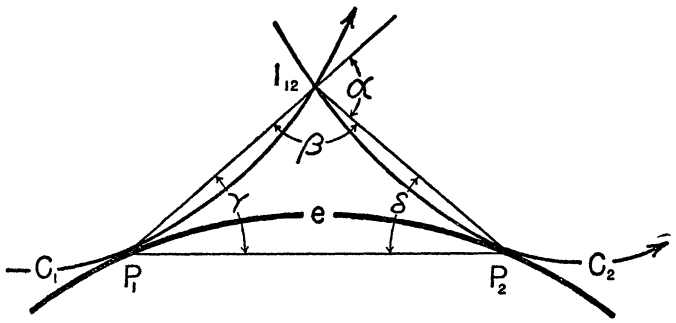
We let e denote the envelope of the family $\{f(x, y, c)=0\}$. We refer to the members $f(x, y, c_1)=0$ and $f(x, y, c_j)=0$ as the curves c_1 and c_j; symbols like these will sometimes represent values of the parameter but more often the curves associated with such values. We let P_j denote the point in which c_j and e meet, I_{jk} that in which c_j and c_k meet. I_{jk} and I_{kj} are, of course, the same point; and assuming the necessary continuity, we see that P_j can be represented equally well by I_{jj}. We shall be dealing with local properties and take c_1, c_2, c_3, c_j, c_k so that they differ from each other by sufficiently little. It is necessary for us to make three basic assumptions about the family $\{f(x, y, c)=0\}$, all geometric and quite natural. First we assume that P_2 lies between P_1 and P_3 on e if $c_1<c_2<c_3$; in other words that, locally at least, there is a direction of advance for the members of the family along the envelope. Second, that the point I_{2k} lies between I_{1k} and I_{3k} on c_k if $c_1<c_2<c_3$; this is simply a way of saying that the approach of I_{jk} along c_k to its limiting position $I_{kk}\equiv P_k$ is to be continuous. Third, that only one branch of c_j passes through P_j, that is, that P_j is not a multiple point with respect to c_j. This last excludes the nodal locus, ordinarily not a part of the envelope, which even analytically complicates the operation of finding the envelope.

Only one branch of c_j passes through P_j, and so we can designate one end (symbolized by c_j+) of the curve as the $+$ or forward end with respect to P_j, the other (symbolized by c_j-) as the $-$ or rearward end. We achieve a certain uniformity by taking this forward end in the general direction of advance along e for every local curve c_j.

We now let c_k approach c_2 and get a limiting form of the relationship conveyed by the second assumption, namely that $I_{22}\equiv P_2$ lies between I_{21} and I_{23} on c_2 if $c_1<c_2<c_3$. That is, I_{21} and I_{23} lie on opposite ends of the curve c_2 emanating from P_2. That is, if c_2- (alternatively c_2+) meets c_1, then c_2+ (alternatively c_2-) meets c_3. But c_2 is a general member of the family and its behavior vis-à-vis curves with larger and smaller parameter values must characterize all local

members. So it is c_1+ (alternatively c_1-) that meets c_2 or, taking into account the ends as well as the curve, it is c_1+ (alternately c_1-) that meets c_2- (alternatively c_2+). In short, c_1 and c_2 meet locally in unlike ends.

Taking the direction of advance along e to the right in the figure, we assume that it is the rear $(-)$ end of c_2 that meets the forward $(+)$ end of c_1—this is no real restriction—and represent the two curves as in the diagram. We draw the lines P_1P_2, P_1I_{12}, and $I_{12}P_2$ and consider the triangle $P_1I_{12}P_2$.



As the curve c_2 moves toward coincidence with c_1, I_{12} approaches P_1 on the one hand and P_2 on the other (definition of envelope), the directed lines $\overrightarrow{P_1I_{12}}$ and $\overrightarrow{I_{12}P_2}$ move into the positions of the forward tangents to these curves at the same point I_{12}, the angle α between the two directed lines approaches zero, β the supplement of α approaches π, and the angles γ and δ individually approach zero. Simultaneously, P_2 approaches P_1 along e and P_1P_2 swings into the position of the tangent to e at P_1, while P_1I_{12} becomes the tangent to c_1 at P_1, so that c_1 and e touch each other at P_1.

The fact that c_1 and c_2 meet in unlike ends is essential to the proof. Otherwise, $\overrightarrow{P_1I_{12}}$ and $\overrightarrow{I_{12}P_2}$ would in the limit be oppositely directed rays of the same line (the tangent to c_1 at I_{12}), α would approach π instead of zero, β would approach zero, $\gamma+\delta$ would approach π, so that γ would be indeterminate and our proof would fail.

This fact actually characterizes the envelope. That is, if we take any other curve g passing through P_1 and making a nonzero angle with e, defining the rear and forward directions of the local members of the family with respect to g, then c_1 and c_2 will meet in like ends. We can see this by appealing to a diagram or, more abstractly, by remembering that otherwise, using the very same reasoning that was used for e, we could prove that g and c_1 touch each other.

ON THE ORTHOCENTROIDAL CIRCLE

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Let H, G, O be the orthocenter, centroid, and circumcenter of triangle ABC. Let the feet of the altitudes be denoted by A_h, B_h, C_h, and the midpoints of the

sides by A_m, B_m, C_m.

If G_a, G_b, G_c are the centroids of triangles HBC, HCA, HAB, then triangle $G_aG_bG_c$ is homothetic to triangle $A_mB_mC_m$ under the homothety $(H, 2/3)$. The lines GG_a, GG_b, GG_c are thus parallel, respectively, to OA_m, OB_m, OC_m, and:

1. *Point G is the orthocenter of triangle $G_aG_bG_c$.*

It follows that $GG_a=(2/3)OA_m=(1/3)AH$. Therefore, if N is the intersection of AG_a and GH, $GN=(1/3)NH$, and we have:

2. *The lines AG_a, BG_b, CG_c are concurrent at N, the center of the nine point circle of triangle ABC.*

Also, since $NG_a=-(1/3)NA$:

3. *Triangle $G_aG_bG_c$ is homothetic to triangle ABC under the homothety $(N, -1/3)$.*

The vertices A, B, C are the orthocenters of triangles HBC, HCA, HAB, and the orthocentroidal circles [1] of these triangles are the circles on AG_a, BG_b, CG_c as diameters. The centers O_a, O_b, O_c of these circles are such that:

4. *Triangle $O_aO_bO_c$ is homothetic to triangle ABC under the homothety $(N, 1/3)$, and the two triangles $G_aG_bG_c$, $O_aO_bO_c$ are symmetric with respect to N.*

As a consequence of this last theorem we have that the circumcenter O_g of triangles $G_aG_bG_c$ is the symmetric of G with respect to N, and is thus the midpoint of GH. This gives us the two following results:

5. *The circumcenter of triangle $G_aG_bG_c$ is the center of the orthocentroidal circle of triangle ABC and is also the orthocenter of triangle $O_aO_bO_c$.*

6. *The centroid of triangle ABC is the circumcenter of triangle $O_aO_bO_c$.*

The circumcircle (O) and the orthocentroidal circle (GH) of triangle ABC have the orthic axis of the triangle for radical axis [2]. Also, the circle (HBC) and the orthocentroidal circle (G_aA) have the orthic axis of triangle HBC as radical axis. But the orthic axes of ABC and HBC pass through the intersection A' of BC and B_hC_h. Now the circumcircles of ABC and HBC are symmetric with respect to BC, and therefore have BC as their radical axis. The point A', then, is the common radical center of the four circles (O), (HBC), (GH), (G_aA). But the radical axis of circles (GH) and (G_aA) is perpendicular to the center line O_gQ_a, and thus parallel to BC. Therefore:

7. *The sides BC, CA, AB of triangle ABC are the radical axes of the orthocentroidal circle of ABC and the orthocentroidal circles of triangles HBC, HCA, HAB.*

The radical axis of the orthocentroidal circles (G_bB), (G_cC) is perpendicular to O_bO_c, and therefore parallel to the altitude AH. Now, by the preceding

theorem, vertex A is the radical center of circles (GH), (G_bB), (G_cC). Therefore AH is the radical axis of the circles (G_bB) and (G_cC). Hence:

8. *The altitudes of triangle ABC are the radical axes of the orthocentroidal circles, taken in pairs, of triangles HBC, HCA, HAB.*

As a final consequence we then have:

9. *The orthocenter H of triangle ABC is the radical center of the orthocentroidal circles of triangles HBC, HCA, HAB.*

In closing we might add that some of the preceding results, which we had communicated to Victor Thébault, have been extended by him to the geometry of the tetrahedron [3].

Notes and References

1. The *orthocentroidal circle* of a triangle is the circle having the join of the orthocenter and centroid of the triangle as diameter. It was so named by Robert Tucker.

2. J. Griffiths, Nouvelles Annales de Mathématiques, 1864, p. 345, and 1865, p. 322. Also this MONTHLY [1947, 538] and [1948, 421].

3. V. Thébault has called the sphere having for diameter the join of the Monge point and the centroid of a tetrahedron, the *orthocentroidal sphere* of the tetrahedron. See Notes mathématiques (42), Mathesis, t. 55, 1945–6, p. 264. Properties of this sphere may be found in V. Thébault, Nouvelles sphères associées au tétraèdre, Comptes Rendus des Séances de l'Académie des Science, t. 224, pp. 1614–1616 and V. Thébault, On the Monge point of the tetrahedron, this MONTHLY [1949, 4–13]. In particular, see sec. 6, part (b), of this last paper. In his paper, Sur le cercle et la sphère de Hagge, Bulletin de la Société Royale des Science de Liége, 1946, pp. 44–52, Thébault has designated as the *Hagge sphere* of a point P, with respect to a tetrahedron $T=ABCD$, the circumsphere of the tetrahedron $A_2B_2C_2D_2$, where A_2, B_2, C_2, D_2 are the transforms of the points A_1, B_1, C_1, D_1 in which the cevians through P cut the circumsphere of T under the homotheties $(G_A, -1/2)$, $(G_B, -1/2)$, $(G_C, -1/2)$, $(G_D, -1/2)$, where G_A, G_B, G_C, G_D are the centroids of the faces BCD, CDA, DAB, ABC of T. When P coincides with the centroid G of T, the Hagge sphere becomes the orthocentroidal sphere of T.

A NOTE ON ORTHOPOLAR TRIANGLES*

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Let ABC be a triangle inscribed in a circle (O) of center O and radius R. If the perpendicular to BC drawn through an arbitrary point A' on the circle (O) cuts the circle again in P and, if one draws an arbitrary chord $B'C'$ perpendicular to AP, then the Simson line of A', with respect to triangle ABC, is parallel to AP and triangles ABC, $A'B'C'$ are orthopolar [1]. The Simson lines of A, B, C for triangle $A'B'C'$ and A', B', C' for triangle ABC are concurrent at the midpoint Ω of the segment HH' joining the orthocenters of the two triangles, and Ω coincides with the orthopole of any side of one triangle with respect to the other [2].

* Translated from the French by Howard Eves.

THEOREM I. *The two bisectors of an angle of either of the triangles ABC, $A'B'C'$ determine on any bisector of the other triangle a segment whose midpoint is on circle (O).*

Because triangles ABC and $A'B'C'$ play a symmetric role and because, keeping the vertex letters of one triangle fixed, we may permute the vertex letters of the other, it suffices to show that the two bisectors of angle A determine on the bisectors of angle A' segments whose midpoints are on circle (O). To this end let Q be the point on (O) diametrically opposite to P. Then AQ and $A'Q$ are respectively parallel to $B'C'$ and BC, and the bisectors of angles A and A' of triangle $AA'Q$ are bisectors of the same angles in the triangles ABC and $A'B'C'$. But the two bisectors of any angle of a triangle determine on any other bisector of the triangle a segment whose midpoint is on the circumcircle of the triangle. This proves our theorem.

THEOREM II. *The centers I, I_a, I_b, I_c and I', I'_a, I'_b, I'_c of the tritangent circles of triangles ABC and $A'B'C'$ are eight points on a common rectangular hyperbola (H).*

In the pencil of rectangular hyperbolas passing through I, I_a, I_b, I_c, the conics consisting of the pairs of bisectors of the angles A, B, C of ABC determine, by Theorem I, on the bisector $A'A'_1$ of angle A' of $A'B'C'$, (A'_1 on circle (O)), a symmetric involution with double point A'_1. The points I' and I'_a correspond in this involution. The rectangular hyperbola (H) which passes through the five points I, I_a, I_b, I_c, I' then passes through I'_a, and similarly through I'_b, I'_c.

THEOREM III. *If we designate by ϕ and ϕ' the foci of the two conics respectively inscribed in triangles ABC and $A'B'C'$ and which have an arbitrary common focus F, the line which joins the foci ϕ_1 and ϕ'_1 of the conics inscribed in ABC and $A'B'C'$ and which have a common focus M, arbitrarily situated on $\phi\phi'$, passes through the point F.*

The point ϕ and ϕ' are conjugate points of F with respect to (H) [3]. On the other hand, the foci ϕ_1 and ϕ'_1 are on the polar of M with respect to (H). The points ϕ_1 and ϕ'_1 are then collinear with F.

THEOREM IV. *If (C_1) and (C'_1) are the conics inscribed in triangles ABC and $A'B'C'$ and having Ω for a common focus, then* (1) *the conics have equal non-focal axes,* (2) *the other foci Ω_1 and Ω'_1 of the conics are symmetric with respect to the circumcenter O* [4], (3) *the distances of the centers K_1 and K'_1 of the conics from the centers N and N' of the nine point circles of the two triangles are equal.*

If d_a, d_b, d_c and d'_a, d'_b, d'_c are the distances of the orthopole Ω from BC, CA, AB and $B'C'$, $C'A'$, $A'B'$, then [5]

$$d_a d_b d_c = d'_a d'_b d'_c$$

and, consequently [6],

$$\beta^2 = 2Rd_ad_bd_c/\omega = 2Rd_a'd_b'd_c'/\omega = \beta'^2,$$

where β and β' are the non-focal semi-axes of (C_1) and (C_1') and ω is the power of Ω with respect to (O). On the other hand, if ω_1 and ω_1' are the powers of Ω_1 and Ω_1' for circle (O), then [7]

$$\omega\omega_1 = 4\beta^2R^2 = 4\beta'^2R^2 = \omega\omega_1',$$

whence $\omega_1=\omega_1'$ and consequently $O\Omega_1=O\Omega_1'$. That Ω_1, O, Ω_1' are collinear follows from Theorem III by taking $\phi\equiv H$, $\phi'\equiv H'$, $F\equiv O$ and $\phi_1\equiv\Omega_1$, $\phi_1'\equiv\Omega_1'$, $M\equiv\Omega$. The fact that $K_1N=K_1'N'$ now follows by elementary geometry, for K_1K_1' and NN' bisect each other.

THEOREM V. *In a triangle ABC with orthocenter H and circumcenter O, the isogonal conjugates F_1, F_2 of the center of the Taylor circle* [8] *in the medial and orthic triangles are symmetric with respect to the nine point center, and HF_2 is equal and parallel to F_1O.*

For the medial and orthic triangles are orthopolar triangles inscribed in the nine point circle, and the center F of the Taylor circle of ABC is the midpoint of the segment joining the orthocenters of the medial and orthic triangles [9].

Part (2) of Theorem IV is a particular case of the following theorem, due to Blanchard (Le Havre), which is scheduled to appear in *Mathesis*. Guillotin (Le Mans) has given a synthetic demonstration which served as the stimulus for the present note.

THEOREM VI. *Being given two orthopolar triangles ABC and $A'B'C'$ inscribed in a circle, if one designates by ϕ and ϕ' the foci of the two conics respectively inscribed in these triangles and which have a common arbitrary focus F, the line joining the foci ϕ_1 and ϕ_1' of conics respectively inscribed in ABC and $A'B'C'$ and which have for common focus the point dividing the segment $\phi\phi'$ in the ratio k, passes through F and, moreover, the point F divides the segment $\phi_1\phi_1'$ in the ratio k.*

References

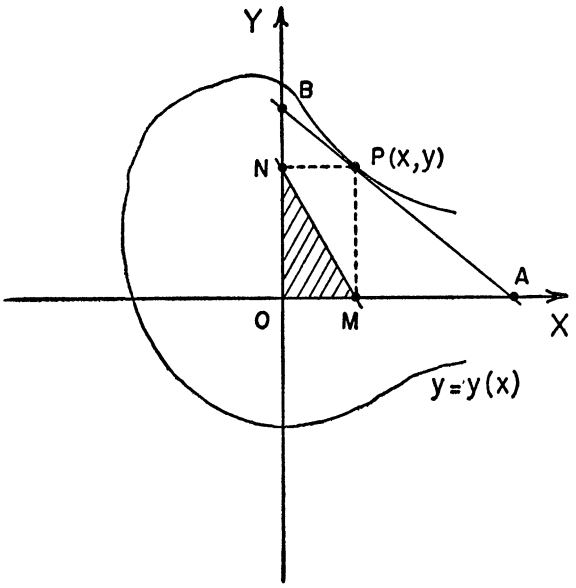
1. V. Thébault, Mathesis, 1932, p. 453.
2. T. Lalescu, Gazeta Matematica (Bucarest), 1915, p. 213.
3. Rouché and de Comberouse, Traité de Géométrie, t. II, p. 613.
4. O. Ramler, this MONTHLY, 1940, pp. 140–145 (using complex coordinates).
5. V. Thébault, Mathesis, 1933 (Supplément) and this MONTHLY, 1946, p. 398, problem 4060.
6. J. Neuberg, Nouvelles Annales de Mathématiques, 1870, p. 65.
7. Laguerre, Nouvelles Annales de Mathématiques, 1879, p. 241.
8. *Cf.* R. A. Johnson, Modern Geometry, p. 277.
9. V. Thébault, Mathesis, 1933, *loc. cit.*

MAXIMA AND MINIMA OF AREAS

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In the accompanying figure we have given the function $y=y(x)$, a point P: (x, y) on the curve, the tangent line to the curve at P cutting the X and Y

axes at A and B respectively, and $OM=x$ and $ON=y$. Let $A_1(x)$ and $A_2(x)$ represent the areas of triangles OAB and OMN respectively. We now define: *a*) a "synonymous" point P at $x=x_1$ denotes a point such that $A_2(x_1)$ and $A_1(x_1)$



are either both at their maximum* values or both at their minimum values; *b*) an "antonymous" point P at $x=x_1$ denotes a point such that either $A_2(x_1)$ is maximum and $A_1(x_1)$ is a minimum or $A_2(x_1)$ is minimum and $A_1(x_1)$ is maximum. We shall prove the following theorems which hold for P anywhere in the plane except on the axes; the latter cases lead to difficulties and trivial cases requiring special studies.

THEOREM 1. *If the solution* $x=x_1$ *of* $A_2'(x)=0$ *is such that* $A_2''(x_1)\neq 0$ *and* $y'''(x_1)$ *exists, then*

(a) $y(x_1)y''(x_1)>0$ *implies that* P *at* $x=x_1$ *is a synonymous point*;

(*b*) $y(x_1)y''(x_1)<0$ *implies that* P *at* $x=x_1$ *is an antonymous point*;

(*c*) $y''(x_1)=0$ *implies that* $A_1(x_1)$ *is neither maximum nor minimum, provided* $y'''(x_1)\neq 0$.

THEOREM 2. *If the solution* $x=x_1$ *of* $A_2'(x_1)=0$ *is such that* $A_2''(x_1)=\cdots=A_2^{(2h-1)}(x_1)=0$, $A_2^{(2h)}(x_1)\neq 0$, $y^{(2h)}(x_1)$ *and* $y^{(2h+1)}(x_1)$ *exist* $(h>1)$, *then* P *at* $x=x_1$ *is a synonymous point.*

Proofs: By means of intercepts of AB, we find that

$$2A_1 = -(xy' - y)^2/y'. \tag{1}$$

* Maxima and minima values for $f(x)$ are here used in the sense that at $x=x_1$, $f'(x_1)=\cdots=f_2^{(2h-1)}(x_1)=0$ and $f^{(2h)}(x_1)\neq 0$.

Obviously,

$$2A_2 = \sum xy, \tag{2}$$

$$2A_2' = \sum xy' + y. \tag{3}$$

Eliminating y and y' from (1), (2), and (3), we find

$$A_1 = 4A_2 + x^2A_2'^2/(A_2 - xA_2'). \tag{4}$$

Differentiating, we find that $A_1'(x_1)=0$ when $A_2'(x_1)=0$ provided $A_2''(x_1)$ exists, which is easily justified by the existence of $y''(x_1)$ and $A_2''=xy''+2y'$. Differentiating again, we find

$$A_1''(x) = 4A_2'(x) + \sum \frac{2}{e_1!e_2!} D^{e_1}(x^2A_2'^2)D^{e_2}(A_2 - xA_2')^{-1}, \quad (e_1 + e_2 = 2), \tag{5}$$

or

$$A_1'(x_1) = 4A_2'(x_1) + 2x_1^2\{A_2'(x_1)\}^2/A_2(x_1), \tag{6}$$

in which $y''(x_1)$ and $y'''(x_1)$ exist by hypothesis

Eliminating x_1 and $A_2(x_1)$ by means of (3) and (2) and replacing one $A_2''(x_1)$ on the right by $\frac{1}{2}(x_1y''(x_1)+2y'(x_1))$, in (6), we obtain

$$A_1'' = 2A_2'yy''/y'^2 \text{ at } x=x_1 \tag{6'}$$

or

$$A_1''A_2'' = 2(A_2''/y')^2yy'' \quad \text{at} \quad x = x_1. \tag{7}$$

Therefore

$$\text{sign } (A_1''A_2'') = \text{sign } (yy'') \quad \text{at} \quad x = x_1.$$

This proves parts (a) and (b) of Theorem 1. In case $y''(x_1)=0$, we differentiate (1) directly and find that

$$2A_1''' = 4y'''(y - xy')/y' \quad \text{at} \quad x = x_1. \tag{8}$$

Now if $y(x_1)-x_1y'(x_1)=0$, we have, with $A_2'(x_1)=0$, the contradiction that $y(x_1)=0$. Hence $A_1'''(x_1)\neq 0$ unless $y'''(x_1)=0$, which proves part (c) of Theorem 1.

We next turn to the case $A_2''(x_1)=\cdots=A_2^{(2h-1)}(x_1)=0$ and $A_2^{(2h)}(x_1)\neq 0$. This also gives us a critical value for $A_2(x_1)$. We employ

$$D^nA_1(x_1) = 4D^nA_2(x_1) + \sum_{i=0}^{n}\binom{n}{i} D^i(x^2A_2'^2)D^{n-i}(A_2-xA_2')^{-1}, (n=3,\cdots,2h). \tag{9}$$

By induction, we find that $D^{n-i}(A_2-xA_2')^{-1}$ is a polynomial expression divided by a power of (A_2-xA_2'). Also, $D^i(x^2A_2'^2)$ is zero at $x=x_1$. Hence

$$A_1^{(n)}(x_1) = 4A_2^{(n)}(x_1), \tag{10}$$

provided $A_2^{(2h)}(x_1)$ and $A_2^{(2h+1)}(x_1)$ exist. But repeated differentiations of (3) equated to zero and the use of the existence of $y^{(2h)}(x_1)$ and $y^{(2h+1)}(x_1)$ justify this provision. Hence (10) implies Theorem 2.

It is interesting to note that, at these critical values, the generating lines AB and MN are parallel.

A SPECIAL TETRAHEDRON

N. A. COURT, University of Oklahoma

The following sections represent a continuation of the paper under the same title that appeared in this MONTHLY, volume 56, May, 1949, page 312.

7. The place of the Monge point on the edge of a tetrahedron $(\sigma\sigma)$. The Monge point of any tetrahedron $(T)=DABC$ is the center of the equilateral hyperboloid (H) determined by the altitudes of (T), and (T) is inscribed in (H).* Now in the case of a doubly special tetrahedron $(\sigma\sigma)$ (§4) the Monge point M lies on an edge, say BC, of (T); the points B, C of (H) are therefore symmetrical with respect to the center M of (H); that is, M coincides with the mid-point A' of the edge BC. Thus: *If the Monge point of a tetrahedron* (T) *lies on an edge of* (T), *then the Monge point coincides with the mid-point of that edge.*

8. The circumcenter of a doubly special tetrahedron $(\sigma\sigma)$. (*a*) The centroid G of any tetrahedron bisects the segment MO determined by the Monge point M and the circumcenter O; and G also bisects the segment determined by the mid-points of any pair of opposite edges. In the case of a tetrahedron $(\sigma\sigma)$ the point M coincides with the mid-point A' of the edge BC (§7); therefore the point O coincides with the mid-point A'' of the opposite edge DA. Thus: *If an edge of a tetrahedron* (T) *passes through the Monge point of* (T), *the opposite edge passes through the circumcenter of* (T).

(*b*) *Conversely, if an edge of a tetrahedron* (T) *passes through the circumcenter of* (T), *then the mid-point of the opposite edges is the Monge point of* (T).

The proof is the same as of the direct proposition, the rôles of the points M, O being interchanged.

(*c*) Thus to construct a tetrahedron $(\sigma\sigma)$ it is necessary and sufficient to take for two of the vertices D, A two diametrically opposite points on a sphere (O), and for the other two vertices B, C any two other points of (O), and the line BC does not have to be perpendicular to DA, as stated in §5. It was erroneous to assume that the Simson lines of the points D_h, A_h for the triangles ABC, DBC, respectively, and the line BC, have a determinate point in common; these three lines coincide.

* H. Schroeter, Flaechen zweiter Ordnung, pp. 203 ff., 1880, George Salmon, Analytic geometry of three dimensions, vol. I, p. 140, fifth edition, 1912.

(*d*) The propositions (*a*) and (*b*) above may be considered as properties of an equilateral hyperboloid, and as such may be considered as generalizations of the properties of four concyclic points of an equilateral hyperbola.*

NOTE ON THE LAURENT SERIES

JOSEPH LEHNER, University of Pennsylvania

1. Introduction. Let $f(x)$ be a uniform analytic function of x, and let it be represented in the region $|x| < 1$ by the convergent power series,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{1}$$

Suppose $x = 1$ is an isolated singular point of f. There is then a Laurent series for $f(x)$:

$$f(x) = \sum_{l=0}^{\infty} \alpha_l (1 - x)^{-l} + \sum_{l=1}^{\infty} \beta_l (1 - x)^l. \tag{2}$$

The Taylor coefficients $\{a_n\}$ completely determine the function f, and therefore determine the Laurent coefficients $\{\alpha_l\}$, $\{\beta_l\}$. If the Taylor coefficients are given, how can one actually calculate the Laurent coefficients? Solution of this problem would enable one to compute the values of $f(x)$ outside of its original region of definition by utilizing the Laurent series.

This problem will be solved in the simplest case in which $x = 1$ is the *only* singularity of $f(x)$. Since f is thereby assumed to be regular at infinity, the $\{\beta_l\}$ necessarily vanish, and we have

$$f(x) = \sum_{0}^{\infty} \alpha_l (1 - x)^{-l}, \qquad x \neq 1. \tag{3}$$

2. Calculation. The obvious procedure is to expand $(1-x)^{-l}$ in powers of x, and rearrange (3) as a power series in x, which can then be compared coefficient-wise with (1), since both expansions represent $f(x)$ in $|x| < 1$. Thus,

$$\begin{aligned} \sum_{0}^{\infty} \alpha_l (1 - x)^{-l} &= \alpha_0 + \sum_{l=1}^{\infty} \alpha_l \sum_{m=0}^{\infty} \binom{l + m - 1}{m} x^m \\ &= \alpha_0 + \sum_{m=0}^{\infty} x^m \sum_{l=1}^{\infty} \binom{l + m - 1}{m} \alpha_l, \quad |x| < 1, \end{aligned}$$

on the assumption that the interchange of summation orders is legitimate. Then, by comparison with (1),

$$a_0 = \alpha_0 + \sum_{1}^{\infty} \alpha_l,$$

* John Casey, Analytic geometry, p. 291, Nos. 6 and 7, sec ed., 1893; Mathesis, 1928, p. 136.

(4)
$$a_n = \sum_{l=1}^{\infty} \binom{l+n-1}{n} \alpha_l, \qquad n \geqq 1.$$

In order to obtain $\{\alpha_l\}$, we should have to solve this system of linear equations in infinitely many variables.

However, this difficulty can easily be avoided. The function $f(x)$ is an entire function of $(1-x)^{-1}$ but may be considered an entire function of $x(1-x)^{-1} = (1-x)^{-1}-1$. Therefore, we have an expansion, valid for $x \neq 1$,

(5)
$$f(x) = e_0 + \sum_{l=1}^{\infty} e_l x^l (1-x)^{-l};$$

and for $|x| < 1$, we expand the denominators,

(6)
$$f(x) = e_0 + \sum_{l=1}^{\infty} e_l x^l \sum_{m=0}^{\infty} \binom{l+m-1}{m} x^m = e_0 + \sum_{n=1}^{\infty} x^n \sum_{l=1}^{n} e_l \binom{n-1}{l-1}.$$

Since the sum in (5) is a power series in $u = x(1-x)^{-1}$ which converges for all finite values of u, we have $\lim |e_l|^{1/l} = 0$, or $|e_l| < \epsilon^l$ for $l > l_0(\epsilon)$. With this estimate it is easy to justify the inversion of summation orders in (6).

Comparing the last series in (6) with (1), we find

(7)
$$a_0 = e_0,$$
$$a_n = \sum_{l=1}^{n} e_l \binom{n-1}{l-1}, \qquad n \geqq 1.$$

This is a linear system which obviously determines $\{e_n\}$ uniquely, since the equations form a triangular set.

Now we wish to find the relation between the $\{e_n\}$ and the $\{\alpha_n\}$. In the series in (5) write

$$x^l = (1 + x - 1)^l = \sum_{j=0}^{l} \binom{l}{j} (x-1)^j;$$

then

$$\begin{aligned} f(x) &= e_0 + \sum_{m=1}^{\infty} e_m \sum_{j=0}^{m} \binom{m}{j} (-1)^j (1-x)^{j-m} \\ &= e_0 + \sum_{m=1}^{\infty} e_m (1-x)^{-m} + \sum_{j=1}^{\infty} (-1)^j \sum_{m=j}^{\infty} \binom{m}{m-j} (1-x)^{j-m} e_m \\ &= e_0 + \sum_{l=1}^{\infty} e_e (1-x)^{-l} + \sum_{l=0}^{\infty} (1-x)^{-l} \sum_{m=l+1}^{\infty} (-1)^{m+l} \binom{m}{l} e_m \\ &= e_0 + \sum_{m=1}^{\infty} (-1)^m e_m + \sum_{l=1}^{\infty} (1-x)^{-l} \sum_{m=l}^{\infty} (-1)^{m+l} \binom{m}{l} e_m. \end{aligned}$$

The last series is in the form (3); hence

$$\alpha_0 = e_0 + \sum_{m=1}^{\infty} (-1)^m e_m,$$

$$\alpha_l = \sum_{m=l}^{\infty} (-1)^{m+l} \binom{m}{l} e_m, \qquad l \geqq 1. \tag{9}$$

The last step is to express the $\{e_n\}$ in terms of the $\{a_n\}$ by means of (7). This is easily done if we make use of the "orthogonality" property

$$\sum_{n=l}^{m} (-1)^{n+m} \binom{n-1}{l-1} \binom{m-1}{n-1} = \delta_{lm}. \tag{10}$$

In order to prove (10) we note that

$$\binom{n-1}{l-1}\binom{m-1}{n-1} = \binom{m-1}{l-1}\binom{m-l}{m-n},$$

so that the sum in (10) equals

$$\binom{m-1}{l-1} \sum_{n=l}^{m} (-1)^{n+m} \binom{m-l}{n-l}.$$

In the above sum we set $p = n-l \geqq 0$; p then runs from 0 to $m-l$. Hence,

$$\sum_{n=l}^{m} (-1)^{n+m} \binom{n-1}{l-1}\binom{m-1}{n-1} = (-1)^{l+m} \binom{m-1}{l-1} \sum_{p=0}^{m-l} (-1)^p \binom{m-l}{p}.$$

The sum in the right member of the last equation vanishes unless $m-l=0$, in which case the right member becomes equal to 1. This proves (10).

Now multiply both members of the second equation of (7) by $(-1)^{n+m}\binom{m-1}{n-1}$ and sum on n, $n=1, 2, \cdots, m$. Interchanging the order of summation and using (10) we get

$$e_m = \sum_{n=1}^{m} (-1)^{n+m} \binom{m-1}{n-1} a_n, \qquad m \geqq 1, \tag{11}$$

and we already had $e_0 = a_0$ from the first equation of (7). Using these results in (9) we obtain the final formula:

$$\alpha_0 = a_0 + \sum_{m=1}^{\infty} \sum_{n=1}^{m} (-1)^n \binom{m-1}{n-1} a_n,$$

$$\alpha_l = \sum_{m=l}^{\infty} \binom{m}{l} \sum_{j=1}^{m} (-1)^{l+j} \binom{m-1}{j-1} a_j, \qquad l \geqq 1. \tag{12}$$

The author wishes to thank the referee, who showed that α_0 could be expressed in the above form.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College and Massachusetts Institute of Technology

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PARTIAL FRACTIONS WITH REPEATED LINEAR OR QUADRATIC FACTORS

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The great tedium involved in separating into partial fractions a rational algebraic function in which the denominator has repeated linear or quadratic factors has long been recognized. It is the purpose of this article to indicate a method which the author has found convenient in working with Laplace transforms where such separation into partial fractions is important from the practical viewpoint.

The best way to present the ideas involved in this labor saving device is by means of an example. We pose the problem:

Separate

$$\frac{3x+1}{(x-2)(x^2+x+1)^3}$$

into partial fractions.

Normally one would write

$$\frac{3x+1}{(x-2)(x^2+x+1)^3} \equiv \frac{A}{x-2} + \frac{Bx+C}{(x^2+x+1)^3} + \frac{Dx+E}{(x^2+x+1)^2} + \frac{Fx+G}{x^2+x+1}.$$

Since A can be calculated immediately (by the usual method of multiplying through in this identity by $(x-2)$ and then letting $x \to 2$) we have 6 coefficients to compute, *i.e.*, we have to solve 6 equations in 6 unknowns. Anyone who has used this method must realize how much labor is involved.

Suppose, however, we consider

$$\frac{3x+1}{(x-2)(x^2+x+a)} \equiv \frac{A}{x-2} + \frac{Bx+C}{x^2+x+a} \tag{1}$$

where a is a parameter and A, B, and C are of course functions of a.

Then

$$A = \frac{7}{a+6}.$$

Multiplying (1) through by x and letting $x\to\infty$ we observe that

$$0 = A + B \quad \text{or} \quad B = \frac{-7}{a+6}.$$

Finally letting $x=0$ in (1) we find

$$-\frac{1}{2a} = -\frac{1}{2}\left(\frac{7}{a+6}\right) + \frac{C}{a}$$

or

$$C = -\frac{1}{2} + \frac{1}{2}\left(\frac{7a}{a+6}\right) = -\frac{1}{2} + \frac{1}{2}\left(\frac{7a+42-42}{a+6}\right)$$
$$= -\frac{1}{2} + \frac{7}{2} - \frac{21}{a+6} = 3 - \frac{21}{a+6}.$$

We now practically have the problem solved.
Differentiating (1) twice with respect to a we get

$$\frac{2(3x+1)}{(x-2)(x^2+x+a)^3} \equiv \frac{A''}{x-2} + \frac{B''x+C''}{x^2+x+a} - \frac{2(B'x+C')}{(x^2+x+a)^2} + \frac{2(Bx+C)}{(x^2+x+a)^3} \tag{2}$$

where the primes denote differentiations with respect to a. In view of the fact that

$$A' = \frac{-7}{(a+6)^2} \qquad B' = \frac{7}{(a+6)^2} \qquad C' = \frac{21}{(a+6)^2},$$
$$A'' = \frac{14}{(a+6)^3} \qquad B'' = \frac{-14}{(a+6)^3} \qquad C'' = \frac{-42}{(a+6)^3}.$$

Putting $a=1$ and using the values thus obtained in (2) we have the result desired.

Further extensions of this and applications to linear repeated factors are obvious.

MORE STRESS ON GENERAL FORMULATIONS IN CALCULUS PROBLEMS

LUISE LANGE, Woodrow Wilson Junior College, Chicago

In many calculus texts problems are formulated too one-sidedly in terms of particular, numerical data rather than in general terms. While pedagogically it may be wise to begin a new type of problem with some numerical examples, it is only the general formulation, and the interpretation of the answer in general

terms, which can give insight into the functional relation between the given and the derived data.

We illustrate our point by some examples from the traditional "minimum fence" and "related time rate" problems. They could readily be multiplied.

1. *Numerical Formulation.* A rectangular lot of given area is to be fenced in on four sides. For what dimensions of the lot will the length of the fence be a minimum? Answer: a square. Same question, but the lot is to be fenced in on only three sides. Answer: (with one side x and two sides y) $x:y=2:1$. Same question, but the lot is to be divided internally by three fences. Answer: (with inside fences parallel to x) $x:y=2:5$. Same question, but the material of one fence costs twice as much as that of the three others. Answer: (with higher-cost material in x direction) $x:y=3:2$.

General formulation. The lot is to be fenced in by n fences (inside or outside) in one direction, and m in the other. The minimum fence is that for which $x:y=m:n$, or $nx=my$; which means, if just as much fence runs in the one direction as in the other. If instead of length it is cost which is to be minimized the condition is the same: the cost is minimum if all fences in one direction cost as much as those in the other.—In fact, all these several questions are one: the minimum of the function $F=c_1x+(c_2A/x)$, which lies at $c_1-c_2(A/x^2)=0$ or $c_1x=c_2(A/x)$. The graphical presentation of this function throws additional light on the question. F is the sum of the ordinates of the straight line $y=c_1x$, which represent the length or cost of the fences in one direction, and of the hyperbola $y=c_2A/x$ representing the length or cost of the fences in the other direction. The minimum of the function F lies at the intersection of the straight line with the hyperbola.

Posed in general formulation the problem of the minimum surface of a closed or open cylindrical can of given volume receives a similarly comprehensive answer. If there are n bases (e.g. inside partitions as well as bottom and top) the minimum condition is found to be $h=nr$, which, substituted into the formula for the lateral surface is seen to mean that the minimum surface is that for which the lateral surface is twice as great as that of all bases.

2. *Numerical formulation.* A street light hangs 18 ft. above the the street. A man 6 ft. tall walks along the street at a rate 3 ft/sec. At what rate is his shadow lengthening? Answer 1.5 ft/sec.

General formulation. The light hangs H ft. high. An object h ft high moves along the street at rate v ft/sec. At what rate is its shadow lengthening? Answer:

$$ds/dt = \frac{h}{H-h}\,v = \frac{1}{H/h-1}\,v.$$

Interpretation: (a) The rate of the shadow's lengthening is proportional to the rate at which the object moves. (b) The factor of proportionality depends only on the ratio H/h. If h gets small in comparison to H the factor approaches zero (e.g. a mouse, or even more so an ant, running along the street push their practically constant shadows ahead of them). As H/h approaches 1 the shadow

lengthens ever more rapidly (e.g. a truck $17\frac{1}{2}$ ft. high under a light 18 ft. high has its shadow lengthening 35 times as fast it drives). (c) The relative rate of the shadow's lengthening, on the other hand,

$$\frac{\dfrac{ds}{dt}}{s} = \frac{v}{x},$$

does not depend on H and h, but is, for a given v, inversely proportional to the distance x from the light. (A mouse and an elephant walking side by side at the rate 2.5 ft/sec. have their respective shadows lengthening at the same rate 10 percent per second when 25 ft. away from the light, and only 1 percent per second when 250 ft. away from the light.)

We still wish to add that positively harmful, in our opinion, are the numerical specifications not only for arbitrary constants but for variables of which the required datum is a function, merely for the purpose of "coming out" with a numerical answer. Example: from a bridge 30 ft. high a boat is towed in by a rope which is pulled in at a rate 4 ft/sec. At what rate is the boat moving *when 40 ft. away from the bridge*? The last specification is pedagogically very bad, because it confuses in the student's mind the basic question as to what is constant and what variable in the problem.

EQUIVALENT PROPERTIES OF A RING

R. A. BEAUMONT, University of Washington

The well-known theorem* which states that if R is a commutative ring which contains no proper divisors of zero, then every polynomial with coefficients in R has at most n zeros in R, is usually accompanied by examples to show that neither hypothesis can be omitted. The fact that the converse holds, although the proof is almost trivial, is not given. It may be of some interest to record the following equivalent properties of a ring.

THEOREM. *In any ring R, the following properties are equivalent.*

(1) *R is commutative and contains no proper divisors of zero.*

(2) *Every polynomial of degree n with coefficients in R has at most n zeros in R.*

(3) *Every linear polynomial with coefficients in R has at most one zero in R.*

(4) *If a, b, c are in R, $b \neq 0$, then $ba = cb$ implies $a = c$.*

Proof. (1) implies (2) is the theorem mentioned above and (2) implies (3). Let a, b, c be elements of R such that $b \neq 0$ and $ba = cb$. Then the linear equation $bx - ba = 0$ has a as a right zero and c as a left zero and $a = c$ by (3). Thus (3) implies (4). If (4) is satisfied, assume that $ba = 0$ where $b \neq 0$. Then $ba = 0 = 0b$ implies $a = 0$ so that R has no proper divisors of zero. Further $b(ab) = (ba)b$ implies $ab = ba$ for all a in R and $b \neq 0$ in R. Since the latter equation is satisfied for $b = 0$, R is commutative. Hence (4) implies (1).

* See for example, B. L. van der Waerden, Modern Algebra, Vol. 1, Ungar (1949), page 65.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 906. *Proposed by Vern Hoggatt, Oregon State College*

Let S be a bounded infinite point set. Show that there exists a polygonal path of arbitrarily small length passing through an infinite number of points of S.

E 907. *Proposed by C. D. Olds, San Jose State College*

A derangement of n ordered letters is a permutation of the n letters which leaves no letter fixed. Find formulas which will give the number of odd and the number of even derangements of n ordered letters.

E 908. *Proposed by R. J. Koch, Tulane University*

Show that $r=1/2$ is the only rational value of r for which the geometric series Σar^n, $a\neq 0$, has the property that there exists a term equal to the sum of all the terms from some point on, and that when $r=1/2$ this property holds for every term.

E 909. *Proposed by W. E. Buker, Perry High School, Pittsburgh, Pa.*

Find positive integers satisfying

$$N = p^2 + k_1^2 = (p+1)^2 + k_2^2 = (p+2)^2 + k_3^2.$$

E 910. *Proposed by R. M. Gordon, Carnegie Institute of Technology*

If $k>0$, $f_i(x)$ are independent, $f_i^{(n)}(x)$ exist and are well behaved, find

$$\lim_{h\to 0} h^{-k} \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1(x+h) & f_2(x+h) & \cdots & f_n(x+h) \\ f_1(x+2h) & f_2(x+2h) & \cdots & f_n(x+2h) \\ \cdot & \cdot & \cdots & \cdot \\ f_1[x+(n-1)h] & f_2[x+(n-1)h] & \cdots & f_n[x+(n-1)h] \end{vmatrix}$$

SOLUTIONS

A Three Man Dice Game

E 850 [1949, 31]. *Proposed by L. C. Hsu, National Tsing-Hua University, Peiping, China*

Three persons, A, B, and C, in rotation throw a pair of dice. If the points 6, 7, 8 are consecutively gotten by A, B, C, then A is declared winner; if the points 7, 8, 6 are consecutively gotten by B, C, A, then B is declared winner; if the points 8, 6, 7 are consecutively gotten by C, A, B, then C is declared winner. Find their chances of winning.

Solution by the Proposer. Let us call the three consecutive throws numbered $3n-2$, $3n-1$, $3n$ the nth cycle with respect to A, those numbered $3n-1$, $3n$, $3n+1$ the nth cycle with respect to B, and those numbered $3n$, $3n+1$, $3n+2$ the nth cycle with respect to C. Define $P_A(n)$, $P_B(n)$, $P_C(n)$ as the probabilities that A, B, C win exactly in their respective nth cycles. Denote by P_A, P_B, P_C the total probabilities of A, B, C winning. It is clear that for $m \neq n$, $P_A(m)$ and $P_A(n)$ are disjoint. Moreover, the events whose probabilities are given by $P_A(n)$, $P_B(m)$, $P_C(k)$ are also incompatible. Therefore

$$P_A = \sum_{n=1}^{\infty} P_A(n), \qquad P_B = \sum_{n=1}^{\infty} P_B(\mathrm{n}), \qquad P_C = \sum_{n=1}^{\infty} P_C(n),$$

where

$$\begin{aligned} P_A(1) &= (5/36)(6/36)(5/36), \\ P_B(1) &= (31/36)(6/36)(5/36)^2, \\ P_C(1) &= (30/36)(5/36)^2(6/36). \end{aligned}$$

Now it is easy to obtain the following equations, for $n \geqq 2$:

$$(1)\ P_A(n) = \left\{1 - \sum_{k=1}^{n-2} P_A(k) - \sum_{k=1}^{n-2} P_B(k) - \sum_{k=1}^{n-2} P_C(k)\right\} (31/36)(5/36)(6/36)(5/36),$$

$$(2)\ P_B(n) = \left\{1 - \sum_{k=1}^{n-1} P_A(k) - \sum_{k=1}^{n-2} P_B(k) - \sum_{k=1}^{n-2} P_C(k)\right\} (31/36)(6/36)(5/36)(5/36),$$

$$(3)\ P_C(n) = \left\{1 - \sum_{k=1}^{n-1} P_A(k) - \sum_{k=1}^{n-1} P_B(k) - \sum_{k=1}^{n-2} P_C(k)\right\} (30/36)(5/36)(5/36)(6/36).$$

From (1) and (2) we then have

$$\sum_{n=2}^{\infty} P_A(n) - \sum_{n=2}^{\infty} P_B(n) = (4650/36^4) \sum_{n=2}^{\infty} P_A(n-1),$$

and from (2) and (3)

$$30 \sum_{n=2}^{\infty} P_B(n) - 31 \sum_{n=2}^{\infty} P_C(n) = (139500/36^4) \sum_{n=2}^{\infty} P_C(n-1).$$

Using the fact that $P_A + P_B + P_C = 1$, we finally get, by solving approximately,

$$P_A \doteq 0.33818, \qquad P_B \doteq 0.33680, \qquad P_C \doteq 0.32502.$$

Other solutions were given by Louis Berkofsky, R. E. Edwards, H. E. H. Greenleaf, J. M. Kingston, Sam Kravitz, Roger Lessard, Bart Park, and L. A. Ringenberg.

Infinite Radicals

E 874 [1949, 404]. *Proposed by C. S. Ogilvy, Trinity College*

If $y_1=x^{1/2}$, $y_2=(x+y_1)^{1/2}$, $\cdots$, $y_m=(x+y_{m-1})^{1/2}$, prove that $\lim_{m\to\infty}y_m$ is integral if and only if x is of the form $n(n-1)$, $n=2, 3, \cdots$, in which case the limit is n.

Solution by C. W. Trigg, Los Angeles City College. The given sequence is monotone increasing and convergent for $x>0$. (See, for example, Aaron Herschfeld, *On Infinite Radicals*, this MONTHLY, pp. 419–29, 1935.) In general, the sequence $y_1=x^{1/p}$, $y_2=(x+y_1)^{1/p}$, $\cdots$, $y_m=(x+y_{m-1})^{1/p}$ is convergent for $p\geqq 2$, since it is convergent for $p=2$. Denote $\lim_{m\to\infty}y_m$ by η, whereupon we have $\eta^p-\eta-x=0$, which has exactly one positive root for $x>0$. For integer values of the limit η, $x=\eta^p-\eta$. For $p=2$, $x=\eta(\eta-1)$; and for $p=3$, $x=(\eta-1)\eta(\eta+1)$.

Also solved by W. D. Berg, D. H. Browne, M. S. Klamkin, Martin Milgram, L. A. Ringenberg, Azriel Rosenfeld, and the proposer.

See E 474 [1942, 197].

Property of an Arithmetic Progression

E 876 [1949, 473]. *Proposed by H. L. Lee, University of Tennessee*

The product of four consecutive terms of an arithmetic progression of integers plus the fourth power of the common difference is a perfect square but in no case a perfect fourth power.

Solution by R. V. Andree, University of Oklahoma. Let a be the first term of the four, and d the common difference. Then

$$a(a+d)(a+2d)(a+3d)+d^4=(a^2+3ad+d^2)^2,$$

a perfect square. However, that it is never a perfect fourth power is false. A counter example is obtained with $a=3$, $d=7$. Other counter examples may be constructed by taking

$$a=u^2-v^2, \qquad d=2uv+3v^2,$$

in which case

$$a(a+d)(a+2d)(a+3d)+d^4=(u^2+3uv+v^2)^4.$$

Also solved by Max Beberman, Daniel Block, Leslie Booth, D. H. Browne, Aaron Buchman, W. E. Buker, J. H. Butchart, E. H. Clarke, E. J. Cogan, W. J. Feeney, W. Fulks, Vern Hoggatt, R. T. Hood, P. W. M. John, Harris Jones, M. S. Klamkin, Frank Kocher, Sam Kravitz, Sidney Kravitz, F. T. Leahy, Jr., Emma Lehmer, D. W. Matlack, F. L. Miksa, E. P. Miles, Jr., T. F.

Mulcrone, C. S. Ogilvy, S. T. Parker, J. W. Ponds, P. W. A. Raine, L. B. Rall, K. S. Rao, R. J. Reid, L. A. Ringenberg, Azriel Rosenfeld, C. M. Sandwick, R. W. Shoemaker, R. V. Simpson, Guy Stevenson, B. M. Stewart, C. W. Trigg, Robert Ullman, G. W. Walker, Lila Walker, Hazel Wilson, and the proposer.

Eleven of these solutions failed to note that the second part of the problem as quoted is incorrect. The failure was due to assuming that a form which is not *formally* a square is never a square in integers. The solution by G. W. Walker constituted a remarkably complete discussion of the problem.

A. Desboves, *Nouv. Ann. Math.* (2), 18, 1879, 269 and (3) 5, 1886, 226–33, showed that $x^2+dxy+by^2=z^2$ has for complete solution

$$x = \pm (q^2 - bp^2), \qquad y = \pm (dp^2 + 2pq), \qquad z = \pm (q^2 + bp^2 + dpq).$$

This yields, essentially, the general solution given above by Andree. See Dickson, *History of the Theory of Numbers*, v. III, pp. 404–407.

Property of Concurrent Cevians

E 879 [1949, 474]. *Proposed by Joseph Langr, Prague, Czechoslovakia*

Let S_1, S_2, S_3 be the midpoints of three concurrent cevians of triangle ABC. Let S_2S_3, S_3S_1, S_1S_2, meet the sides BC, CA, AB in A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3 respectively. Show that (1) A_2, A_3; B_3, B_1; C_1, C_2 are isotomic points on the segments BC, CA, AB, (2) A_1, B_2, C_3 are collinear, (3) A_2, A_3, B_3, B_1, C_1, C_2 lie on a conic.

Solution by J. H. Butchart, Arizona State College at Flagstaff. Triangles ABC and $S_1S_2S_3$ are perspective. Hence A_1, B_2, C_3 are collinear. The triangle formed by joining the midpoints A', B', C' of the sides of triangle ABC is circumscribed about triangle $S_1S_2S_3$ and is inscribed and perspective with triangle ABC. Hence triangles $A'B'C'$, $S_1S_2S_3$ are perspective. Then the pencil $S_1(S_2S_3, A'B')$ is harmonic and A' is the midpoint of A_2A_3. The conics through the fixed points B_1, B_3, C_2, C_1 establish an involution on the line BC, which, for the degenerate cases, pairs B with C and A_1 with its isotomic conjugate. Hence A_2, A_3 are determined by the same conic.

Also solved by L. M. Kelly, Roscoe Woods, and the proposer.

Editorial Note. The perspectivity of triangles $A'B'C'$ and $S_1S_2S_3$ can be established by Ceva's theorem, for

$$(B'S_1 \cdot C'S_2 \cdot A'S_3)/(S_1C' \cdot S_2A' \cdot S_3B') = (BL \cdot CM \cdot AN)/(LC \cdot MA \cdot NB) = 1,$$

where L, M, N are the feet, on the sides BC, CA, AB, of the given concurrent cevians. Also, part (3) follows from part (1) by Carnot's theorem.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4385. *Proposed by P. Ungar, University College, London, England*

The three-digit sections of the sequence 1110001011 represent all three-digit numbers in the binary system exactly once each. For a given positive integer n an analogous sequence is obtained in the following manner: write down n 1's to begin with, and in each subsequent place write 0 unless the n-digit section thus completed occurs previously, in which case put 1. Show that the resulting sequence of 2^n+n-1 digits has the same property as the case $n=3$ cited at the outset.

4386. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In any triangle the Monge circle (orthoptic locus) of the inscribed conic concentric with the circumcircle is orthogonal to the polar circle.

4387. *Proposed by Paul Bateman, Institute for Advanced Study*

If $\sigma_r(n)$ denotes the sum of the rth powers of the divisors of the positive integer n, prove that

$$\sigma_r(n)\sigma_r(m) = \sum_{d\,|\,(n,m)} d^r\sigma_r(nm/d^2),$$

where d runs through all the common divisors of n and m.

4388. *Proposed by Paul Erdös, Syracuse University and W. H. J. Fuchs, Cornell University*

Let

$$f(z) = \prod_{i=1}^{n} (z - z_i), \qquad |z_i| \leqq 1.$$

Consider the set $|f(z)| \leqq 1$. Prove that it consists of at most $n-1$ components.

4389. *Proposed by F. J. Dyson, Institute for Advanced Study*

Given N numbers a_m satisfying the N equations

$$\sum_{m=1}^{N} \frac{a_m}{m+n} = \frac{4}{2n+1}, \qquad n = 1, 2, \cdots, N.$$

prove that

$$\sum_{m=1}^{N} \frac{a_m}{2m+1} = 1 - \frac{1}{(2N+1)^2}.$$

This problem arose in an investigation of the diffraction of light (H. Levine and J. Schwinger, *Physical Review*, 74, 970 (1948)).

SOLUTIONS

A Property of the Orthocentric Tetrahedron

4266 [1947, 479]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a tetrahedron $ABCD$ and a sphere (S). If the polar planes of the vertices A, B, C, D with respect to (S) and the corresponding planes tangent to the circumsphere at A, B, C, D cut each other, respectively, on the faces BCD, CDA, DAB, ABC, the tetrahedron is orthocentric. Establish a converse theorem.

*Solution by the Proposer.** In the coaxal pencil of spheres determined by the circumsphere (O) of the tetrahedron $ABCD$ and the given sphere (S), let (Σ) be the sphere which separates A and B harmonically. The polar plane of the vertex A with respect to (Σ) contains the line Δ_a of the plane BCD in which the polar planes of A with respect to (O) and (S) intersect (since the polar planes of a point with respect to the spheres of a coaxal pencil form a coaxal pencil). Likewise the polar plane of A with respect to (Σ) contains the point B, and so it coincides with BCD. Hence (Σ) separates A and C, and A and D, harmonically. The same reasoning holds for the other vertices B, C, D. Therefore (Σ) is the conjugate sphere of the tetrahedron $ABCD$ and the given tetrahedron is orthocentric.

Conversely, in every orthocentric tetrahedron the property of (S) holds for every sphere of the coaxal pencil determined by the circumsphere and the conjugate sphere.

Difference of Consecutive Cubes

4299 [1948, 321]. *Proposed by R. C. Lyness, Crakemarsh, Uttoxeter, Staffordshire, England*

If the difference between two consecutive cubes is a square, then it is the square of the sum of two successive squares.

Solution by Ernest Trost, Technikum Winterthur, Switzerland. We have to solve the Diophantine equation

$$(x+1)^3 - x^3 = 3x^2 + 3x + 1 = y^2. \tag{1}$$

Multiplication by 4 gives

$$3(2x+1)^2 = (2y-1)(2y+1).$$

* Translation by W. E. Byrne, Virginia Military Institute.

Since $2y-1$ and $2y+1$ are relatively prime, we obtain the following two cases, where m and n are relatively prime odd integers:

$$\text{(a)}\quad \begin{aligned} 2y-1 &= 3m^2 \\ 2y+1 &= n^2, \end{aligned} \qquad \text{(b)}\quad \begin{aligned} 2y-1 &= m^2 \\ 2y+1 &= 3n^2. \end{aligned}$$

The first case is impossible because we have $n^2-3m^2=2$, which contradicts the fact that 2 is a quadratic non-residue of 3. Putting $m=2k+1$ in the second case we get

$$2y = 4k^2+4k+2 = 2\{(k+1)^2+k^2\},$$

which proves the theorem.

Also solved by Murray Barbour, W. H. Benson, D. A. Darling, Fritz Herzog, Alexander Kevitz, Roger Lessard, Leo Moser, C. S. Ogilvy, and the Proposer.

Editorial Note. Several solvers obtained explicit forms for the letters involved. These may be indicated briefly as follows.

Let $r(=2+\sqrt{3})$ and $1/r$ be the roots of $R^2-4R+1=0$, and let integers a_n and b_n be given by $2a_n=r^n+r^{-n}$, $2\sqrt{3}b_n=r^n-r^{-n}$. Then $a_n^2-3b_n^2=1$, and it is well known that a_n, b_n, $(n=0, 1, 2, \cdots)$ give the only integral solutions of $a^2-3b^2=1$. Thus the solutions of (1) are given by

$$x_n = (b_{2n+1}-1)/2, \qquad y_n = a_{2n+1}/2,$$

where alternate a's and b's are taken in order to obtain integral values of x_n, y_n. Direct calculation will verify:

$$y_n = z_n^2+(z_n+1)^2, \qquad z_n = (3b_n+a_n-1)/2,$$

where z_n is always integral.

With initial values $x_0=0$, $y_0=1$, $z_0=0$, $x_1=7$, $y_1=13$, $z_1=2$, it is easy to show that

$$\begin{aligned} x_{n+1} &= 14x_n - x_{n-1} + 6, \\ y_{n+1} &= 14y_n - y_{n-1}, \\ z_{n+1} &= 4z_n - z_{n-1} + 1. \end{aligned}$$

Many of these results have previously been noted, without proof, by V. Thébault [1949, 174].

Factors of $x^{2^n}+y^{2^n}$

4302 [1948, 369]. *Proposed by Joseph Rosenbaum, The Milford School, Connecticut*

Prove that if x and y have no common factor then every odd factor of $x^{2^n}+y^{2^n}$, where n is a positive integer, is of the form $2^{n+1}m+1$.

Solution by J. B. Kelly, University of Wisconsin. Let p be an odd prime such that

$$x^{2^n} + y^{2^n} \equiv 0 \pmod{p}.$$

Since $(x, y) = 1$, we have $x \not\equiv 0$, $y \not\equiv 0 \pmod{p}$. Then

$$(xy^{-1})^{2^n} \equiv -1 \pmod{p},$$

where y^{-1} is the inverse of y in the multiplication group of residues modulo p. Therefore the congruence

$$u^{2^n} \equiv -1 \pmod{p} \tag{1}$$

has a solution. According to Euler's criterion,

$$(-1)^{(p-1)/d} \equiv 1 \pmod{p}, \tag{2}$$

where $d = (p-1, 2^n)$. It follows from (2) that $(p-1)/d$ is even. Then from the definition of d it follows that $p-1$ is divisible by 2^{n+1}, whence

$$p = 2^{n+1}m + 1. \tag{3}$$

Finally, since every product of numbers of the form (3) is also of this form, the proposed theorem is proved.

Also solved by Leo Moser, Robert Seamons, and the Proposer. The Proposer notes the corollary that there are infinitely many primes of the form $2^k m+1$, because every prime factor of

$$(n!)^{2^{k-1}} + 1$$

is of the form $2^k m+1$ and is greater than n.

Coefficients of Bernoulli Polynomials of the Second Kind

4303 [1948, 369]. *Proposed by G. T. Williams, Cambridge, Massachusetts*

If

$$\theta_n = \int_0^1 x(x-1)\cdots(x-n+1)dx,$$

and

$$\phi_n = \int_0^1 x(x+1)\cdots(x+n-1)dx,$$

show that $(-1)^n\theta_n = \phi_n - n\phi_{n-1}$ and find ϕ_n in terms of θ_n.

Solution by Ragnar Dybvik, Levanger, Norway. We have

$$\phi_n - n\phi_{n-1} = \int_0^1 x(x+1)\cdots(x+n-1)dx - n\int_0^1 x(x+1)\cdots(x+n-2)dx$$

$$= \int_0^1 (x-1)x(x+1)\cdots(x+n-2)dx.$$

Setting $x = 1-t$, we get

$$\phi_n - n\phi_{n-1} = -\int_1^0 (-t)(-t+1)\cdots(-t+n-1)dt$$

(1)

$$= (-1)^n \int_0^1 t(t-1)\cdots(t-n+1)dt = (-1)^n\theta_n$$

as required. Further

$$\theta_1 = \phi_1 = \int_0^1 x dx = \tfrac{1}{2}$$

and from (1) we have

$$\frac{\phi_i}{i!} - \frac{\phi_{i-1}}{(i-1)!} = \frac{(-1)^i\theta_i}{i!}.$$

Summing for $i = 2, 3, \cdots, n$, we get

$$\frac{\phi_n}{n!} - \frac{\phi_1}{1!} = \sum_{i=2}^{n} (-1)^i \frac{\theta_i}{i!},$$

whence

$$\phi_n = n!/2 + n!\sum_{i=2}^{n} (-1)^i\theta_i/i!. \tag{2}$$

Also solved by Joshua Barlaz, J. H. Curtiss, M. S. Klamkin, Max LeLeiko, Roger Lessard, Jules Lieblein, Norman Miller, W. O. Pennell, Robert Seamons, Chih-Yi Wang, and the Proposer.

Editorial Note. LeLeiko points out that $\theta_n = b_n n!$, where b_n is the general coefficient of the Bernoulli polynomial of the second kind. See Jordan, *Calculus of Finite Differences*, 1947, p. 249. Using a result from p. 266 we have

$$\sum_{i=0}^{n-1} (-1)^i \frac{\theta_i}{(n-i)i!} = 0,$$

where $\theta_0 = \phi_0 = 1$, by definition. From this result and (1) we derive

$$\phi_n = n!\sum_{i=0}^{n-1} \frac{\phi_i}{(n-i)(n-i+1)i!}.$$

Wang obtains the following form for ϕ_n,

$$\phi_n = n!\sum_{i=0}^{n}\binom{n-1}{i-1}\theta_i/i!.$$

Partitions

4304 [1948, 369]. *Proposed by H. D. Grossman, New York City*

From the vertices of a regular n-gon three are chosen to be the vertices of a triangle. Prove that the number of essentially different possible scalene triangles is the integer nearest to $(n-3)^2/12$. Compare 3893 [1940, 664].

Solution by Norman Miller, Queen's University, Kingston, Ontario. All the possible types of triangles are included among the $N=(n-1)(n-2)/2$ triangles with a common vertex A. Each of the S types of scalene triangles is counted 6 times in the number N and each of the I types of isosceles, non-equilateral triangles is counted 3 times in N. An equilateral triangle occurs if and only if $n\equiv 0 \pmod 3$. The values of I are readily found on consideration of the possible lengths of the equal sides. The cases are now as follows:

1) $n\equiv \pm 1 \pmod 6$. $I=(n-1)/2$ and $6S+3I=N$.
2) $n\equiv 3 \pmod 6$. $I=(n-1)/2-1$ and $6S+3I+1=N$.
3) $n\equiv \pm 2 \pmod 6$. $I=(n-2)/2$ and $6S+3I=N$.
4) $n\equiv 0 \pmod 6$. $I=(n-2)/2-1$ and $6S+3I+1=N$.

The equations in these four cases yield respectively the following values of S: $(n-3)^2/12-\frac{1}{3}$, $(n-3)^2/12$, $(n-3)^2/12-1/12$, $(n-3)^2/12+\frac{1}{4}$. This proves the theorem.

Note by the Proposer. There is an intimate connection between the present problem 4304, Anning's problem 3893 [1940, 664], and problem 25 from Polya and Szegö, *Aufgaben und Lehrsätze, I*, pp. 4, 157. By reading lattice-point graphs of partitions both horizontally and vertically, we can establish the equality of the number of partitions of n into three $\left\{\begin{array}{l}\text{(a) positive, unequal}\\ \text{(b) positive}\\ \text{(c) non-negative}\end{array}\right\}$ parts and the number of $\left\{\begin{array}{l}\text{(a') positive solutions}\\ \text{(b') solutions with } x, y \text{ non-negative and } z \text{ positive}\\ \text{(c') non-negative solutions}\end{array}\right\}$ of the Diophantine equation $x+2y+3z=n$. A partition of type (a) can be reduced to one of type (b) by subtracting 2 from the largest part and 1 from the next largest, thereby reducing n by 3. The new partition can be again reduced to one of type (c) by subtracting unity from each part, again reducing n by 3. These operations are reversible.

The present problem is equivalent to one of partitions of type (a); 3893 is equivalent to one of partitions of type (b); Polya and Szegö's problem, to one of partitions of type (c). Hence Polya and Szegö's $(n+3)^2/12$ becomes in turn Anning's $n^2/12$ and the present $(n-3)^2/12$.

Also solved by Murray Barbour, John Kelly, Roger Lessard, Leo Moser, and W. E. Patten.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Fluid Dynamics. By V. L. Streeter, New York, McGraw-Hill 1948. 11+263 pages $5.00.

This book is a significant contribution to graduate or senior courses in fluid mechanics. Not requiring the reader to possess a knowledge of mathematics beyond the elementary differential and integral calculus, sufficient advanced mathematical theory is developed in an adequate manner before its use. Before developing details, the author outlines in general statements the theory to be discussed. Careful summaries are drawn which collect the basic assumptions used and the results obtained.

Three-dimensional flow examples are preceded by the foundations of ideal fluid theory. Complex variables are used in the application of conformal mapping to two-dimensional flow. Development of vortex theory is followed by examples. After deriving the Navier-Stokes equations, they are used in examples which include the laminar boundary layer problem.

There is a representative collection of problems. In general the first exercises are routine manipulation. The later ones are of a more difficult nature which when solved contribute significantly to the understanding of the theoretical discussions.

The book is written carefully with an accurate and concise presentation of its material.

E. A. TRABANT

Integraltafel, Erster Teil: Unbestimmte Integrale. By W. Gröbner and N. Hofreiter. Vienna, Springer-Verlag, 1949. 8+166 pages. $5.40.

The authors have taken great pains to make this collection of nearly 1700 indefinite integrals as accurate and complete as possible. The collection contains all the elementary integrals which are commonly found in mathematical handbooks as well as integrals of greater complexity, including elliptic and hyperelliptic integrals and integrals of the elliptic functions of Weierstrass and Jacobi. In many cases (for example, $\int \sin^n x\,dx$) the authors not only give the customary reduction formulas but also express the integrals in their ultimate series form. The value of the book is enhanced by short summaries of all the methods which are applicable in special cases of frequent occurrence. The handwritten text reproduced on $8\frac{1}{2}\times 12$ pages is particularly attractive and legible. This book should prove of great value to mathematicians, physicists, and engineers.

H. D. LARSEN

Computation Curves for Compressible Fluid Problems. By C. L. Dailey and F. C. Wood. New York, John Wiley and Sons, Inc., 1949. 10+32 pages+32 charts. Not bound. $2.00.

The title of this collection of 32 charts correctly characterizes the book and indicates its practical everyday usage by the engineer. In the introductory 32 pages the authors set out to "explain every relation presented in this text in such a way that anyone using the curves will be able to understand their meaning and limitations without having to refer to any other source." Since acquaintance with detailed thermodynamic relations for a perfect gas and with the circulation theorem is assumed, this commendable effort succeeds only in part.

The charts fall into three categories: I. So-called "Energy Relations" applicable to approximately one-dimensional steady-flow problems with and without heat addition; II. So-called "Plane-Shock Relations" applicable locally across any shock wave; III. "Conical-Shock Relations" summarizing the results of numerical solutions for the steady flow field around an infinite cone placed axially in a supersonic stream.

The basic independent variable in all the charts is the Mach number, local or free-stream. This fact provides a unified treatment but limits somewhat the applicability of the curves since often enough pressure, velocity direction, velocity ratio to the critical speed of sound, M^*, and velocity ratio to the maximum possible speed, $\bar{w}$, are more convenient as independent variables. It is to be hoped that the loose-leaf form of the book will allow additions of other helpful charts found, say, in the publications of the National Advisory Committee for Aeronautics.

As a matter of fact, the theory of Part III uses $\bar{w}$ as a variable, and the absence of an appropriate chart forces the reader repeatedly to raise quantities near unity to the power $\gamma/\gamma-1=7/2$. Also, Chart 4 in Part III is included so that $\bar{w}$ values on the surface of the cone could be expressed in terms of the Mach number on the cone surface. Actually, the relation represented by this chart is a completely general "point relation" valid anywhere and should be included with similar relations in Part I.

In view of the wide range of topics covered, the brief introductory derivations of the various relations understandably falter in places. The questionable derivation of the energy equation will not satisfy the careful student. The derivation of the Prandtl-Meyer relations is awkward and does not have the advantage of physical plausibility. All of the derivations assume steady flow while such a limitation is not necessary for most of the relations in Part II.

In summary, this publication of charts so useful to the engineer working with compressible fluids should be welcome by the profession.

M. V. MORKOVIN

NEW BOOKS RECEIVED

Figures of Fantasy. By A. N. Aheart, New York, Exposition, 1949. 54 pp. $2.00.

Konforme Abbildung. By A. Betz. Berlin, Springer, 1948. 8+539 pp. DM 36.

Introduction to Mathematics. 2nd Edition. By H. R. Cooley, D. Gans, M. Kline, and H. E. Wahlert. Boston, Houghton Mifflin, 1949. 22+170 pp. $4.25.

The Mathematics of Great Amateurs. By J. L. Coolidge. Oxford University Press, 1949. 8+211 pp. $6.00.

Fundamental Algebra with Practical Applications. By R. L. Erickson. New York, McGraw-Hill, 1949. 12+317 pp. $2.80.

Integralgleichungen, Einfughrung in Lehre und Gebrauch. By G. Hamel. Berlin, Springer-Verlag, 1949. 8+166 pp. DM 15.60.

Analytic Geometry. By L. M. Kells and H. C. Stotz. New York, Prentice-Hall, 1949. 8+280 pp. $2.85.

Einführung in die höhere Mathematik. By H. V. Mangoldt. Completely revised and enlarged by K. Knopp. 9th Edition. Stuttgart, Hirzel, 1948. 3 vols. Vol. 1, 16+585 pp. Vol. 2, 16+634 pp. Vol. 3, 16+618 pp. DM 24.80 each.

Practical Mathematics. By C. I. Palmer and S. F. Bibb. Part I, Arithmetic with Applications. 5th Edition. New York, McGraw-Hill, 1949. 12+179 pp. $1.80.

Darstellende Geometrie. (Teubners Mathematische Leitfäden, Volume 2). 2d Edition. By H. von Sanden. Leipzig, Teubner, 1949. 107 pp. $1.65.

Praktische Mathematik. (Teubners Mathematische Leitfäden, volume 44.). By H. von Sanden. Leipzig, Teubner, 1948. 99 pp. $1.00.

Equazioni Differenziali nel Campo Reale, part 2, 2nd Edition. By G. Sansone. Bologna, Zanichelli, 1949. 16+475 pp. 4000 lire.

Tables of the Bessel Functions of the First Kind of Orders, Fifty-two through Sixty-three. (The Annals of the Computation Laboratory of Harvard University, vol. 11.) Cambridge, Harvard University Press, 1949. 12+544 pp. $8.00.

Philosophy of Mathematics and Natural Science. By H. Weyl. Revised and augmented English edition based on a translation by Olaf Helmer. Princeton University Press, 1949. 10+311 pp. $5.00.

An Introduction to the Theory of Mechanics. By K. E. Bullen. Science Press, Sydney, Australia, 1949. xvi+368 pp.

The Axonometric Method of Descriptive Geometry. By W. H. Roever. Lithoprinted by Edwards Brothers, Inc., Ann Arbor, Michigan, 1949. For sale by Washington University Bookstore. 75 pp. $3.00.

Fundamentals of Symbolic Logic. By A. Ambrose and M. Lazerowitz. New York, Rinehart Publishers, 1948. 10+310 pp. $5.00.

Tables of Elementary Functions, 2d Edition. By F. Emde. Leipzig, Teubner, 1948. 12+181 pp. $3.50.

Tables of Higher Functions, 4th Revised Edition. By Jahnke and Emde. Leipzig, Teubner, 1948. 12+300 pp. $3.60.

Mathematics Dictionary, Revised Edition. By G. and R. C. James. New York, Van Nostrand, 1949. 6+432 pp. $7.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1948–49

Mathematics Club, Stanford University

The Stanford *Mathematics Club* held seven meetings during the academic year 1948–49. The following papers were presented:

The solution of Poncelet's problem, by Prof. D. C. Olds

Some fixed point theorems, by Dr. P. Garabedian

Complex moments, by Joe Ullman

Harmonic webs, by Hans Haegi

The gamma function, by John Luke

Space filling curves, by Kenneth Cooke

Progression-free sequences, by Dr. D. C. Spencer

Dr. G. Szego gave a talk on his trip to Europe.

The annual picnic was held at Alum Rock Park.

Officers for 1948–49 were: President, Sherman Lehman; Secretary, Robert Stalley; Treasurer, John Bennett.

Undergraduate Mathematics Club, University of Rochester

The end of the academic year, 1948–49, marks the first year of the newly organized *Undergraduate Mathematics Club* of the University of Rochester. The theme for the year was: that mathematics is not primarily a tool for other sciences but rather a distinct field of investigation.

The papers heard by the members were:

Diophantine equations, by Adnah Kostenbauder

What is a quaternion?, by Nathan Rickless

Almost every number contains an eight, by Norma Schuller

Mathematical fallacies, by Fred Straub

Boolean algebra, by Morten Kenner

Surface area of a cylinder, by Frank Hahn.

Prizes for the best papers were awarded to Adnah Kostenbauder and Fred Straub.

A great deal of interest was aroused by a game analogous to the popular radio game "Twenty Questions." In the game as played by the Mathematics Club, the topic always affiliates itself with something of a mathematical nature, such as geometric figures, equations, sets, *etc.* There is no limit on the number of questions to be asked but the investigating group must arrive at the solution in less than three minutes. The only restriction placed upon the questions is that they must be so directed as to be answered by a *yes* or *no* or in some cases by an indefinite answer of *sometimes* or *possibly*.

The newly elected officers are: Chairman, Fred Straub; Vice-Chairman, Carl Whiteman; Secretary-Treasurer, Frank Hahn; Assistant Secretary-Treasurer, Mary Chapman; Advisors, Dr. Bernstein and Mr. Gunderson.

Kappa Mu Epsilon, Chicago Teachers College

The papers presented to the *Illinois Gamma* chapter of *Kappa Mu Epsilon* during 1948–49 were:

The mystical significance of numbers, by Dorothy Dahlberg

Ramifications in cryptography, by Ramona Goldblatt

Pythagorean number oddities, by Kathryn Graham.

Fifty-seven members attended the annual banquet at which entertainment was furnished by students.

The officers for 1949–50 are: President, Marceline Kostris; Vice-President, Robert McNamara; Secretary, Lois DeGrange; Treasurer, Eleanor Outly; Sponsor, Dr. J. W. Sachs; Corresponding-Secretary, Dr. J. J. Urbancek.

Mathematics Club, Montana State University

The following papers were presented at meetings of the Montana State University *Mathematics Club* the past year:

Bi-products of mathematics, by Dr. A. S. Merrill

The duo-decimal system, by William Shipman

Trisection of angles, by James Wright

The philosophy of mathematics, by Dr. T. G. Ostrom

How to make an atomic bomb, by Dr. C. R. Jeppesen

How to extract cube roots, by Marybelle Fry

Men of mathematics, by Laurence Zimmerman

History of numbers, by Joseph Kratofil.

Informal discussions and refreshments followed the addresses.

A social meeting was held each quarter—a Christmas party in the Fall, Dr. and Mrs. Merrill entertained at a valentine party during the Winter, and the annual picnic with the Chemistry Club was held in the Spring.

The *Pi Mu Epsilon* prizes were awarded at an early meeting.

The officers for next year are: President, Marybelle Fry; Vice-President, Evan Rempel; Secretary-Treasurer, Joseph Kratofil.

Mathematics Club, Immaculate Heart College

The following papers were presented by student members of the club:

Topology, by Marie Kreuper and Dolores Barenberg

The transcendental and imaginary, by Mary Parker

Mathematics as related to other great fields of knowledge, by Mary Gleason

History of linear and quadratic equations, 1550–60, by Mary Dominguez

Historical problems in algebra, by Maureen Brennan, Patricia Hanson, Joan Rae, Glenna Christie, Colleene Harron and Virginia Pritchard

History of the quartic equation and methods of solution, by Dolores Barenberg.

Following each paper a period was devoted to mathematical recreations after which refreshments were served.

Officers for 1948–49 were: President, Doris Mills; Secretary-Treasurer, Marie Kreuper.

Mathematics Club, Purdue University

Monthly meetings were held throughout the year during which the following topics were discussed:

The development of Hilbert spaces, by Roger Burt

The solution of a partial difference equation, by Eugene Usdin

Probability in continuum, by Bertrom Kastant

Probate analysis, by Robert Carson

Summability of divergent series, by Robert Derflinger

Symbolic logic, by Sakari Jutila

The solution of the partial difference equation for Pascal's triangle, by O. Moan

History of the calculus, by Prof. A. Rosenthal.

Mathematics Club, Bowdoin College

The Bowdoin College *Mathematics Club* held nine regular meetings during the academic year 1948–49. During the year the following papers were prepared and discussed:

Perspective triangles, by Dr. S. Hammond

Finite numbers, by Reed Dawson

New aspects of non-Euclidian geometry, by C. Carruthers

Elementary discussion of harmonic functions, by W. Gager

General inversion, by Demetriardies

Dynamic symmetry, by Thorndike

Mathematics of code analysis, by Dr. R. Brumbaugh

The development and significance of the Laplace transform, by D. Union.

Refreshments were served at the last meeting of each semester.

The officers of the club for the year were: President, Charles Carruthers (1st semester), William Gager (2nd semester); Secretary, Alfred Waxler; Treasurer, Donald Union.

Kappa Mu Epsilon, Albion College

The *Michigan Alpha* chapter of *Kappa Mu Epsilon*, Albion College, held seven meetings during the school year 1948–49. The following papers were presented:

Identification of conic sections by invariants, by Ralph Powers

Algebra and the abacus, by Richard Vetter

Contributions to mathematics by ancient Egyptians, by Richard Carver

Fallacies in mathematics, by Wendell Martin

Linkages, by Harold Wakelin

LaGrange and Laplace, by Jean Hayward

Flatland, by Betty Marshall
Proofs of the Pythagorean theorem, by Betty McIllvenan
Non-Euclidean geometry, by Barbara Barnes
Women in mathematics, by Charlotte Phelps.

Officers elected for 1949–50 are: President, Raymond Gillespie; Vice-President, Albert Foster; Secretary-Treasurer, Patricia Collins; Faculty Sponsor, Prof. H. Larsen.

Mathematics Club, Case Institute of Technology

The *Mathematics Club* of Case Institute of Technology enjoyed a year of varied meetings with rather more than usual emphasis on student participation. The following papers were presented:

Plasticity, by Deonisie Trifan
Matrix algebra, by Ernest Leach
The Whitney ham sandwich problem, by Charles Saltzer
Group theory, by Frank Palermo
Exterior ballistics, by Richard Varga
Projective geometry, by Arthur Shaw.

Officers for the year 1948–49 were: President, Ernest Leach; Adviser, Prof. M. Morris.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

GUIDANCE PAMPHLET FOR MATHEMATICS STUDENTS

A guidance pamphlet entitled *Why Study Mathematics* has been published recently by the Canadian Mathematical Congress. It has been prepared for the use of secondary school teachers and students and should be especially helpful to those who are concerned with vocational guidance. Copies may be obtained from the Canadian Mathematical Congress, Engineering Building, McGill University, Montreal, Canada.

PI MU EPSILON JOURNAL

The honorary mathematical fraternity, Pi Mu Epsilon, is now publishing a journal entitled *Pi Mu Epsilon Journal*. The first number indicates that this journal will be of interest to undergraduate as well as graduate members of the fraternity. Inquiries about the journal should be addressed to the Editor, Ruth W. Stokes, 15 Smith Hall, Syracuse University, Syracuse 10, New York.

STATISTICAL SUMMER SESSION IN BERKELEY, CALIFORNIA

Following the established pattern, there will be held this year a Statistical Summer Session at the University of California, Berkeley. The faculty will include W. G. Cochran of Johns Hopkins University, Benjamin Epstein of Wayne University, E. L. Lehmann of the University of California, Paul Lévy of the École Polytechnique, Paris, France, and G. E. Noether of New York University.

Courses will be offered on both the graduate and the undergraduate levels. The graduate courses, all given during the First Summer Session, June 19 to July 29, are meant primarily for students who either have already obtained their Ph.D. degree or are working toward it. No specific prerequisites to graduate courses will be required. The graduate program includes (1) a course on design of experiments and a seminar on analysis of variance by W. G. Cochran, (2) a course on theory of estimation by E. L. Lehmann, and (3) a course and a seminar on random variables and random functions by Paul Lévy.

Inquiries should be addressed to the Office of the Summer Sessions, 1A Administration Building, University of California, Berkeley 4, California.

PERSONAL ITEMS

Cambridge University has announced that the 1949 award of the Adams Prize is divided equally among four men: J. C. Burkill, Fellow and Lecturer of Peterhouse and Lecturer in Mathematics in the University of Cambridge; S. Chandrasekhar, Distinguished Service Professor of the University of Chicago and astrophysicist at the Yerkes Observatory; W. K. Hayman, Lecturer in Mathematics at University College of the Southwest of England, Exeter, and this year Lecturer in Mathematics at Brown University; and J. M. Whittaker, Professor of Mathematics at Liverpool University.

Baylor University reports: Mr. Dale Maness has been appointed to an assistant professorship; Assistant Professor Eugene McLachlan has been granted a year's leave of absence to continue graduate study at Rice Institute.

Catholic University of America announces: Professor O. J. Ramler has been appointed Head of the Department of Mathematics; Mr. Edgar Barrett and Mrs. Paul Nesbeda have been appointed to instructorships.

At Duke University the following appointments to instructorships have been made: Dr. Mary E. Estill of the University of Texas, Mr. L. M. Fulton Jr. and Dr. Walter Rudin, formerly graduate students at Duke University.

Emory University makes the following announcements: Assistant Professor M. B. Sledd has been promoted to an associate professorship; Mr. Earl Carroll, University of Illinois, and Mr. Donald Holmes, Purdue University, have been appointed to instructorships; Professor Douglas Rumble has retired with the title of Professor Emeritus.

Florida State University reports the following: Mr. J. P. Gill formerly at the University of Texas has been appointed to an assistant professorship; Mr. R. W. Bagley and Mr. K. W. Martin have been appointed to instructorships.

At Johns Hopkins University: Dr. Ion Carstoiu of the University of Paris has been appointed to an assistant professorship; Mr. A. C. Smith, formerly of Stonehurst College, Lancashire, England, has been appointed to an instructorship.

Kansas State College announces: Instructor Marilynn Spanglet has been promoted to an assistant professorship; Professor A. E. White is now teaching on a part-time basis.

Lehigh University announces the following promotions and appointments: Assistant Professor F. S. Beale has been promoted to an associate professorship; Assistants B. C. Kenny and Michael Tikson have been promoted to instructorships; Assistant Professor Jacob Samoloff of the University of New Brunswick has been appointed to an instructorship.

The National Bureau of Standards announces: Professor M. R. Hestenes of the University of California at Los Angeles is a visiting staff member at the Institute for the year 1949–1950; Mr. John Todd has been appointed chief of the Computation Laboratory of the Bureau in Washington; Mr. H. E. Salzer, formerly with the New York Unit of the Computation Laboratory, has joined the Washington staff.

North Carolina State College announces the promotions of Assistant Professors C. L. Carroll, Jr. and P. E. Lewis to associate professorships.

Purdue University reports: Associate Professor I. W. Burr has been promoted to a professorship; Dr. E. A. Trabant has been promoted to an assistant professorship; Professor R. J. Duffin, Carnegie Institute of Technology, served as visiting professor for the first semester of 1949–1950; Dr. Jacob Korevaar, Mathematical Center, Amsterdam, Holland, has been appointed Visiting Lecturer for 1949–1950; Dr. Gordon Overholtzer of the University of Kansas and Dr. N. P. Yeardley of the University of Cincinnati have been appointed to assistant professorships; Dr. C. C. Goldman, University of Cincinnati, and Mr. Gustave Rabson of the University of Michigan have been appointed to instructorships; Assistant Professor W. H. Hardman, Acting Professor C. H. Skinner and Instructor J. H. Shock have retired.

Rutgers University announces the promotion of Assistant Instructor G. Y. Cherlin to an instructorship and the appointment of C. N. Campopiano, K. T. Lee and Herman Munick to teaching assistantships.

Syracuse University makes the following announcements: Instructor Eckford Cohen has been appointed to an assistant professorship; Associate Professor Lipman Bers is on leave of absence and is at the Institute for Advanced Study.

At the University of Arizona: Miss Elizabeth A. Button, assistant at the University of Wisconsin, Mr. C. B. Harper, teaching fellow at the University of Michigan, Mr. J. P. Middlekauff, graduate student of Stanford University, and Mr. E. D. Stalley, graduate assistant at Stanford University, have been appointed to instructorships.

The University of Houston reports the following promotions: Instructors

C. P. Benner and R. F. Ray to assistant professorships; Assistant Instructors Dean O. Gray and June P. Wood to instructorships.

The University of Manitoba announces: Professor J. W. Lawson has been appointed Acting Dean of Arts and Sciences; Professor W. H. McEwen has been appointed Dean of the Faculty of Graduate Studies and Research; Mr. H. R. Coish of the National Research Council, Chalk River Development, has been appointed to an assistant professorship.

At the University of New Hampshire: Mr. S. R. Peterson, Mr. F. J. Robinson and Mr. C. W. Schenck have been appointed to instructorships.

The University of Oklahoma makes the following announcements: Associate Professor Casper Goffman has been promoted to a professorship; Mr. P. W. M. John, Mr. D. A. Gorsline, Mr. Gene Levy of the College of the Ozarks and Mr. C. S. Williams have been appointed to instructorships; Mrs. Lyla Tisdale, formerly at Southern Methodist University, has been appointed Special Instructor.

The University of Pennsylvania reports: Dr. Bernard Epstein has been promoted to an assistant professorship; Dr. M. I. Aissen, Stanford University, Dr. H. E. Campbell, University of Wisconsin, and Dr. Anne M. Whitney have been appointed to instructorships.

The University of Rochester announces: Dr. Norman Gunderson has been promoted to an assistant professorship; Professor Waldimir Seidel has returned after a year's leave of absence; Mr. Warren Stenberg of the University of California at Berkeley has been appointed to an instructorship; Mr. Lester Arnold, Mr. David Barton, Mr. A. E. Danese and Mr. John Mairhuber have been appointed to part-time instructorships; Dr. S. L. Crump who is associated with the Atomic Project at the University Hospital is teaching part-time.

The University of Saskatchewan announces the appointments of Dr. Albert Edrei, formerly of Farouk 1st University, Alexandria, Egypt, and Mr. E. I. Gale of the University of New Brunswick as Special Lecturers.

The University of Toronto reports the following: Assistant Professor W. J. R. Crosby of the University of Saskatchewan and Instructor D. A. S. Fraser of Princeton University have been appointed to assistant professorships; Mr. H. L. Seal, actuary, National Life Assurance Company, has been appointed Special Lecturer; Dr. R. G. Stanton, formerly lecturer at the University of Michigan, has been appointed Lecturer.

The University of Utah announces: Dr. R. N. Thomas, formerly of Harvard University and the Institute for Advanced Study, has been appointed to an associate professorship in mathematical astronomy; Miss Rachel S. Hodges of the University of Wisconsin, Mr. R. E. Janssen of the University of Illinois and Mr. H. C. Mayer of the University of Minnesota have been appointed to instructorships; Associate Professor C. J. Thorne has returned after a year's leave of absence; Professor C. R. Wylie Jr. has been elected Chairman of the Mathematics Division of the American Society for Engineering Education for the current year.

At the University of Western Ontario: Professor H. R. Kingston, head of the Department of Mathematics, has been promoted to the position of Dean of the College of Arts and Science; Associate Professor R. H. Cole has returned after a year's leave of absence; Professor Arthur Woods has retired.

The University of Wisconsin reports: Assistant Professor R. H. Bing, now at the University of Virginia, has been promoted to an associate professorship; Assistant Professor R. C. Buck of Brown University has been appointed to an associate professorship; Mr. R. G. D. Steel of Iowa State Teachers College has been appointed to an assistant professorship; Professor H. W. March has retired with the title of Professor Emeritus.

Instructor Tyler Allhands of the State College of Washington has been appointed to an assistant professorship at Pacific University.

Instructor Helmut Aulbach of Syracuse University has been appointed to an assistant professorship at Triple Cities.

Mr. Earle Canfield of Drake University has been promoted to an assistant professorship.

Mr. P. L. Chessin of Columbia University has been appointed to an instructorship at Cooper Union.

Mr. D. M. Clarke has accepted an instructorship at Gonzaga University.

Mr. F. B. Crippen of Rutgers University has been appointed to an instructorship at Fordham University.

Mr. W. S. Doss has been appointed to an instructorship at Springfield College.

Miss Katherine Garland of the University of Denver has been promoted to an instructorship.

Associate Professor R. E. Gilman of Brown University has been promoted to a professorship.

Mr. W. A. Glenn, formerly graduate student at the University of New Brunswick, has been appointed Lecturer at Mount Allison University, New Brunswick, Canada.

Dr. B. H. Gundlach, formerly at Ruston Academy, Havana, Cuba, has been appointed to an assistant professorship at the University of Arkansas.

Mr. Raymond Gutzman of Fenn College has been appointed to an assistant professorship at the Colorado School of Mines.

Dr. H. L. Harter of Purdue University has been appointed to an assistant professorship at Michigan State College.

Dr. T. E. Hull has received an appointment as instructor at the University of British Columbia.

Dr. C. C. Hurd, who has been a research head at Oak Ridge, Tennessee, has accepted an appointment as director of the Applied Science Department of International Business Machines Corporation.

Professor J. L. Jones of the University of Toledo has retired.

Associate Professor W. Kozekiewicz has been appointed to a professorship at the University of Montreal.

Associate Professor Eugene Leimanis of the University of Riga, Latvia, has accepted an appointment as assistant professor at the University of British Columbia.

Professor W. R. Longley of Yale University has been appointed Visiting Professor of Mathematics at Haverford College for the second semester 1949–1950.

Professor W. V. Parker of the University of Georgia has been appointed to the position of Head Professor of Mathematics at Alabama Polytechnic Institute.

Dr. T. D. Reynolds of Duke University is now Director of Instruction, Buncombe County, North Carolina, Schools.

Professor C. K. Robbins of Purdue University has accepted the editorship of the Department of Mathematical Miscellany of the *Mathematics Magazine.*

Professor W. W. Rogosinski of King's College, University of Durham, England, is Visiting Professor of Mathematics at the Oklahoma Agricultural and Mechanical College for the second semester, 1949–1950.

Assistant Professor Andrew Sobczyk of Boston University has been promoted to an associate professorship.

Assistant Professor R. D. Specht of the University of Wisconsin is now with the Rand Corporation.

Assistant Professor Irving Sussman of the University of California at Berkeley has accepted a professorship at St. Mary's College of California.

Associate Professor Ethel Sutherland has been promoted to a professorship at Longwood College, formerly named State Teachers College, Farmville, Virginia.

Dr. Sylvan Wallach of Johns Hopkins University is now with the Westinghouse Company in Pittsburgh.

Professor Alexander Weinstein has accepted a joint appointment by the Department of Mathematics of the University of Maryland and the Institute for Fluid Dynamics and Applied Mathematics.

Instructor G. C. Zader of the Citadel has been promoted to an assistant professorship.

Professor F. F. Decker of Syracuse University died November 28, 1949. He was a charter member of the Association.

Professor A. E. Mallory of Colorado State College of Education died June 1949.

Professor Emeritus Virgil Snyder of Cornell University died January 4, 1950.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-THIRD ANNUAL MEETING OF THE ASSOCIATION

The thirty-third annual meeting of the Mathematical Association of America was held at Columbia University, New York City, on Friday, December 30, 1949, in conjunction with the annual meetings of the American Mathematical Society and the American Association for the Advancement of Science. A total of seven hundred and eighty persons were registered at Earl Hall, including the following four hundred and twelve members of the Association:

C. R. ADAMS, Brown University
R. P. AGNEW, Cornell University
L. V. AHLFORS, Harvard University
M. I. AISSEN, University of Pennsylvania
A. A. ALBERT, University of Chicago
R. G. ALBERT, Brown University
C. B. ALLENDOERFER, Haverford College
R. D. ANDERSON, University of Pennsylvania
R. LUCILE ANDERSON, Hunter College
R. G. ARCHIBALD, Queens College
K. J. ARNOLD, University of Wisconsin
L. A. AROIAN, Hunter College
MAX ASTRACHAN, U.S.A.F. Institute of Technology
SILVIO AURORA, Columbia University
FRANK AYRES, Dickinson College
W. L. AYRES, Purdue University
FRANCES E. BAKER, Vassar College
JOSHUA BARLAZ, Rutgers University
P. T. BATEMAN, Institute for Advanced Study
M. R. BATES, Union College
HELEN P. BEARD, Sophie Newcomb College
RALPH BEATLEY, Harvard University
F. S. BECKMAN, Pratt Institute
F. P. BEER, Lehigh University
E. G. BEGLE, Yale University
A. A. BENNETT, Brown University
MARTIN BERMAN, University of Cincinnati
R. R. BERNARD, Yale University
W. W. BESSELL, U. S. Military Academy
BARBARA B. BETTS, D. C. Heath & Company
R. H. BING, University of Virginia
GARRETT BIRKHOFF, Harvard University
ERWIN BISER, Rutgers University
D. W. BLACKETT, Princeton University
ARCHIE BLAKE, Office of Army Surgeon General
W. E. BLEICK, U.S.N. Postgraduate School
H. F. BOHNENBLUST, California Institute of Technology
GRACE L. BOLTON, Barnard College
T. A. BOTTS, University of Virginia
J. G. BOWKER, Middlebury College
C. B. BOYER, Brooklyn College
A. D. BRADLEY, Hunter College
RICHARD BRAUER, University of Michigan
N. A. BRIGHAM, University of Maryland
H. W. BRINKMANN, Swarthmore College
PAUL BROCK, Reeves Instrument Corporation
A. B. BROWN, Queens College
R. H. BRUCK, University of Wisconsin
H. D. BRUNK, Rice Institute
W. J. BRUNS, Syracuse University
C. T. BUMER, Clark University
L. H. BUNYAN, Rutgers University
R. S. BURINGTON, Bureau of Ordnance
L. J. BURTON, Bryn Mawr College
L. E. BUSH, College of St. Thomas
J. H. BUSHEY, Hunter College
JEWELL HUGHES BUSHEY, Hunter College
S. S. CAIRNS, University of Illinois
J. W. CALKIN, Los Alamos Scientific Laboratory
ETHEL B. CALLAHAN, Hartwick College
MILDRED E. CARLEN, Brown University
L. VIRGINIA CARLTON, Northwestern State College
J. O. CHELLEVOLD, Lehigh University
SARVADAMAN CHOWLA, University of Kansas
RANDOLPH CHURCH, U.S.N. Postgraduate School
F. E. CLARK, Tulane University
G. R. CLEMENTS, U. S. Naval Academy
E. J. COGAN, Pennsylvania State College
A. C. COHEN, University of Georgia
H. J. COHEN, Tulane University
L. W. COHEN, Queens College
R. M. COHN, Rutgers University
NANCY COLE, Syracuse University
J. B. COLEMAN, Columbia, S. C.
H. R. COOLEY, New York University
T. F. COPE, Queens College
A. H. COPELAND, University of Michigan
BYRON COSBY, State University of Iowa

R. R. COVEYOU, Oak Ridge National Laboratory
W. H. H. COWLES, Pratt Institute
V. F. COWLING, University of Kentucky
E. H. CRISLER, West Virginia University
E. H. CUTLER, Lehigh University
J. M. DANSKIN, Office of the Chief of Naval Operations
J. E. DARRAUGH, Case Institute of Technology
CONSTANCE H. DAVIS, South Orange, N. J.
D. R. DAVIS, New Jersey State Teachers College
A. H. DIAMOND, Oklahoma A & M College
MARY P. DOLCIANI, Vassar College
H. L. DORWART, Trinity College
R. H. DOWNING, U.S.A.F. Institute of Technology
ARNOLD DRESDEN, Swarthmore College
NELSON DUNFORD, Yale University
P. S. DWYER, University of Michigan
J. N. EASTHAM, Cooper Union
W. F. EBERLEIN, University of Wisconsin
SAMUEL EILENBERG, Columbia University
CAROLYN EISELE, Hunter College
PAUL ERDÖS, University of Illinois
R. W. ERICKSON, Mississippi State College for Women
W. H. FAGERSTROM, City College of the City of New York
F. D. FAULKNER, University of Michigan
J. M. FELD, Queens College
WILLIAM FELLER, Cornell University
W. E. FERGUSON, Connecticut College for Women
F. A. FICKEN, University of Tennessee
N. J. FINE, University of Pennsylvania
C. H. FISCHER, University of Michigan
EDWARD FLEISHER, Brooklyn College
E. E. FLOYD, University of Virginia
TOMLINSON FORT, University of Georgia
R. M. FOSTER, Polytechnic Institute of Brooklyn
J. H. FOUNTAIN, University of Buffalo
J. S. FRAME, Michigan State College
W. M. FRANK, Yeshiva University
ORRIN FRINK, Pennsylvania State College
H. M. GEHMAN, University of Buffalo
ABE GELBART, Syracuse University
B. H. GERE, Hamilton College
F. J. GERST, Loyola University
J. C. GIBSON, Rensselaer Polytechnic Institute
B. P. GILL, City College of the City of New York
R. E. GILMAN, Brown University
J. W. GIVENS, University of Tennessee
A. M. GLEASON, Harvard University
H. E. GOHEEN, Syracuse University
MICHAEL GOLDBERG, Bureau of Ordnance
A. W. GOODMAN, University of Kentucky
D. B. GOODNER, Florida State University
R. D. GORDON, University of Buffalo
W. O. GORDON, Pennsylvania State College
ARTHUR GRAD, Office of Naval Research
H. S. GRANT, Rutgers University
J. B. GREELEY, Utica College
HARRIET M. GRIFFIN, Brooklyn College
GEORGE GROSSMAN, Polytechnic Institute of Brooklyn
C. C. GROVE, City College of the City of New York
H. T. GUARD, Colorado A & M College
R. W. HAMMING, Bell Telephone Laboratories
HELEN J. HAND, D'Youville College
FRANK HARARY, University of Michigan
G. E. HAY, University of Michigan
F. S. HAWTHORNE, Hofstra College
KATHARINE E. HAZARD, New Jersey College for Women
G. A. HEDLUND, Yale University
C. E. HEILMAN, Elizabethtown College
M. H. HEINS, Brown University
G. C. HELME, Pratt Institute
R. G. HELSEL, Ohio State University
A. S. HENDLER, Rensselaer Polytechnic Institute
R. E. HENRY, Newark College, Rutgers University
I. N. HERSTEIN, University of Kansas
M. R. HESTENES, University of California at Los Angeles
T. H. HILDEBRANDT, University of Michigan
J. C. HILFERTY, St. Thomas More High School
L. S. HILL, Hunter College
EINAR HILLE, Yale University
T. R. HOLLCROFT, Wells College
H. K. HOLT, Union College
E. MARIE HOVE, Hofstra College
C. C. HSIUNG, University of Wisconsin
R. C. HUFFER, Beloit College
RALPH HULL, Purdue University
M. GWENETH HUMPHREYS, Randolph-Macon Woman's College
T. R. HUMPHREYS, New Jersey State Teachers College
MILDRED HUNT, Illinois Wesleyan University

SOLOMON HURWITZ, City College of the City of New York
W. A. HURWITZ, Cornell University
L. C. HUTCHINSON, Polytechnic Institute of Brooklyn
M. A. HYMAN, Naval Ordnance Laboratory
J. E. IKENBERRY, Madison College
EUGENE ISAACSON, New York University
E. D. JENKINS, Kent State University
WALTER JENNINGS, U.S.N. Postgraduate School
FRITZ JOHN, New York University
R. A. JOHNSON, Brooklyn College
F. E. JOHNSTON, George Washington University
A. W. JONES, Rensselaer Polytechnic Institute
B. W. JONES, University of Colorado
O. C. JUELICH, Hofstra College
MARK KAC, Cornell University
W. C. KALINOWSKI, St. John's University
AIDA KALISH, Polytechnic Institute of Brooklyn
SIDNEY KAPLAN, Naval Ordnance Laboratory
IRVING KAPLANSKY, University of Chicago
GILBERT KASKEY, University of Delaware
EDWARD KASNER, Columbia University
E. S. KENNEDY, Brown University
D. E. KIBBEY, Syracuse University
H. L. KINSOLVING, U. S. Naval Academy
A. R. KIRBY, Fordham University
S. C. KLEENE, University of Wisconsin
J. R. KLINE, University of Pennsylvania
MORRIS KLINE, New York University
R. J. KOHLMEYER, Pratt Institute
JOHN KRONSBEIN, Evansville College
R. R. KUEBLER, Dickinson College
A. W. LANDERS, Brooklyn College
MARY K. LANDERS, Hunter College
R. E. LANGER, University of Wisconsin
J. C. LANZ, Hershey Junior College
E. H. LARGUIER, Spring Hill College
H. D. LARSEN, Albion College
V. V. LATSHAW, Lehigh University
SOLOMON LEFSCHETZ, Princeton University
C. H. LEHMANN, Cooper Union
MARGUERITE LEHR, Bryn Mawr College
H. R. LEIFER, Veterans Administration
MAX LELEIKO, Rutgers University
TADEUSZ LESER, University of Kentucky
CAROLINE A. LESTER, New York State College for Teachers
M. E. LEVENSON, Brooklyn College
W. J. LEVEQUE, University of Michigan
A. J. LEWIS, University of Denver
D. C. LEWIS, Johns Hopkins University
SALLY I. LIPSEY, Hunter College
H. D. LIPSICH, University of Cincinnati
CHARLES LOEWNER, Syracuse University
E. R. LORCH, Columbia University
R. B. LOWE, Polytechnic Institute of Brooklyn
C. I. LUBIN, University of Cincinnati
ELIZABETH C. LUKACS, China Lake, California
L. A. MACCOLL, Bell Telephone Laboratories
C. C. MACDUFFEE, University of Wisconsin
D. M. MACEWEN, City College of the City of New York
H. M. MACNEILLE, American Mathematical Society
NATHANIEL MACON, University of North Carolina
D. MAY HICKEY MARIA, Brooklyn College
R. H. MARQUIS, Ohio University
W. T. MARTIN, Massachusetts Institute of Technology
KENNETH MAY, Carleton College
E. D. MCCARTHY, University of Detroit
N. H. MCCOY, Smith College
S. W. MCCUSKEY, Case Institute of Technology
E. J. MCSHANE, Institute for Advanced Study
P. C. MEADOWS, Carroll College
A. E. MEDER, Rutgers University
L. E. MEHLENBACHER, University of Detroit
RUDOLF MEYER, University of Buffalo
E. J. MILES, Yale University
E. P. MILES, Alabama Polytechnic Institute
JOSEPH MILKMAN, U. S. Naval Academy
FREDERIC H. MILLER, Cooper Union
W. H. MILLS, Yale University
P. D. MINTON, University of North Carolina
H. J. MISER, Headquarters, U.S.A.F.
JOSEPHINE M. Mitchell, University of Illinois
HARRIET F. MONTAGUE, University of Buffalo
MABEL D. MONTGOMERY, University of Buffalo
C. N. MOORE, University of Cincinnati
T. W. MOORE, U. S. Naval Academy
MARSTON MORSE, Institute for Advanced Study
A. J. MORTOLA, City College of the City of New York
W. R. MURRAY, Franklin and Marshall College
D. J. MYATT, Antioch College
D. S. NATHAN, City College of the City of New York
A. B. NEALE, Naval Aircraft Factory, Philadelphia
C. A. NELSON, Rutgers University
F. E. NEMMERS, State University of Iowa

PAUL NESBEDA, Catholic University of America
J. J. NEWMAN, Harvard University
C. V. NEWSOM, University of the State of New York
HELEN K. NICKERSON, Wheaton College
O. M. NIKODYM, Kenyon College
M. A. NORDGAARD, Upsala College
P. B. NORMAN, Wagner College
G. G. O'BRIEN, Washington Missionary College
EUGENE ODIN, Arma Corporation, New York
L. F. OLLMANN, Hofstra College
ANNE FRANCES O'NEILL, Smith College
E. R. OTT, Rutgers University
J. C. OXTOBY, Bryn Mawr College
F. W. PERKINS, Dartmouth College
C. L. PERRY, University of Arkansas
G. W. PETRIE, Lehigh University
B. J. PETTIS, Institute for Advanced Study
H. R. PHALEN, College of William and Mary
C. R. PHELPS, Rutgers University
C. G. PHIPPS, University of Florida
A. E. PITCHER, Lehigh University
J. C. POLLEY, Wabash College
WALTER PRENOWITZ, Brooklyn College
G. B. PRICE, University of Kansas
K. S. PURDUE, Virginia Military Institute
TIBOR RADO, Ohio State University
LEILA R. RAINES, Norwood, Rhode Island
GORDON RAISBECK, Bell Telephone Laboratories
O. J. RAMLER, Catholic University of America
J. F. RANDOLPH, University of Rochester
W. R. RANSOM, Tufts College
L. T. RATNER, Vanderbilt University
G. E. RAYNOR, Lehigh University
R. W. RECTOR, U. S. Naval Academy
C. J. REES, University of Delaware
MINA S. REES, Office of Naval Research
C. F. REHBERG, New York University
I. M. REINER, Danville Community College
IRVING REINER, University of Illinois
ERIC REISSNER, Massachusetts Institute of Technology
C. N. REYNOLDS, West Virginia
D. P. RICHARDSON, University of Arkansas
MOSES RICHARDSON, Brooklyn College
C. C. RICHTMEYER, Central Michigan College
P. R. RIDER, Washington University
R. F. RINEHART, Research and Development Board
J. F. RITT, Columbia University
E. K. RITTER, University of Michigan
J. H. ROBERTS, Duke University
G. N. ROBINSON, New Bedford, Massachusetts
SELBY ROBINSON, City College of the City of New York
V. N. ROBINSON, U. S. Naval Academy
G. B. ROBISON, Cornell University
C. A. ROGERS, Institute for Advanced Study
IRA ROSENBAUM, University of Miami
P. C. ROSENBLOOM, Syracuse University
ARTHUR ROSENTHAL, Purdue University
J. B. ROSSER, National Bureau of Standards
M. F. ROSSKOPF, Syracuse University
CHARLES SALKIND, Polytechnic Institute of Brooklyn
CHARLES SALTZER, Case Institute of Technology
R. G. SANGER, Kansas State College
ARTHUR SARD, Queens College
A. C. SCHAEFFER, Purdue University
R. D. SCHAFER, University of Pennsylvania
ROBERT SCHATTEN, University of Kansas
HENRY SCHEFFE, Columbia University
EDITH R. SCHNECKENBURGER, University of Buffalo
K. C. SCHRAUT, University of Dayton
AUGUSTA L. SCHURRER, University of Wisconsin
ABRAHAM SCHWARTZ, City College of the City of New York
WLADIMIR SEIDEL, University of Rochester
I. M. SHEFFER, Pennsylvania State College
L. W. SHERIDAN, College of St. Thomas
L. L. SILVERMAN, Dartmouth College
JAMES SINGER, Brooklyn College
SISTER MARY CORDIA, College of Notre Dame of Maryland
SISTER MARY ESTHER, Mundelein College
L. L. SMAIL, Lehigh University
ROBERT EDWARD SMITH, College of William and Mary
W. M. SMITH, Lafayette College
ERNST SNAPPER, Princeton University
W. S. SNYDER, University of Tennessee
ANDREW SOBCZYK, Boston University
ELIZABETH S. SOKOLNIKOFF, University of Wisconsin
J. L. SOLOMON, University of Toronto
D. E. SOUTH, University of Kentucky
W. K. SPEARS, Purdue University
VIVIAN E. SPENCER, U. S. Bureau of the Census
E. R. STABLER, Hofstra College
E. P. STARKE, Rutgers University
J. R. K. STAUFFER, Rhode Island State College
RUTH W. STOKES, Syracuse University

R. R. STOLL, Lehigh University
E. A. STURLEY, Allegheny College
GABOR SZEGO, Stanford University
J. S. TAYLOR, University of Pittsburgh
MILDRED E. TAYLOR, Mary Baldwin College
A. J. TERZUOLI, Polytechnic Institute of Brooklyn
FEODOR THEILHEIMER, Naval Ordnance Laboratory
J. M. THOMAS, Duke University
C. B. TOMPKINS, George Washington University
D. L. THOMSEN, Haverford College
W. J. THRON, Washington University
MICHAEL TIKSON, Lehigh University
G. L. TILLER, Utica College
JOHN TODD, National Bureau of Standards
L. F. TOLLE, St. John's University
MARIAN M. TORREY, Goucher College
J. I. TRACEY, Yale University
A. W. TUCKER, Princeton University
J. W. TUKEY, Princeton University
ANNITA TULLER, Hunter College
J. L. ULLMAN, University of Michigan
F. E. ULRICH, Rice Institute
E. P. VANCE, Oberlin College
HENRY VAN ENGEN, Iowa State Teachers College
HELEN E. VAN SANT, Beaver College
S. I. VROOMAN, Rensselaer Polytechnic Institute
G. L. WALKER, Cornell University
S. E. WALKLEY, University of Illinois
J. L. WALSH, Harvard University
R. M. WALTER, Rutgers University
W. G. WARNOCK, Rensselaer Polytechnic Institute
G. C. WEBBER, University of Delaware
MARGARET C. WEEBER, Teachers College of Connecticut
J. V. WEHAUSEN, Office of Naval Research
LOUIS WEISNER, Hunter College
MARIE J. WEISS, Sophie Newcomb College
J. G. WENDEL, Yale University
F. J. WEYL, Bureau of Ordnance
M. E. WHITE, Hunter College
D. E. WHITFORD, Polytechnic Institute of Brooklyn
P. M. WHITMAN, Johns Hopkins University
D. R. WHITNEY, Ohio State University
G. T. WHYBURN, University of Virginia
W. M. WHYBURN, University of North Carolina
D. V. WIDDER, Harvard University
ALBERT WILANSKY, Lehigh University
G. T. WILLIAMS, Long Island, New York
MARY E. WILLIAMS, Skidmore College
FLORENCE A. WIRSCHING, Purdue University
Y. K. WONG, Institute for Advanced Study
MARGARET Y. WOODBRIDGE, Brooklyn College
FRANCES M. WRIGHT, Triple Cities College
MARIE A. WURSTER, Temple University
BERTRAM YOOD, Cornell University
J. W. T. YOUNGS, Indiana University
OSCAR ZARISKI, Harvard University
J. A. ZILBER, Columbia University
H. J. ZIMMERBERG, Rutgers University
ANTONI ZYGMUND, University of Chicago

Sessions of the Association were held on Friday morning and afternoon in Room 301, Pupin Physics Laboratories of Columbia University. President R. E. Langer presided at the morning session and Professor R. M. Foster at the afternoon session. The Program Committee for the meeting consisted of R. M. Foster, chairman, Jewell H. Bushey, T. F. Cope, and Henry Scheffe.

FIRST SESSION OF THE ASSOCIATION

"Mathematical Biology," by Professor Nicholas Rashevsky, University of Chicago.

"Some remarks on Hartogs' Theorem for Several Complex Variables," by Professor W. T. Martin, Massachusetts Institute of Technology.

"The Pupil's Advocate," by Professor Ralph Beatley, Harvard University.

SECOND SESSION OF THE ASSOCIATION

"Some Applications of Matrices in the Theory of Equations," by Professor C. C. MacDuffee, University of Wisconsin.

"Diffusion Processes and the Theory of Evolution," by Professor William Feller, Cornell University.

"The Education of a Scientific Generalist," by Professor J. W. Tukey, Princeton University.

MEETING OF THE BOARD OF GOVERNORS

The Board met on Thursday afternoon in John Jay Hall of Columbia University. Sixteen members of the Board were present. Among the more important items of business transacted were the following:

The Board voted to approve the appointment by President Langer of the following Nominating Committee for 1950: R. M. Foster, chairman, R. L. Jeffery, J. C. Oxtoby.

A ballot by the Board resulted in the reelection of J. F. Randolph as a member of the Finance Committee for 1950–1953.

It was voted to hold the 1950 Annual Meeting at the University of Florida, Gainesville, Florida, on Saturday, December 30, 1950 and to accept invitations for the 1951 Annual Meeting to be held at Brown University, Providence, Rhode Island and for the 1953 Summer Meeting to be held at Laval University, Quebec, Canada. This latter meeting is to be held in conjunction with the Canadian Mathematical Congress and the one hundredth anniversary celebration of the charter of Laval University.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

President Langer presided at the annual business meeting which was held on Friday at 2:30 in Room 301 Pupin Physics Laboratories.

The Secretary announced the results of the balloting for officers, in which 917 votes were cast. L. M. Graves of the University of Chicago was elected First Vice-President for the two year term 1950–1951. M. R. Hestenes of the University of California at Los Angeles and Marie J. Weiss of Sophie Newcomb College were elected Governors for the three-year term 1950–1952.

Upon motion duly seconded and carried unanimously, it was voted that the By-Laws of the Association be amended by adding the following sentence to Article VII, Dues, Section I: "The Board of Governors may authorize the admission to membership of individuals and classes of applicants without payment of the initiation fee."

It was announced that the Board of Governors had authorized remission of the initiation fee for (1) members of the National Council of Teachers of Mathematics and (2) regularly enrolled undergraduate students in colleges and universities of the United States and Canada whose applications for membership are properly certified by the chairman of the appropriate department of mathematics.

MEETINGS OF OTHER ORGANIZATIONS

The sessions of the American Mathematical Society began on the morning of Tuesday, December 27 and continued through Thursday afternoon. The Josiah

Willard Gibbs Lecture was delivered by Professor Norbert Wiener of the Massachusetts Institute of Technology on "The Mathematics of Sensory Prosthesis." Professor S. Chowla of the University of Kansas spoke on "The Riemann Zeta and Allied Functions," and Professor L. V. Ahlfors of Harvard University spoke on "The Classification of Open Riemann Surfaces."

Section A of the American Association for the Advancement of Science met on Tuesday afternoon, at which time Professor G. T. Whyburn of the University of Virginia delivered his retiring Vice-Presidential address: "The Open Mapping Medium in Topological Analysis."

During the week of December 26–31, the American Association for the Advancement of Science and its affiliated societies met in New York City, as did the American Economic Association and its affiliated societies. Among the other societies holding meetings in New York City during the week were: The American Statistical Association, The Econometric Society, The Institute of Mathematical Statistics, and The National Council of Teachers of Mathematics.

On Tuesday afternoon the American Statistical Association met at the Biltmore Hotel for a discussion of "The Pre-statistics Mathematics Course." Speakers were Professors Mark Kac of Cornell University and W. G. Madow of the University of Illinois.

Kappa Mu Epsilon, national honorary mathematics fraternity, held a luncheon meeting on Friday in Butler Hall.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the Meeting consisted of E. R. Lorch, chairman, Grace Bolton, T. F. Cope, H. M. Gehman, B. P. Gill, T. R. Hollcroft, S. B. Littauer, Walter Prenowitz.

Hotel headquarters for the mathematical organizations was the Hotel Governor Clinton. Registration and recreation headquarters were situated in the lobby of Earl Hall on the Columbia University campus.

A tea for members of the mathematical organizations and their guests was given by Columbia University on Tuesday afternoon from 4:00 to 6:00 in the Men's Faculty Club.

On Wednesday evening there was a Musicale in the Casa Italiana under the auspices of the Columbia University Music Department. Participants were E. S. Anderson, bass-baritone, and Harold Triggs, pianist.

A tour of the Museum of Modern Art was arranged for the ladies on the afternoon of Thursday.

A joint dinner for members of the mathematical organizations and their guests was held at 6:30 P.M. on Thursday in the Dining Room of John Jay Hall, Columbia University. Professor P. A. Smith acted as toastmaster. Vice-President G. B. Pegram of Columbia University welcomed the guests on behalf of the University. Professor J. L. Walsh described the plans for the 1950 International Congress of Mathematicians. Dr. H. M. MacNeille discussed the functions of his new office as Executive Director of the Society. Professor Jerzy

Neyman spoke on behalf of the Institute of Mathematical Statistics. On behalf of the Association, President R. E. Langer described some of the problems of collegiate mathematics, especially in connection with the training of teachers.

Through a motion prepared by Professor M. R. Hestenes, which was enthusiastically adopted by those present, the mathematical organizations expressed their sincere appreciation to Columbia University and its staff and to all others who assisted in the arrangements for the meetings. Everyone agreed with the concluding sentence of the motion: "From every point of view the meetings have been a complete success."

H. M. GEHMAN, *Secretary-Treasurer*

THE MAY MEETING OF THE KENTUCKY SECTION

The Kentucky Section of the Mathematical Association of America held its annual meeting at Centre College, Danville, Kentucky, on Saturday, May 14, 1949. The Chairman, Professor W. J. Robinson, Centre College, presided.

Sixty-two persons attended the meeting, including the following twenty members of the Association: D. F. Atkins, M. C. Brown, H. H. Downing, Clarence Ford, Aughtum S. Howard, Sister Charles Mary, W. L. Moore, D. J. Myatt, R. S. Park, D. W. Pugsley, F. M. Pulliam, S. L. Riggs, G. G. Roberts, W. J. Robinson, Florence V. Rohde, W. C. Royster, Klaus Schocken, D. E. South, Guy Stevenson, Jean M. Wyre.

The nominating committee presented the following slate of nominees: Chairman, Dr. H. H. Downing, University of Kentucky; Secretary-Treasurer, Dr. Aughtum S. Howard, Kentucky Wesleyan College. These were duly elected.

The following papers were presented at the meeting:

1. *From spherical trigonometry to plane trigonometry*, by Professor H. H. Downing, University of Kentucky.

Professor Downing showed how the formulas relating to a plane triangle could be derived from the formulas relating to a spherical triangle. The vertices of the spherical triangle were held fixed, and the center of the sphere was allowed to recede to infinity. The spherical triangle thus became in the limit a plane triangle, the sides (arcs) and the angles of the spherical triangle became the sides (straight line segments) and angles of the plane triangle. Use was made of simple theorems on limits including

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \quad \text{and} \quad \lim_{R \to \infty,\, \alpha \to 0} (R\alpha) = C$$

where R = the radius of the sphere, α = a central angle, and C is the chord subtending the angle α.

2. *The fourth dimension and the Bible*, by Professor Beulah Graham, Campbellsville College, introduced by the Secretary.

This paper constituted a review of the book *The Fourth Dimension and the Bible*, by Anthony Granville. It dealt with the concepts of higher spaces and their remarkable agreement with Bible passages, and emphasized the possibility that the higher-dimensional spaces are the heavens to which we ascend after death. Instead of being restricted to three degrees of freedom, we would be able to move in many independent directions.

3. *On pseudo-doubly periodic functions*, by Mr. D. F. Atkins, University of Kentucky.

The paper considers functions of a complex variable, $f(z)$, which satisfy the equations

$$f(z + w_i) = f(z) + h_i(z), \qquad (i = 1, 2). \tag{1}$$

Among others, the following theorems result: If $f_1(z)$ and $f_2(z)$ satisfy (1), then $f_1(z) - f_2(z)$ is an elliptic function. A necessary and sufficient condition for the existence of a solution is that

$$h_1(z + w_2) - h_1(z) = h_2(z + w_2) - h_2(z).$$

The equations (1) are transformed into $\phi(z+1) = \phi(z)$, $\phi(z+w) = \phi(z) + h(z)$, and a $\phi(z)$ which satisfies these equations is written. It is also shown that if $h(z)$ is entire and $\int_0^1 h(z)dz = 0$, the $\phi(z)$ is entire.

4. *The use of the analytical triangle in curve tracing*, by Miss Virginia Baskett, University of Kentucky, introduced by the Secretary.

The speaker discussed the use of the analytical triangle in sketching algebraic curves. Methods for finding the intercepts and asymptotes of the curve were illustrated, and the determination of approximations to the shape of the curve at the origin and at the points at infinity on the axes were discussed.

5. *Sequential analysis*, by Professor Guy Stevenson, University of Louisville.

This paper gave a brief account of the development of the sequential analysis method developed by Wald, Wolfowitz, and others, together with a short account of its principles applied to sampling rechniques.

6. *What are colleges doing to meet the needs of freshmen in mathematics?* by Sister M. Elizabeth Frisch, Villa Madonna College, introduced by the Secretary.

What colleges can do to meet the needs of freshmen in mathematics is a vital question for every mathematician and teacher. Students entering college today do not have the background in mathematics that was required of them twenty years ago. Hence colleges have been required to change the standards and to supply two alternatives: first, use a double-track plan, College Algebra for those whose ability and future outlook already indicate that they should take it, and a five hour course in general mathematics for the rest; and a second alternative, eliminate the requirements of mathematics for all freshmen. Entrance examinations and placement tests are of help in the proper grouping of students.

7. *Some uses of vectors in elementary courses in mathematics*, by Professor W. J. Robinson, Centre College.

This paper discussed the development of elementary courses in mathematics, and in particular, in analytic geometry, by the use of vectors.

SALLIE E. PENCE, *Secretary*

THE MAY MEETING OF THE NEBRASKA SECTION

The twenty-fifth annual meeting of the Nebraska section of the Mathematical Association of America was held at the University of Nebraska in Lincoln, Nebraska, on Saturday, May 7, 1949. Two sessions were held, at which Professor H. L. Rice presided.

Thirty-seven persons attended the meetings, including the following twenty members of the Association: M. A. Basoco, A. K. Bettinger, W. C. Brenke, C. C Camp, F. Marion Clarke, Helen E. Clarkson, H. M. Cox, J. M. Earl, M. G. Gaba, C. B. Gass, R. E. Heath, B. W. Jones, L. M. Larsen, W. G. Leavitt, E.

J. Lowry, Walter Lyche, Florence E. Poole, H. L. Rice, Lulu L. Runge, and C. J. Tegels.

At the business meeting held after the morning session the following were elected as officers for the coming year: Chairman, C. B. Gass, Nebraska Wesleyan University; Vice-Chairman, H. L. Rice, University of Omaha; Secretary, Lulu L. Runge, University of Nebraska. The meetings for next year are to be held on May 6, 1950, at Nebraska Wesleyan University, Lincoln, Nebraska.

Professor B. W. Jones of the University of Colorado was the guest speaker and presented papers at both the morning and afternoon sessions.

The following short papers were presented:

1. *On systems of linear differential equations*, by W. G. Leavitt, University of Nebraska.

Any system of linear differential equations may be written in matric form. The author showed the utility of this form both for its conciseness and for the opportunity it affords of employing many theorems from the abstract theory of matrices. As an illustration, the matrix notation of a linear transformation was given. Certain results were stated, also, from the asymptotic theory of differential equations involving a parameter.

2. *Approximation of a discontinuous function*, by Professor J. M. Earl, University of Omaha.

A general method of least mth power approximation by polynomials to a discontinuous function was illustrated by showing that if the function $f(x)$ has a finite discontinuity at $x=0$, then under proper restrictions on the continuity of $f(x)$ elsewhere in the interval $(-1, 1)$, the magnitude of the expression $x^2|f(x)-P_n(x)|$ does not exceed n^{-1} uniformly on $(-1, 1)$. The polynomial $P_n(x)$ is determined so as to minimize the integral of the square of that expression from -1 to 1. The method applies to the least mth power problem when the function $f(x)$ to be approximated has a finite number of discontinuities and the multiplier x^2 is replaced by a suitable restricted non-negative function.

3. *An application of continued fractions to solutions of equations*, by Prof. B. W. Jones, University of Colorado.

The applicable properties of simple and nearest integer continued fractions were developed. The technique of using such fractions for finding the solutions of equations was explained, and it was shown that use of the nearest integer continued fraction has some advantages over Horner's method.

4. *Simplified procedure for astrofix by commutation*, by Professor O. C. Collins, University of Nebraska, introduced by Prof. H. L. Rice.

Further simplification was reported in the process of obtaining a determinate "fix" from the observed altitudes of two stars. The appropriate intersection of the two circles of equal altitudes was found by computation. The arrangement of the solution is of such simplicity that it will offer an effective alternative to the St. Hilaire intercept procedure as soon as adequate tables of precomputed data are provided. The use of this method might be preferred in any situation where the navigator wished to avoid the plotting of intercepts and of lines of position together with the use of dead reckoning or of an assumed position.

5. *On a special class of topological operators of the form* $t(u\cdot v)$, by Edwin Halfar, University of Nebraska.

It was shown that certain operators of the form $t(u\cdot v)$, where t is of the form dcd, can be expressed as a sum $dcdu+dcdv$.

6. *On automorphisms of extension fields*, by F. Marion Clarke, University of Nebraska.

The partitioning of a finite, separable, normal extension of a ground field which is induced by the group of relative automorphisms of the extension field over the ground field was demonstrated, first with respect to subfields lying between the ground field and the extension field and corresponding to the subgroups of the automorphism group; and second, with respect to mutually exclusive classes of conjugate elements corresponding to co-sets of these subgroups. The classical theory for polynomials in one variable defining the field extension was generalized to polynomials in n variables.

7. *On a theorem of Liouville*, By W. T. Lenser, University of Nebraska, introduced by Professor H. L. Rice.

A theorem of Liouville concerning the integration in finite terms of an elementary function, with certain extensions and applications, was presented.

8. *Some applications of the finite Fourier transformation*, by M. D. Lamoree, University of Nebraska, introduced by Prof. H. L. Rice.

The speaker illustrated the use of the finite Fourier sine transformation by solving a boundary value problem involving one-dimensional flow of heat in a bar.

9. *Mathematical aesthetics*, by F. Marion Clarke, University of Nebraska.

The number ϕ is defined as the positive root of the equation $x^2=x+1$. Its role as an aesthetic measure alike for the art creator and the art critic is traced from the works of the Pythagoreans to those of a modern French school.

10. *Less about more or more about less*, by Professor B. W. Jones, University of Colorado.

The main ideas of this paper was that the primary aim of a teacher of mathematics should be to cultivate understanding and comprehension in his students. The urge should not be to cover a certain amount of material but to instill fundamental ideas. This tendency to teach less and less about more and more should be reversed. In mathematics and in life the emphasis seems to be on techniques—learning how to do things—rather than on appreciating the reasons for things and their meanings. The primary use of applications should be to cultivate this understanding. Let the student know fundamental principles, and he can make his own applications. In applying these ideas the teacher will notice that the student's methods of expression must not be scorned, and that individual differences must be provided for; such a program requires a superior teacher who will develop means of testing for understanding. An experiment for testing the validity of these proposals was suggested.

11. *Analysis of variance*, by Professor C. M. Harsh, Department of Psychology, University of Nebraska, introduced by Professor H. L. Rice.

Preliminary investigations in the sciences often seek evidence as to which of several possible factors is influencing measurements. If data is grouped according to suspected factors, it is easy to show by algebra that the total variance of scores can be analyzed into variance within group and variance between groups. From each component the variance of a parent population is estimated, and the diversity of these estimates is tested by Fisher's F-ratio for samples from a normal population. More complex designs permit similar tests of interaction between several factors.

12. *A norm for frequencies under the new grading system of the University of Nebraska*, by Professor C. C. Camp.

A previous study of all-university grades for 1938–41 predicted the percentages, 2.6, 9.5, 17.2., 20, 18.4, 13.6, 8.7, 5, 4.9 for the new grades 9, 8, 7, 6, 5, 4, 3, 2, 1 which now replace the old grade intervals of 95 or above, 90–94, 85–89, 80–84, 75–79, 70–74, 65–69, 60–64, and *F* (below passing). The group of 40342 grades for the first semester 1948–49 for courses giving from 1 to 5 hours credit showed the actual percentage frequencies of 2.7, 10.5, 17.8, 22.9, 18.4, 12.8, 6.5, 4,4.4, respectively. This distribution was fitted by a Charlier type *A* and Pearson types I, III. The last gave calculated frequencies of 3.3, 9.5, 17.9 21.8, 19.1, 13.7, 7.9, 4, 2.8. If grades are weighted by hours credit the skewness changes from −.5 to −.7, and the actual frequency percentages become 2.7, 10.4, 17.1, 22.2, 17.8, 13.1, 6.9, 4.6, 5.1, while the average grade changes from 5.45 to 5.38.

LULU L. RUNGE, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at Gustavus Adolphus College in St. Peter, Minnesota, on Saturday, May 8, 1949. Sessions were held in the forenoon, at luncheon, and in the afternoon. Professors H. M. Anderson, J. M. H. Olmsted and W. R. McEwen (Chairman of the Section) presided at the respective sessions.

Seventy-six persons attended the meeting, including the following fifty-two members of the Association: H. M. Anderson, F. J. Arena, H. H. Barnett, J. E. Bearman, S. Louise Beasley, Sister Ada Marie Boehm, K. H. Bracewell, W. H. Bussey, L. E. Bush, R. H. Cameron, E. J. Camp, C. S. Carlson, Elizabeth Carlson, Helen Engebretson, I. C. Fischer, Gladys Gibbens, Sister Mary Seraphim Gibbons, C. H. Gingrich, K. L. Hankerson, W. L. Hart, Charles Hatfield, Jr., W. C. Kalinowski, G. K. Kalisch, P. G. Kirmser, Sister M. Thomas a Kempis Kloyd, W. S. Loud, H. B. MacDougal, Kenneth May, H. C. Mayer, Jr., W. H. McBride, W. R. McEwen, A. G. Montgomery, W. D. Morgan, M. J. Norris, F. R. Ohnsorg, J. M. H. Olmsted, J. C. Peterson, G. C. Priester, Dorothy V. Schrader, Sister M. Leontius Schulte, L. W. Sheridan, F. C. Smith, A. H. Speltz, Esther R. Steinberg, A. G. Swanson, F. J. Taylor, Takashi Terami, Ella Thorp, H. L. Turrittin, O. E. Walder, K. W. Wegner, and Irene L. Wente.

The nominating committee presented the following slate of nominees for the coming year: Chairman, Kenneth May, Carleton College; Secretary, L. E. Bush, College of St. Thomas; Executive Committee, W. R. McEwen, Duluth Branch, University of Minnesota; R. C. Staley, University of North Dakota; Sister Mary Seraphim Gibbons, College of St. Catherine. These were duly elected.

At the business meeting the following resolutions were adopted: (1) That the Chairman of the Section appoint a committee to investigate the possibility of instituting and financing a high school mathematics contest under the sponsorship of the Section; (2) That the Chairman appoint a committee to cooperate with the Minnesota Council of Teachers of Mathematics, and if possible arrange for joint meetings of the Section with the Minnesota Council; (3) That the by-laws of the Section be amended so as to include North Dakota, Manitoba and Saskatchewan in the territory of the Section.

By invitation of the Executive Committee, Professor S. E. Warschawski delivered an address at the morning session. The title of his address was *Geo-*

metrical Aspects of Function Theory. Professor Warschawski discussed some of the classical problems of geometrical function theory, the Riemann mapping theorem, and the uniformization of algebraic functions. L. Bieberbach's variational method for determining the mapping function of a Jordan domain upon a circle by minimizing the area was outlined, and the convergence of the process was indicated. Simple examples illustrating the problem of uniformization of algebraic functions were given, and the use of conformal mapping in the solution of this problem was explained.

The following eight short papers were presented:

1. *An engineering application of a Volterra integral equation,* by Professor P. G. Kirmser, University of Minnesota.

In certain experiments it is sometimes necessary to measure phenomena occurring at some distance from the observer. One method of doing this is to convert the measurement into an electric signal which is then conducted through a wire to an oscilloscope, where the input electric signal is reproduced with some distortion, the magnitude of distortion depending on the conditions under which the oscilloscope is used. Since the observer sees only the distorted response function, he is interested in determining what was actually measured.

The input and response functions are related by a complicated linear differential equation with constant coefficients, the exact form of which it is impractical to obtain. However, if the response to a known input signal is known, the response to an arbitrary input signal can be calculated by superposition. The result is Duhamel's integral. In this case, however, it is the response that is known, and Duhamel's integral becomes a Volterra integral equation, which can be solved by a method of successive approximations.

2. *The probability that* $2+2=4$, by Professor W. S. Loud, University of Minnesota.

This paper deals with rounding-off error in addition. The result of the addition of n numbers is to be rounded off to a given number of decimal places. We ask the probability that the correct result is obtained if the addends are rounded off to that accuracy before addition. The addends are considered to have a uniform distribution. The result is independent of the particular accuracy used, so for purposes of analysis we may use nearest integer accuracy. As an example, for $n=2$, we would ask what is the probability that the sum of two numbers each between 1.5 and 2.5 should be between 3.5 and 4.5; that is, what is the probability that $2+2=4$. This is easily computed by geometric methods, and is found to be 0.75. The result for n numbers is obtained by Fourier transforms, and is $2\pi^{-1}\int_0^\infty (t^{-1}\sin t)^{n+1}dt$.

3. *Heart-shaped curves,* by Professor W. H. Bussey, University of Minnesota.

In the March 21, 1949 issue of *Life* magazine there was an article about Dr. Jekuthiel Ginsberg and *Scripta Mathematica.* One of the illustrations of "beauty in mathematics" was a heart-shaped curve. It has been copied from *Scripta Mathematica* which has taken it from El-Milick's *Elements d'Algebre Ornamentale. Life* said that the equation of the curve was $y=\sqrt{x^2}+\sqrt{a^2-x^2}$. This was obviously a misprint. It was corrected in a later issue of the magazine. Mr. Bussey said that, before he read that correction or looked for the correct equation elsewhere, he made two guesses as to what the equation was intended to be. The first one, which turned out to be correct, was $y=\sqrt[3]{x^2}\pm\sqrt{a^2-x^2}$. The graph of this equation looks just about like the ace of hearts of an ordinary deck of playing cards. The second guess, based partly on the assumption that the curve as it appeared in *Life* might not have been carefully drawn, was $y=|x|\pm\sqrt{a^2-x^2}$, with the absolute value of x as an interpretation of the positive square root of x^2. Mr. Bussey showed that

the graph of this equation is a heart-shaped curve; and he called it a "valentine heart" because it is decidedly more like the hearts on commercial valentine cards than is El-Milick's curve or the ordinary ace of hearts. Then he showed how it was related to certain ellipse, and said that the equation and similar equations related to other ellipses might well be used as formulas for the making of valentine hearts of somewhat different shapes.

4. *Some early mathematical and astronomical tables*, by Mr. W. D. Morgan, St. Paul, Minnesota.

The purpose of this paper was to describe briefly some of the mathematical and astronomical tables of the medieval and renaissance periods. The discussion was accompanied by an exhibit of several of these tables, both in type and manuscript. A very interesting item which was shown was Ptolemy's *Table of Chords*, published in the Leichtenstein edition of the *Almagest*, Venice, 1515. This is the earliest trigonometrical table to appear in print. There was also shown a printed edition of the *Alphonsine Tables*, Venice, 1483. This is the first of ten published editions of this work which appeared between the dates 1483 and 1649. A thirteenth century vellum manuscript of the *Toledo Tables* was also displayed. These tables antedated the *Alphonsine Tables* by about two centuries, but were never printed.

5. *The average sum of the digits of integers*, by Professors L. E. Bush and M. J. Norris, College of St. Thomas.

Let $S(N, r)$ be the sum of the digits, for radix r, of the non-negative integers less than N. Let $A(N, r)$ be $S(N, r)/N$. For fixed r, L. E. Bush has proved that $A(N, r)$ is asymptotic to $(r-1)\log N/2\log r$ (this MONTHLY, vol. 47, 1940, pp. 154–156). In this paper it is proved that the graph of $A(N, r)$ never goes above the chords for the curve $y=(r-1)\log N/2\log r$ which join the points whose abscissas are successive powers of r, and never goes below such chords by more than $r/8$. Modifications of the proof yield sharper limits than $r/8$. For example, for $r=10$ the maximum dip below the chords was reduced to 0.8972. Again for $r=10$, by actually locating the value of N between successive powers of 10 at which the maximum dip is obtained, the size of the dip can be shown to be less than 100/117. No better bound is possible.

6. *Axiomatic definitions of determinants*, by Professor G. K. Kalisch, University of Minnesota.

Various axiomatic definitions of determinants are discussed (cf. E. Artin, *Galois Theory*, 2nd ed., p. 11; C. C. MacDuffee, *Vectors and Matrices*, p. 50). In particular, the Weierstrass definition is proved to be equivalent to two coordinate free definitions in terms of linear transformations of a vector space $V_n(F)$ of n dimensions over a field F into itself. The first of the two definitions mentioned is $D(AB)=D(A)D(B)$, $D(aI)=a^nD(I)$ for a in F (this formulation is valid only for algebraically closed F; for general F the extensibility of D over $V_n(F)$ to $\overline{D}$ over $V_n(\overline{F})$ satisfies the equations above). The second definition is as follows: $D(AB)=D(A)D(B)$; $D(a)=a$ if $n=1$, for a in F; $D(A)=D(B)D(C)$ if A is the direct sum $B+C$. In both formulations, "trivial" solutions ($D\equiv 0$ and $D\equiv 1$) are to be excluded. In conclusion it is shown that the existence of a determinant function over $V_n(F)$ if F is a skew field implies the commutativity of F.

7. *Transformations in elementary analytic geometry*, by Professor Kenneth May, Carleton College.

The idea of a transformation of coördinates or a change of variables is fundamental in analytic geometry, and indeed in all analysis, but in the elementary courses it appears rather late as a device for graphing the general conic. Since the formulas for translation follow immediately from the definition of coördinates, and those for rotation require nothing more than trigonometry, transformations could be introduced early in the course and used to simplify proofs and deepen the student's appreciation of the subject. For example, it is easy to prove that by appropriate transla-

tion and rotation any linear equation in x and y can be reduced to either $x=0$ or $y=0$. Then its locus is a straight line, since it is congruent to an axis. This observation indicates that the usual proofs based on the constancy of the slope are redundant. Having proved by transformations that the locus of a linear equation is a straight line, it can easily be shown that any straight line has an equation of this form and that the slope of a straight line is a constant. Of course, the treatment by transformations brings out automatically the relations between the coefficients of the equation and the position of the line.

8. *A graphic approach to a problem in multiple regression*, by Professor L. W. Sheridan, St. Thomas College.

A variable X admitting to one of two alternatives, may be estimated graphically by plotting pairs of observed values of the independent variables, and indicating which of the two alternatives occurred. The data in each such scattergram may then be grouped into cells having any convenient number of observations, and probabilities computed for each cell. Based on the observed probabilities, contour lines then drawn give secondary variables, such as $Y_1=f(X_1, X_2)$, $Y_2=f(X_3, X_4)$, etc. A single variable is obtained by plotting a pair of Y values in the same manner, and continuing the process. This single variable, when plotted against observed values of X, and represented by a smooth curve, may be used to estimate the value of X sought in the problem. An advantage of this method is that it gives a measure of the probability of occurrence of the event.

L. E. BUSH, *Secretary*

CALENDAR OF FUTURE MEETINGS

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December 30, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Southern Illinois University, Carbondale, May 12–13, 1950.

INDIANA, Wabash College, Crawfordsville, April 29, 1950.

IOWA, State University of Iowa, Iowa City, April 21–22, 1950.

KANSAS, April 22, 1950.

KENTUCKY, University of Kentucky, Lexington, April 29, 1950.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Richmond, May, 1950.

METROPOLITAN NEW YORK, City College, April 1, 1950.

MICHIGAN, University of Michigan, Ann Arbor, March 25, 1950.

MINNESOTA, Macalester College, St. Paul, May 6, 1950.

MISSOURI, Washington University, St. Louis, March 24–25, 1950.

NEBRASKA, Nebraska Wesleyan University, Lincoln, May 6, 1950.

NORTHERN CALIFORNIA

OHIO, Denison University, Granville, April 22, 1950.

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PACIFIC NORTHWEST, University of Washington, Seattle, June, 1950.

PHILADELPHIA

ROCKY MOUNTAIN, University of Denver, April 28–29, 1950.

SOUTHEASTERN, University of Florida, Gainesville, April 7–8, 1950.

SOUTHERN CALIFORNIA, Immaculate Heart College, Hollywood, March 11, 1950.

SOUTHWESTERN, Spring, 1950.

TEXAS, Abilene, April 14–15, 1950.

UPPER NEW YORK STATE, Syracuse University, April 22, 1950.

WISCONSIN, Marquette University, Milwaukee, May 13, 1950.

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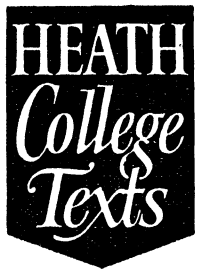
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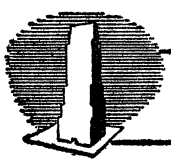
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APRIL 1950

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THE ARCHITECTURE OF MATHEMATICS*

NICHOLAS BOURBAKI†

1. Mathematic or mathematics? To present a view of the entire field of mathematical science as it exists,—this is an enterprise which presents, at first sight, almost insurmountable difficulties, on account of the extent and the varied character of the subject. As is the case in all other sciences, the number of mathematicians and the number of works devoted to mathematics have greatly increased since the end of the 19th century. The memoirs in pure mathematics published in the world during a normal year cover several thousands of pages. Of course, not all of this material is of equal value; but, after full allowance has been made for the unavoidable tares, it remains true nevertheless that mathematical science is enriched each year by a mass of new results, that it spreads and branches out steadily into theories, which are subjected to modifications based on new foundations, compared and combined with one another. No mathematician, even were he to devote all his time to the task, would be able to follow all the details of this development. Many mathematicians take up quarters in a corner of the domain of mathematics, which they do not intend to leave; not only do they ignore almost completely what does not concern their special field, but they are unable to understand the language and the terminology used by colleagues who are working in a corner remote from their own. Even among those who have the widest training, there are none who do not feel lost in certain regions of the immense world of mathematics; those who, like Poincaré or Hilbert, put the seal of their genius on almost every domain, constitute a very great exception even among the men of greatest accomplishment.

It must therefore be out of the question to give to the uninitiated an exact picture of that which the mathematicians themselves can not conceive in its totality. Nevertheless it is legitimate to ask whether this exuberant proliferation makes for the development of a strongly constructed organism, acquiring ever greater cohesion and unity with its new growths, or whether it is the external manifestation of a tendency towards a progressive splintering, inherent in the very nature of mathematics, whether the domain of mathematics is not becoming a tower of Babel, in which autonomous disciplines are being more and more widely separated from one another, not only in their aims, but also in their methods and even in their language. In other words, do we have today a mathematic or do we have several mathematics?

Although this question is perhaps of greater urgency now than ever before, it is by no means a new one; it has been asked almost from the very beginning of mathematical science. Indeed, quite apart from applied mathematics, there has

* Authorized translation by Arnold Dresden of a chapter in "Les grands courants de la pensée mathématique," edited by F. Le Lionnais (Cahiers du Sud, 1948).

† "Professor N. Bourbaki, formerly of the Royal Poldavian Academy, now residing in Nancy, France, is the author of a comprehensive treatise of modern mathematics, in course of publication under the title *Eléments de Mathématique* (Hermann et Cie, Paris 1939–), of which ten volumes have appeared so far."

always existed a dualism between the origins of geometry and of arithmetic (certainly in their elementary aspects), since the latter was at the start a science of discrete magnitude, while the former has always been a science of continuous extent; these two aspects have brought about two points of view which have been in opposition to each other since the discovery of irrationals. Indeed, it is exactly this discovery which defeated the first attempt to unify the science, *viz.*, the arithmetization of the Pythagoreans ("everything is number").

It would carry us too far if we were to attempt to follow the vicissitudes of the unitary conception of mathematics from the period of Pythagoras to the present time. Moreover this task would suit a philosopher better than a mathematician; for it is a common characteristic of the various attempts to integrate the whole of mathematics into a coherent whole—whether we think of Plato, of Descartes or of Leibnitz, of arithmetization, or of the logistics of the 19th century—that they have all been made in connection with a philosophical system, more or less wide in scope; always starting from *a priori* views concerning the relations of mathematics with the twofold universe of the external world and the world of thought. We can do no better on this point than to refer the reader to the historical and critical study of L. Brunschvicg [1]. Our task is a more modest and a less extensive one; we shall not undertake to examine the relations of mathematics to reality or to the great categories of thought; we intend to remain within the field of mathematics and we shall look for an answer to the question which we have raised, by analyzing the procedures of mathematics themselves.

2. Logical formalism and axiomatic method. After the more or less evident bankruptcy of the different systems, to which we have referred above, it looked, at the beginning of the present century as if the attempt had just about been abandoned to conceive of mathematics as a science characterized by a definitely specified purpose and method; instead there was a tendency to look upon mathematics as "a collection of disciplines based on particular, exactly specified concepts," interrelated by "a thousand roads of communication," allowing the methods of any one of these disciplines to fertilize one or more of the others [1, page 447]. Today, we believe however that the internal evolution of mathematical science has, in spite of appearance, brought about a closer unity among its different parts, so as to create something like a central nucleus that is more coherent than it has ever been. The essential aspect of this evolution has been the systematic study of the relations existing between different mathematical théories, and which has led to what is generally known as the "axiomatic method."

The words "formalism" and "formalistic method" are also often used; but it is important to be on one's guard from the start against the confusion which may be caused by the use of these ill-defined words, and which is but too frequently made use of by the opponents of the axiomatic method. Everyone knows that superficially mathematics appears as this "long chain of reasons" of

which Descartes spoke; every mathematical theory is a concatenation of propositions, each one derived from the preceding ones in conformity with the rules of a logical system, which is essentially the one codified, since the time of Aristotle, under the name of "formal logic," conveniently adapted to the particular aims of the mathematician. It is therefore a meaningless truism to say that this "deductive reasoning" is a unifying principle for mathematics. So superficial a remark can certainly not account for the evident complexity of different mathematical theories, not any more than one could, for example, unite physics and biology into a single science on the ground that both use the experimental method. The method of reasoning by means of chains of syllogisms is nothing but a transforming mechanism, applicable just as well to one set of premises as to another; it could not serve therefore to characterize these premises. In other words, it is the external form which the mathematician gives to his thought, the vehicle which makes it accessible to others,* in short, the language suited to mathematics; this is all, no further significance should be attached to it. To lay down the rules of this language, to set up its vocabulary and to clarify its syntax, all that is indeed extremely useful; indeed this constitutes one aspect of the axiomatic method, the one that can properly be called logical formalism (or "logistics" as it is sometimes called). But we emphasize that it is but one aspect of this method, indeed the least interesting one.

What the axiomatic method sets as its essential aim, is exactly that which logical formalism by itself can not supply, namely the profound intelligibility of mathematics. Just as the experimental method starts from the *a priori* belief in the permanence of natural laws, so the axiomatic method has its cornerstone in the conviction that, not only is mathematics not a randomly developing concatenation of syllogisms, but neither is it a collection of more or less "astute" tricks, arrived at by lucky combinations, in which purely technical cleverness wins the day. Where the superficial observer sees only two, or several, quite distinct theories, lending one another "unexpected support" [1, page 446] through the intervention of a mathematician of genius, the axiomatic method teaches us to look for the deep-lying reasons for such a discovery, to find the common ideas of these theories, buried under the accumulation of details properly belonging to each of them, to bring these ideas forward and to put them in their proper light.

3. The notion of structure. In what form can this be done? It is here that the axiomatic method comes closest to the experimental method. Like the latter drawing its strength from the source of Cartesianism, it will "divide the difficulties in order to overcome them better." It will try, in the demonstrations of a theory, to separate out the principal mainsprings of its arguments; then, taking each of these separately and formulating it in abstract form, it will develop

* Indeed every mathematician knows that a proof has not really been "understood" if one has done nothing more than verifying step by step the correctness of the deductions of which it is composed, and has not tried to gain a clear insight into the ideas which have led to the construction of this particular chain of deductions in preference to every other one.

the consequences which follow from it alone. Returning after that to the theory under consideration, it will recombine the component elements, which had previously been separated out, and it will inquire how these different components influence one another. There is indeed nothing new in this classical going to-and-fro between analysis and synthesis; the originality of the method lies entirely in the way in which it is applied.

In order to illustrate the procedure which we have just sketched, by an example, we shall take one of the oldest (and also one of the simplest) of axiomatic theories, *viz.* that of the "abstract groups." Let us consider for example, the three following operations: 1. the addition of real numbers, their sum (positive negative or zero) being defined in the usual manner; 2. the multiplication of integers "modulo a prime number p," (where the elements under consideration are the whole numbers $1, 2, \cdots, p-1$) and the "product" of two of these numbers is, by agreement, defined as the remainder of the division of their usual product by p; 3. the "composition" of displacements in three-dimensional Euclidean space, the "resultant" (or "product") of two displacements S, T (taken in this order) being defined as the displacement obtained by carrying out first the displacement T and then the displacement S. In each of these three theories, one makes correspond, by means of a procedure defined for each theory, to two elements x, y (taken in that order) of the set under consideration (in the first case the set of real numbers, in the second the set of numbers $1, 2, \cdots, p-1$, in the third the set of all displacements) a well-determined third element; we shall agree to designate this third element in all three cases by $x\tau y$ (this will be the sum of x and y if x and y are real numbers, their product "modulo p" if they are integers $\leqq p-1$, their resultant if they are displacements). If we now examine the various properties of this "operation" in each of the three theories, we discover a remarkable parallelism; but, in each of the separate theories, the properties are interconnected, and an analysis of their logical connections leads us to select a small number of them which are independent (*i.e.*, none of them is a logical consequence of all the others). For example,* one can take the three following, which we shall express by means of our symbolic notation, common to the three theories, but which it would be very easy to translate into the particular language of each of them:

(a) For all elements x, y, z, one has $x\tau(y\tau z)=(x\tau y)\tau z$ ("associativity" of the operation $x\tau y$);

(b) There exists an element e, such that for every element x, one has $e\tau x=x\tau e=x$ (for the addition of real numbers, it is the number 0; for multiplication "modulo p," it is the number 1; for the composition of displacements, it is the "identical" displacement, which leaves every point of space fixed);

(c) Corresponding to every element x, there exists an element x' such that $x\tau x'=x'\tau x=e$ (for the addition of real numbers x' is the number $-x$; for the

* There is nothing absolute in this choice; several systems of axioms are known which are "equivalent" to the one which we are stating explicitly, the axioms of each of these systems being logical consequences of the axioms of any other one.

composition of displacements, x' is the "inverse" displacement of x, *i.e.* the displacement which replaces each point that had been displaced by x to its original position; for multiplication "modulo p," the existence of x' follows from a very simple arithmetic argument.*

It follows then that the properties which can be expressed in the same way in the three theories, by means of the common notation, are consequences of the three preceding ones. Let us try to show, for example that from $x\tau y = x\tau z$ follows $y = z$; one could do this in each of the theories by a reasoning peculiar to it. But, we can proceed as follows by a method that is applicable in all cases: from the relation $x\tau y = x\tau z$ we derive (x' having the meaning which was defined above) $x'\tau(x\tau y) = x'\tau(x\tau z)$; thence by applying (a), $(x'\tau x)\tau y = (x'\tau x)\tau z$; by means of (c), this relation takes the form $e\tau y = e\tau z$, and finally, by applying (b), $y = z$, which was to be proved. In this reasoning the nature of the elements x, y, z under consideration has been left completely out of account; we have not been concerned to know whether they are real numbers, or integers $\leqq p-1$, or displacements; the only premise that was of importance was that the operation $x\tau y$ on these elements has the properties (a), (b), and (c). Even if it were only to avoid irksome repetitions, it is readily seen that it would be convenient to develop once and for all the logical consequences of the three properties (a), (b), (c) only. For linguistic convenience, it is of course desirable to adopt a common terminology for the three sets. One says that a set in which an operation $x\tau y$ has been defined which has the three properties (a), (b), (c) is provided with a group structure (or, briefly, that it is a group); the properties (a), (b), (c) are called the axioms of** the group structures, and the development of their consequences constitutes setting up the axiomatic theory of groups.

It can now be made clear what is to be understood, in general, by a mathematical structure. The common character of the different concepts designated by this generic name, is that they can be applied to sets of elements whose nature† has not been specified; to define a structure, one takes as given one or

* We observe that the remainders left when the numbers $x, x^2, \cdots, x^n, \cdots$ are divided by p, can not all be distinct; by expressing the fact that two of these remainders are equal, one shows easily that a power x^m of x exists which has a remainder equal to 1; if now x' is the remainder of the division of x^{m-1} by p, we conclude that the product "modulo p" of x and x' is equal to 1.

** It goes without saying that there is no longer any connection between this interpretation of the word "axiom" and its traditional meaning of "evident truth."

† We take here a naive point of view and do not deal with the thorny questions, half philosophical, half mathematical, raised by the problem of the "nature" of the mathematical "beings" or "objects." Suffice it to say that the axiomatic studies of the nineteenth and twentieth centuries have gradually replaced the initial pluralism of the mental representation of these "beings"—thought of at first as ideal "abstractions" of sense experiences and retaining all their heterogeneity—by an unitary concept, gradually reducing all the mathematical notions, first to the concept of the natural number and then, in a second stage, to the notion of set. This latter concept, considered for a long time as "primitive" and "undefinable," has been the object of endless polemics, as a result of its extremely general character and on account of the very vague type of mental representation which it calls forth; the difficulties did not disappear until the notion of set itself disappeared (and with it all the metaphysical pseudo-problems concerning mathematical "beings" in the light of the recent work on logical formalism. From this new point of view, mathematical

several relations, into which these elements enter* (in the case of groups, this was the relation $z=xry$ between three arbitrary elements); then one postulates that the given relation, or relations, satisfy certain conditions (which are explicitly stated and which are the axioms of the structure under consideration.)† To set up the axiomatic theory of a given structure, amounts to the deduction of the logical consequences of the axioms of the structure, excluding every other hypothesis on the elements under consideration (in particular, every hypotheses as to their own nature).

4. The great types of structures. The relations which form the starting point for the definition of a structure can be of very different characters. The one which occurs in the group structure is what one calls a "law of composition," *i.e.*, a relation between three elements which determines the third uniquely as a function of the first two. When the relations which enter the definition of a structure are "laws of composition," the corresponding structure is called an algebraic structure (for example, a field structure is defined by two laws of composition, with suitable axioms: the addition and multiplication of real numbers define a field structure on the set of these numbers).

Another important type is furnished by the structures defined by an order relation; this is a relation between two elements x, y which is expressed most frequently in the form "x is at most equal to y," and which we shall represent in general by $x\mathrm{R}y$. It is not at all supposed here that it determines one of the two elements x, y uniquely as a function of the other; the axioms to which it is subjected are the following: (a) for every x we have $x\mathrm{R}x$; (b) from the relations $x\mathrm{R}y$ and $y\mathrm{R}x$ follows $x=y$; (c) the relations $x\mathrm{R}y$ and $y\mathrm{R}z$ have as a consequence $x\mathrm{R}z$. An obvious example of a set with a structure of this kind is the set of integers (or that of real numbers), when the symbol R is replaced by the symbol $\leqq$. But it must be observed that we have not included among the axioms the following property, which seems to be inseparable from the popular notion of "order," "for every pair of elements x and y, either $x\mathrm{R}y$ or $y\mathrm{R}x$ holds." In other words, the case in which x and y are incomparable is not excluded. This may seem paradoxical at first sight, but it is easy to give examples of very important order structures, in which such a phenomenon appears. This is what happens when X and Y denote parts of the same set and the relation $X\mathrm{R}Y$ is interpreted to mean "X is contained in Y"; again when x and y are positive integers and $x\mathrm{R}y$ means

structures become, properly speaking, the only "objects" of mathematics. The reader will find fuller developments of this point in articles by J. Dieudonné [2] and H. Cartan [3].

* In effect, this definition of structures is not sufficiently general for the needs of mathematics; it is also necessary to consider the case in which the relations which define a structure hold not between elements of the set under consideration, but also between parts of this set and even, more generally, between elements of sets of still higher "degree" in the terminology of the "hierarchy of types." For further details on this point, see [4].

† Strictly speaking, one should, in the case of groups, count among the axioms, besides properties (a), (b), (c) stated above, the fact that the relation $z=xry$ determines one and only one z when x and y are given; one usually considers this property as tacitly implied by the form in which the relation is written.

"x divides y"; also if $f(x)$ and $g(x)$ are real-valued functions defined on an interval $a \leqq x \leqq b$, while $f(x)\mathrm{R}g(x)$ is interpreted to mean "for every x, $f(x) \leqq g(x)$." These examples also give an indication of the great variety of domains in which order structures appear and thus point to the interest attached to their study.

We want to say a few words about a third large type of structures, *viz.* topological structures (or topologies): they furnish an abstract mathematical formulation of the intuitive concepts of neighborhood, limit and continuity, to which we are led by our idea of space. The degree of abstraction required for the formulation of the axioms of such a structure is decidedly greater than it was in the preceding examples; the character of the present article makes it necessary to refer interested readers to special treatises. See, for example, [5].

5. The standardization of mathematical technique. We have probably said enough to enable the reader to form a fairly accurate idea of the axiomatic method. It should be clear from what precedes that its most striking feature is to effect a considerable economy of thought. The "structures" are tools for the mathematician; as soon as he has recognized among the elements, which he is studying, relations which satisfy the axioms of a known type, he has at his disposal immediately the entire arsenal of general theorems which belong to the structures of that type. Previously, on the other hand, he was obliged to forge for himself the means of attack on his problems; their power depended on his personal talents and they were often loaded down with restrictive hypotheses, resulting from the peculiarities of the problem that was being studied. One could say that the axiomatic method is nothing but the "Taylor system" for mathematics.

This is however, a very poor analogy; the mathematician does not work like a machine, nor as the workingman on a moving belt; we can not over-emphasize the fundamental role played in his research by a special intuition,* which is not the popular sense-intuition, but rather a kind of direct divination (ahead of all reasoning) of the normal behavior, which he seems to have the right to expect of mathematical beings, with whom a long acquaintance has made him as familiar as with the beings of the real world. Now, each structure carries with it its own language, freighted with special intuitive references derived from the theories from which the axiomatic analysis described above has derived the structure. And, for the research worker who suddenly discovers this structure in the phenomena which he is studying, it is like a sudden modulation which orients at one stroke in an unexpected direction the intuitive course of his thought, and which illumines with a new light the mathematical landscape in which he is moving about. Let us think—to take an old example—of the progress made at the beginning of the nineteenth century by the geometric representation of imaginaries. From our point of view, this amounted to discovering in the set of complex numbers a well-known topological structure, that of the Euclidean plane, with all the possibilities for applications which this in-

* Like all intuitions, this one also is frequently wrong.

volved; in the hands of Gauss, Abel, Cauchy and Riemann, it gave new life to analysis in less than a century. Such examples have occurred repeatedly during the last fifty years; Hilbert space, and more generally, functional spaces, establishing topological structures in sets whose elements are no longer points, but functions; the theory of the Hensel p-adic numbers, where, in a still more astounding way, topology invades a region which had been until then the domain *par excellence* of the discrete, of the discontinuous, *viz.* the set of whole numbers; Haar measure, which enlarged enormously the field of application of the concept of integral, and made possible a very profound analysis of the properties of continuous groups;—all of these are decisive instances of mathematical progress, of turning points at which a stroke of genius brought about a new orientation of a theory, by revealing the existence in it of a structure which did not *a priori* seem to play a part in it.

What all this amounts to is that mathematics has less than ever been reduced to a purely mechanical game of isolated formulas; more than ever does intuition dominate in the genesis of discoveries. But henceforth, it possesses the powerful tools furnished by the theory of the great types of structures; in a single view, it sweeps over immense domains, now unified by the axiomatic method, but which were formerly in a completely chaotic state.

6. A general survey. Let us now try, guided by the axiomatic concept, to look over the whole of the mathematical universe. It is clear that we shall no longer recognize the traditional order of things, which, just like the first nomenclatures of animal species, restricted itself to placing side by side the theories which showed greatest external similarity. In place of the sharply bounded compartments of algebra, of analysis, of the theory of numbers, and of geometry, we shall see, for example, that the theory of prime numbers is a close neighbor of the theory of algebraic curves, or, that Euclidean geometry borders on the theory of integral equations. The organizing principle will be the concept of a hierarchy of structures, going from the simple to the complex, from the general to the particular.

At the center of our universe are found the great types of structures, of which the principal ones were mentioned above; they might be called the mother-structures. A considerable diversity exists in each of these types; one has to distinguish between the most general structure of the type under consideration, with the smallest number of axioms, and those which are obtained by enriching the type with supplementary axioms, from each of which comes a harvest of new consequences. Thus, the theory of groups contains, beyond the general conclusions valid for all groups and depending only on the axioms enunciated above, a particular theory of finite groups (obtained by adding the axiom that the number of elements of the group is finite), a particular theory of abelian groups (in which $x\tau y = y\tau x$ for every x and y), as well as a theory of finite abelian groups (where these two axioms are supposed to hold simultaneously). Similarly, in the theory of ordered sets, one notices in particular those sets

(as for example, the set of integers, or of real numbers) in which any two elements are comparable, and which are called totally ordered. Among the latter, further attention is given to the sets which are called well-ordered (in which, as in the set of integers greater than 0, every subset has a "least element"). There is an analogous gradation among topological structures.

Beyond this first nucleus, appear the structures which might be called multiple structures. They involve two or more of the great mother-structures simultaneously not in simple juxtaposition (which would not produce anything new), but combined organically by one or more axioms which set up a connection between them. Thus, one has topological algebra. This is a study of structures in which occur at the same time, one or more laws of composition and a topology, connected by the condition that the algebraic operations be (for the topology under consideration) continuous functions of the elements on which they operate. Not less important is algebraic topology, in which certain sets of points in space, defined by topological properties (simplexes, cycles, etc.) are themselves taken as elements on which laws of composition operate. The combination of order structures and algebraic structures is also fertile in results, leading, in one direction to the theory of divisibility and of ideals, and in another to integration and to the "spectral theory" or operators, in which topology also joins in.

Farther along we come finally to the theories properly called particular. In these the elements of the sets under consideration, which, in the general structures have remained entirely indeterminate, obtain a more definitely characterized individuality. At this point we merge with the theories of classical mathematics, the analysis of functions of a real or complex variable, differential geometry, algebraic geometry, theory of numbers. But they have no longer their former autonomy; they have become crossroads, where several more general mathematical structures meet and react upon one another.

To maintain a correct perspective, we must at once add to this rapid sketch, the remark that it has to be looked upon as only a very rough approximation of the actual state of mathematics, as it exists; the sketch is schematic, and idealized as well as frozen.

Schematic—because in the actual procedures, things do not happen in as simple and as systematic a manner as has been described above. There occur, among other things, unexpected reverse movements, in which a specialized theory, such as the theory of real numbers, lends indispensable aid in the construction of a general theory like topology or integration.

Idealized—because it is far from true that in all fields of mathematics, the role of each of the great structures is clearly recognized and marked off; in certain theories (for example in the theory of numbers), there remain numerous isolated results, which it has thus far not been possible to classify, nor to connect in a satisfactory way with known structures.

Finally *frozen*,—for nothing is farther from the axiomatic method than a static conception of the science. We do not want to lead the reader to think that we claim to have traced out a definitive state of the science. The structures

are not immutable, neither in number nor in their essential contents. It is quite possible that the future development of mathematics may increase the number of fundamental structures, revealing the fruitfulness of new axioms, or of new combinations of axioms. We can look forward to important progress from the invention of structures, by considering the progress which has resulted from actually known structures. On the other hand, these are by no means finished edifices; it would indeed be very surprising if all the essence had already been extracted from their principles. Thus, with these indispensable qualifications, we can become better aware of the internal life of mathematics, of its unity as well as of its diversity. It is like a big city, whose outlying districts and suburbs encroach incessantly, and in a somewhat chaotic manner, on the surrounding country, while the center is rebuilt from time to time, each time in accordance with a more clearly conceived plan and a more majestic order, tearing down the old sections with their labyrinths of alleys, and projecting towards the periphery new avenues, more direct, broader and more commodious.

7. Return to the past and conclusion. The concept which we have tried to present in the above paragraphs, was not formed all at once; rather is it a stage in an evolution, which has been in progress for more than a half-century, and which has not escaped serious opposition, among philosophers as well as among mathematicians themselves. Many of the latter have been unwilling for a long time to see in axiomatics anything else than futile logical hairsplitting not capable of fructifying any theory whatever. This critical attitude can probably be accounted for by a purely historical accident. The first axiomatic treatments and those which caused the greatest stir (those of arithmetic by Dedekind and Peano, those of Euclidean geometry by Hilbert) dealt with univalent theories, *i.e.*, theories which are entirely determined by their complete system of axioms; for this reason they could not be applied to any theory except the one from which they had been extracted (quite contrary to what we have seen, for instance, for the theory of groups). If the same had been true for all other structures, the reproach of sterility brought against the axiomatic method, would have been fully justified.* But the further development of the method has revealed its power; and the repugnance which it still meets here and there, can only be explained by the natural difficulty of the mind to admit, in dealing with a concrete problem, that a form of intuition, which is not suggested directly by the given elements (and which often can be arrived at only by a higher and frequently difficult stage of abstraction), can turn out to be equally fruitful.

As concerns the objections of the philosophers, they are related to a domain, on which for reasons of inadequate competence we must guard ourselves from

* There also occurred, especially at the beginning of axiomatics, a whole crop of monster-structures, entirely without applications; their sole merit was that of showing the exact bearing of each axiom, by observing what happened if one omitted or changed it. There was of course a temptation to conclude that these were the only results that could be expected from the axiomatic method.

entering; the great problem of the relations between the empirical world and the mathematical world.* That there is an intimate connection between experimental phenomena and mathematical structures, seems to be fully confirmed in the most unexpected manner by the recent discoveries of contemporary physics. But we are completely ignorant as to the underlying reasons for this fact (supposing that one could indeed attribute a meaning to these words) and we shall perhaps always remain ignorant of them. There certainly is one observation which might lead the philosophers to greater circumspection on this point in the future: before the revolutionary developments of modern physics, a great deal of effort was spent on trying to derive mathematics from experimental truths, especially from immediate space intuitions. But, on the one hand, quantum physics has shown that this macroscopic intuition of reality covered microscopic phenomena of a totally different nature, connected with fields of mathematics which had certainly not been thought of for the purpose of applications to experimental science. And, on the other hand, the axiomatic method has shown that the "truths" from which it was hoped to develop mathematics, were but special aspects of general concepts, whose significance was not limited to these domains. Hence it turned out, after all was said and done, that this intimate connection, of which we were asked to admire the harmonious inner necessity, was nothing more than a fortuitous contact of two disciplines whose real connections are much more deeply hidden than could have been supposed *a priori*.

From the axiomatic point of view, mathematics appears thus as a storehouse of abstract forms—the mathematical structures; and it so happens—without our knowing why—that certain aspects of empirical reality fit themselves into these forms, as if through a kind of preadaptation. Of course, it can not be denied that most of these forms had originally a very definite intuitive content; but, it is exactly by deliberately throwing out this content, that it has been possible to give these forms all the power which they were capable of displaying and to prepare them for new interpretations and for the development of their full power.

It is only in this sense of the word "form" that one can call the axiomatic method a "formalism." The unity which it gives to mathematics is not the armor of formal logic, the unity of a lifeless skeleton; it is the nutritive fluid of an organism at the height of its development, the supple and fertile research instrument to which all the great mathematical thinkers since Gauss have contributed, all those who, in the words of Lejeune-Dirichlet, have always labored to "substitute ideas for calculations."

* We do not consider here the objections which have arisen from the application of the rules of formal logic to the reasoning in axiomatic theories; these are connected with logical difficulties encountered in the theory of sets. Suffice it to point out that these difficulties can be overcome in a way which leaves neither the slightest qualms nor any doubt as to the correctness of the reasoning; [2] and [3] are valuable references for this point.

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THE GEOMETRIES OF THE THERMAL AND GRAVITATIONAL FIELDS*

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1. Introduction. In physics, geometry is—or should be—what you measure it. Deviations from the geometry or kinematics you expect to find, on *a priori* or other grounds, may lead to the discovery or better understanding of some hitherto overlooked element in the realm under examination. To such deviations may be attributed the discovery by the ancients of the curvature of the earth's surface, by Kepler of the laws of planetary motion, and by Einstein of the special and general theories of relativity. Perhaps the most spectacular in its impact on the modern mind, because we are still immersed in the intellectual milieu in which it arose, is Einstein's geometrization of the universal force of gravitation. Here the recognition of the appropriateness of a certain Riemannian kinematics made possible the incorporation of gravitation into the geometrical structure of space-time, and by identification of inertial and gravitational mass made tautological the law which states their equivalence.

To illustrate the relationship between geometry and physical law, I have elected here first to discuss in some detail the problem of the geometry exhibited by a "hot plate." We shall find, as could be foreseen, that the geometry disclosed by measurement depends not on the state of the plate alone, but also on the method of measurement, *e.g.*, whether by purely optical means or by the use of measuring rods which are brought into thermal equilibrium with the plate before being read. The arbitrariness (within, however, fairly closely prescribed limits) of the geometry thus found will support us in the common-sense view that the plate is actually flat, and that the observed deviations are to be attributed to quite well understood forces (thermal stresses) which act differ-

* Based on an address "Geometry and Physical Space" before the Southern California Sections of the Mathematical Association of America and the American Association of Physics Teachers, held at the University of Redlands, Redlands, California, on March 13, 1948.

entially on the various instruments which may be employed in the measurement.

Leaving the rather prosaic domain of common sense, we next consider the extension of this problem to the 3-dimensional thermal field. Here it will be suggested that the physical laws of heat conduction may (at least approximately) be given a purely geometrical formulation. But again the strong dependence of the geometry thus disclosed on the particular method of measurement will prevent us from taking our proposed theory too seriously.

Finally, we shall contrast the geometry of the thermal field with the geometry of the gravitational field in order to see why gravitational, as opposed to thermal, forces may satisfactorily be incorporated into the geometrical structure of space and time. This comparison will be carried out in terms of a sort of special relativity caricature of the general relativity theory of gravitation, which bears a striking formal relationship to the thermal geometry—and which, I hold, presents the elements of the Einstein theory necessary to our purposes without involving us in the mathematical intricacies of the full theory.*

2. The hot plate. Let us suppose we have before us a flat sheet of metal, of uniform thickness t, and that we can by judicious use of heat sources, constant temperature baths and the like produce on the plate any temperature distribution consistent with the laws of heat conduction. Let us further suppose the plate is so constrained that it cannot stretch or buckle, despite the thermal stresses induced in it by the treatment, so that it can be said to remain flat by ordinary laboratory criteria.

Confining our attention to stationary states, and considering heat losses from the free faces of the plate as negligible, the temperature $T(x, y)$ at any point $P(x, y)$ of the plate is governed by the equation

$$k\nabla^2 T + p = 0, \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{1.1}$$

k is the thermal conductivity of the plate material and $p(x, y)$ is the source density of heat at $P(x, y)$, *i.e.*, $ptdxdy$ is the rate at which heat is being supplied to the plate through an area $dxdy$ about P. Heat flows past P in the direction of the negative gradient $-\nabla T$ of T, and the rate of flow across unit length normal to this direction is t times the magnitude of the flux vector

$$\mathbf{f} = -k\nabla T. \tag{1.11}$$

We propose now to consider the geometry of the plate as revealed by measurements made as follows: a short graduated metal rod, made of a material of thermal coefficient of expansion h, is placed upon the plate, and distances are measured *after the rod has come into thermal equilibrium with the plate* in the neighborhood of the point $P(x, y)$ of interest. But then a linear element of

* For a more searching account of the epistemological background of the present paper see the author's chapter "Geometry as a Branch of Physics" in the forthcoming volume, Albert Einstein: Philosopher-Scientist, of The Library of Living Philosophers.

"true" length ds going out from the point P(x, y) on the plate will be measured as having a length

$$d\sigma = \frac{ds}{1 + hT(x, y)};$$

in terms of Cartesian coordinates x, y the linear element

$$d\sigma^2 = \frac{dx^2 + dy^2}{[1 + hT(x, y)]^2}. \tag{1.2}$$

Now this linear element (1.2) may (but need not) be considered as defining a 2-dimensional curved surface, in terms of the "isothermal" coordinates x, y. The Gaussian or total curvature K of this hypothetical surface is readily computed to be*

$$K = h(1 + hT)\nabla^2 T - h^2(\nabla T)^2, \tag{1.3}$$

where ∇T is the ordinary gradient $(\partial T/\partial x, \partial T/\partial y)$ of T, and $(\nabla T)^2$ is the square of its magnitude in the Euclidean sense. On taking the physical laws expressed by equations (1.1), (1.11) into account, the Gaussian curvature of the hot plate, as implied by the prescribed measuring procedure, is

$$K = -\frac{h}{k}(1 + hT)p - \left(\frac{h}{k}\right)^2 f^2. \tag{1.31}$$

From this it follows that in any region in which no heat is extracted from the plate ($p \geqq 0$) the "curvature" of the plate is negative; if in particular $p = 0$

$$K = -\left(\frac{h}{k} f\right)^2 \tag{1.32}$$

and the quantity

$$R = |K|^{-1/2} = \frac{k}{h|f|}, \tag{1.33}$$

which has the physical dimensions of a length and may conveniently be called the "radius of curvature" of the plate at P(x, y), is inversely proportional to the heat flux through P. The case in which $p = 0$ and the curvature is constant over the whole plate leads to the hyperbolic geometry of the Poincaré half-plane; this case has been given a very elegant and detailed treatment in this MONTHLY by E. W. Barankin.**

Returning to the general case, the lines which play the role of the straights of euclidean geometry are the geodesics, defined as the extremals $y = F(x)$ of the

* Cf. L. P. Eisenhart's Introduction to Differential Geometry, p. 154 (Princeton 1940); quoted hereafter as IDG.

** "Heat Flow and Non-Euclidean Geometry," this MONTHLY vol. 49, 1942, pp. 4–14.

variation principle

$$\delta \int d\sigma \equiv \delta \int \frac{(1 + F'^2)^{1/2}}{1 + hT(x, F)}\, dx = 0. \tag{1.4}$$

These geodesics are convex toward the side on which T increases, as can be seen intuitively from the fact that for sufficiently short geodesic arcs the integral in (1.4) above must be minimal, and since the measuring rod expands more on the side toward which T is increasing, the measured distance along an arc bowed slightly in this direction will be less than along the chord joining its endpoints. From the standpoint of the measured geometry the Euclidean straight

$$x \sin a - y \cos a = \text{const.}$$

of slope tan a is actually "curved"; its "geodesic curvature" is found to be*

$$\kappa_g = h(\nabla T)_n = -\frac{h}{k} f_n,$$

where $f_n = f_y \cos a - f_x \sin a$ is the heat flux across unit length of the straight. We have here a geometrical interpretation of the flux of heat in terms of the geodesic curvature of the Euclidean straights; the components $-f_y$, f_x of the flux, rotated through $\pi/2$, are proportional to the geodesic curvature of the lines of parameter x, y, respectively, through the point P of interest.

We have thus shown that the geometry of the heated plate is to some extent arbitrary, depending as it does on the coefficient of expansion h of the measuring instrument, and is, in particular, flat for an ideal rod for which $h = 0$. The geometry found is in fact interpretable as that of a curved surface, of curvature defined by Equation (1.3). In order to gain some appreciation of the magnitudes involved, consider the case of a steel plate ($k = 0.11$ cal/cm sec deg C) measured with a steel rod ($h = 0.11 \times 10^{-4}$/deg C). At a point P at which the flux f is 1 cal/cm^2 sec the Gaussian curvature K of the plate is -10^{-8} cm^{-2}, or its equivalent radius of curvature $R = 10^4$ cm$\sim$320 ft; the maximum geodesic curvature of an Euclidean straight through P is then 10^{-4} cm^{-1}, or the radius of geodesic curvature of the straight is 10^4 cm.

3. The thermal field in space. We turn next to the generalization of the above problem to 3-dimensional space, where our predilection for considering the measured space as imbedded in a higher-dimensional flat space will surely lose some of its appeal. Imagine, then, a material of heat conductivity k filling some part of our flat physical space, which is described by cartesian coordinates $(x_i) = (x, y, z)$. Suppose, by some stretch of the imagination, that it is possible to explore this thermal material with a short measuring rod, of thermal coefficient of expansion h. The temperature $T(xyz)$ at any point P(xyz) in the medium will again be governed by the fundamental equations (1.1), (1.11), where ∇ is

* IDG, p. 186.

now the 3-dimensional gradient operator $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ and $p(xyz)$ is the source strength per unit volume at P.

The measured geometry at the thermal field will then be defined by the linear element

$$d\sigma^2 = \frac{dx^2 + dy^2 + dz^2}{[1 + hT(xyz)]^2}. \tag{2.1}$$

The geodesics $\delta \int d\sigma = 0$ are now space-curves, with respect to which an euclidean straight has a point P of it a geodesic curvature

$$\kappa_g = h(\nabla T)_n = -\frac{h}{k} f_n, \tag{2.2}$$

where f_n is the component of the flux at P normal to the given straight.

The intrinsic geometry of such a 3-dimensional Riemannian space is no longer, as in the 2-dimensional case treated above, characterized by a single scalar curvature K. There is instead at each point P a curvature $K(\mathrm{P}, \mathbf{a})$ associated with each direction $\mathbf{a} = (a_i)$ through P, whose geometrical interpretation is as follows: Construct through P the 1-parameter family of geodesics orthogonal to the given direction $\mathbf{a}$, thus forming a 2-dimensional surface through P; then $K(\mathrm{P}, \mathbf{a})$ is the Gaussian curvature of this surface at P. On taking the direction coefficients a_i as normalized in the euclidean sense $\sum a_i^2 = 1$, the directional curvature of the space defined by the linear element (2.1) is*

$$K(\mathrm{P}, \mathbf{a}) = h(1 + hT)[\nabla^2 T - \sum T_{ij} a_i a_j] - h^2(\nabla T)^2, \tag{2.3}$$

where $T_{ij} = \partial^2 T/\partial x_i \partial x_j$. The directional quality of this curvature is here due to its dependence on the second derivative

$$\frac{d^2T}{dn^2} = \sum T_{ij} a_i a_j$$

of the temperature along the normal $\mathbf{a}$ to the geodesic surface of which it is the Gaussian curvature.

It will be most convenient for our purposes to introduce here an intrinsic "mean curvature" $\overline{K}(\mathrm{P})$ which is independent of direction, defined as the mean of the directional curvatures (2.3) associated with any triplet $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ of three mutually perpendicular directions. It follows that

$$\overline{K}(P) = \tfrac{2}{3} h(1 + hT)\nabla^2 T - h^2(\nabla T)^2, \tag{2.31}$$

as may most readily be seen by computing the mean in that Cartesian system whose axes are in the directions defined by the triplet; $\overline{K}$ is accordingly a scalar, independent of the particular triplet chosen.

* As may be computed as in L. P. Eisenhart's Riemannian Geometry, p. 79 (Princeton 1926). These directional curvatures are expressible in terms of the components a_i and the six linearly independent components of the Riemann-Christoffel tensor R_{ijkl}, or the equivalent six components of the Ricci tensor $R_{jk} \equiv R^i_{jki}$. The intrinsic mean curvature $\overline{K}$ introduced below is equal to $-1/6$ the scalar contraction R^i_i of the Ricci tensor.

A geometrical interpretation of the physical laws (1.1), (1.11) of heat conduction can now be obtained in terms of the mean curvature $\overline{K}$, whose value is here found to be

$$\overline{K} = -\frac{2h}{3k}(1 + hT)p - \left(\frac{h}{k}\right)^2 f^2. \tag{2.32}$$

In source-free regions this reduces to

$$\overline{K} = -\left(\frac{h}{k}\right)^2 f^2; \tag{2.33}$$

that is, the mean curvature at such a point P is, as in the 2-dimensional case (1.32), negative and proportional to the square of the heat flux f past P. And as in the previous case, the deviation from flatness in the measured geometry can, in principle, be determined by intrinsic measurements alone, without reference to any hypothetical imbedding space. Thus the surface S and the volume V of a "sphere" of measured (geodesic) radius ρ, *i.e.*, the locus of all points at a given geodesic distance ρ from P, differ from their flat values $4\pi\rho^2$, $4\pi\rho^3/3$ by amounts which depend on the intrinsic curvature at points within the sphere. For a small sphere of radius $\rho \ll 1/|\overline{K}|^{1/2}$ these measures are given by expressions whose leading terms are

$$\begin{cases} S = 4\pi\rho^2(1 - \overline{K}\rho^2/3 + \cdots), \\ V = \frac{4}{3}\pi\rho^2(1 - \overline{K}\rho^2/5 + \cdots), \end{cases} \tag{2.4}$$

where $\overline{K}$ is the value of the mean curvature at the center P of the sphere; to the order here considered these expressions are identical with those for a sphere of geodesic radius ρ in a spherical or hyperbolic 3-space of constant curvature $\overline{K}$.

As an example of the type of geometry to which one is led, consider the case of a spherically symmetric thermal field generated by a point source of heat of strength P cal/sec at the origin $r=0$. The temperature resulting at a cartesian distance r from the point source is

$$T = \frac{P}{4\pi kr}, \tag{2.5}$$

and the mean curvature there is

$$\overline{K} = -\frac{a^2}{r^4}, \quad \text{where} \quad a = \frac{hP}{4\pi k}. \tag{2.51'}$$

It may be noted that, because of the symmetry, any geodesic in this field is a "plane" curve, in the sense that it must lie completely in a cartesian plane, such as $y=0$, passing through the origin. But the curvature of this geodesic meridian surface, as implied by the measuring technique, is found to be a/r^3, and the mean curvature $\overline{K}$ at P$(r, 0, 0)$ is the mean of the curvature a/r^3 of two such

meridians $y=0$, $z=0$ and the curvature $-2a/r^3-3a^2/r^4$ of the surface generated by all geodesics tangent at P to the cartesian plane $x=$const.

A more interesting case, because it leads to the classical non-euclidean geometries, is that in which there is a spherical region of radius r_1 at all points of which heat is supplied to the medium at the constant rate of p cal/cm^3 sec. The temperature at any point at cartesian distance r from the source is found, on integrating the heat conduction equation (1.1), to be

$$T=\begin{cases} -\dfrac{p}{6k}r^2, & 0\leqq r\leqq r_1, \\ -\dfrac{p}{3k}r_1^2\left(\dfrac{3}{2}-\dfrac{r_1}{r}\right), & r_1\leqq r, \end{cases} \tag{2.6}$$

where the constant of integration is so chosen that $T=0$ at the center. Note that the outside field is, as it should be, that for a point source of strength $P=4\pi r_1^3 p/3$.

Now within the sphere $r=r_1$, the mean curvature $\overline{K}$ and radius of curvature $\overline{R}$ have the constant values

$$\overline{K}=-\frac{2}{3}\frac{hp}{k}, \qquad \overline{R}=\frac{1}{|\overline{K}|^{1/2}}=\left|\frac{3k}{2hp}\right|^{1/2}. \tag{2.61}$$

But also within the source the directiona' curvature $K(\mathrm{P}, \mathbf{a})$ defined by equation (2.3) is constant and equal to $\overline{K}$, and is accordingly independent of both position and orientation. The geometry revealed by measurement within the cartesian sphere $r=r_1$ is therefore one of the classical non-euclidean geometries, the hyperbolic geometry of Bolyai and Lobachewski if the central region is a source ($p>0$, whence $K<0$), and the spherical geometry of Riemann if it is a sink ($p<0$, whence $K>0$).

This circumstance merits some further examination. In either case the measured geometry is a true non-homaloidal congruence geometry; each configuration of point, incident direction and incident 2-spread of directions containing the given direction, is equivalent to every other such configuration, and may be transformed into it by a motion of the space. The case $p>0$ has been cited by Poincaré in illustration of the 3-dimensional hyperbolic geometry, for which it offers a finite euclidean model.* For let us suppose that the cartesian radius of the source region is

$$r_1=\left(\frac{6k}{hp}\right)^{1/2}=2R$$

(which for unit source-strength $p=1$ in the example $h/k=10^{-4}$ considered above, is around 245 cm), then at the boundary $r=r_1$ of the source the rule will have shrunk to zero length, admitting such an outrageous extrapolation of the linear

* H. Poincaré, Science and Hypothesis, pp. 76–78 (Science Press 1929).

expansion law! The measured radius of the source will in consequence become infinite, as can be seen by considering the integral of $d\sigma$ between $r=0$ and $r=r_1$. The cartesian sphere $r \leqq r_1$, supplied with the metric

$$d\sigma^2 = \frac{dx^2 + dy^2 + dz^2}{[1 - (r/r_1)^2]^2},$$

thus presents a finite model of the open hyperbolic geometry, in which by its own measuring standards and procedures the volume is infinite and in which every full geodesic is of infinite length.

The case $p<0$, in which there is a spherical region of radius r_1 within which heat is abstracted from the medium at the constant rate p cal/cm^3 sec, offers on the other hand a model of a finite space of constant positive curvature which is represented by the entire flat cartesian space. For here the measured radius of the sink is

$$\rho_1 = \int_0^{r_1} \frac{dr}{1 + r^2/4R^2} = 2R \tan^{-1} \frac{r_1}{2R},$$

and this approaches the finite limit πR as $r_1 \to \infty$. The measured length of the full Euclidean straight, and therefore of every other full geodesic of the space, is thus $2\pi R$.

4. Geometrization of the physical law. The results obtained in the above sections present a possible geometrical interpretation of the physical laws of heat conduction. Thus in a source-free region the mean curvature $\overline{K}$ at a point P is a measure of the heat flux past P, and the direction of flow is that of a cartesian straight whose geodesic curvature κ_g is zero, as follows from equations (2.33), (2.2). So far as the temperature is concerned, we may think of it as prescribed either by

(a) the physical law (1.1),

$$k\nabla^2 T + p = 0,$$

with appropriate boundary conditions, or by

(b) the requirement that the mean curvature [eq. (2.32)]

$$\overline{K} = -\frac{2h}{3k}(1 + hT)p - \left(\frac{h}{k}\right)^2 f^2.$$

This second viewpoint is, however, but a clumsy circumlocution of the first; it is a geometrical interpretation of the physical law, but in no way a modification of it.

A far more complete and satisfactory geometrization of the physical law would be possible had it turned out that the mean curvature $\overline{K}(\mathrm{P})$ of the medium, under the adopted measuring prescription, were determined directly in terms of the given source distribution $p(\mathrm{P})$, without involving explicitly either

T or its gradient, as is in fact the case for a uniform source distribution, equation (2.61). And because it is by following this line of attack that the general theory of relativity can be evolved from the classical theory of gravitation, we shall consider the corresponding modification of the theory of heat conduction. It is to be expected that here, as well as there, the resulting geometrized theory will constitute a factual modification, rather than a mere geometrical interpretation, of the "classical" theory.

We note that for sufficiently weak thermal fields, say in which the temperature differences are such that $h\Delta T \ll 1$, the first term $-2hp/3k$ in the mean curvature $\overline{K}$, eq. (2.32), is predominant; the remaining two terms are, in regions in which heat is being supplied, of the relative order $h\Delta T$. Under these conditions the mean curvature is given approximately by

$$\overline{K} = -\frac{2h}{3k}p; \tag{3.1}$$

we propose to effect the complete geometrization sought above by replacing the "classical" equation (1.1) *by the equation* (3.1), which requires that the mean curvature $\overline{K}$ of space at any point P be rigorously proportional to the strength p of the source density at P. That this geometrization is a factual, and not merely a formal, modification of the equation of heat conduction is apparent on noting that the mean curvature $\overline{K}$ at a source-free point P vanishes, and is not, as in equation (2.33), a measure of the heat flux past P.

It will now be found convenient to replace T by the dependent variable U defined by

$$1 + hT = U^{-2}, \tag{3.2}$$

i.e., to take the linear element (2.1) of the space in the form

$$d\sigma^2 = U^4(dx^2 + dy^2 + dz^2). \tag{3.21}$$

The mean curvature is then

$$\overline{K} = -\frac{4}{3}U^{-5}\nabla^2 U, \tag{3.22}$$

and the proposed field equation (3.1) for heat conduction

$$U^{-5}\nabla^2 U = \frac{h}{2k}p. \tag{3.23}$$

The case of an isotropic point-source of heat will illustrate the kind of deviations from the "classical" theory, developed in Section 3 above, which are here to be expected. In this case $p=0$ except at the origin, and the general solution of equation (3.23) satisfying this condition is

$$U = A + \frac{B}{r} \cdot \tag{3.3}$$

The natural normalization $U=1$ at $r=\infty$ requires that $A=1$, and the approximate correspondence with the classical theory in regions $r \gg 1$ of weak fields that $B=-hP/8\pi k$ where P is the strength of the source. The linear element of the physical geometry is then

$$d\sigma^2 = \left(1 - \frac{a}{2r}\right)^4 [dx^2 + dy^2 + dz^2], \tag{3.31}$$

where a is again $hP/4\pi k$.

We have here an amusing coincidence which is worthy of note: the spatial geometry defined by this $d\sigma^2$ is precisely that of the spatial section $t=$const. of a point-mass

$$m = -ac^2/G \tag{3.32}$$

in the general relativity theory of gravitation.* Here a sink of unit strength $P=-1$ cal/sec corresponds, in the example $h/k=10^{-4}$ cm sec/cal contemplated above, to a mass $m=10^{23}$ gm, *i.e.*, the thermal distortion caused by a sink of unit strength in steel is geometrically equivalent to the gravitational distortion caused by an asteroid some 100 miles in radius! Also, the singularity of $d\sigma^2$ at $r=a/2$ ($\sim 4\times 10^{-6}$ cm for a unit source) is at least superficially analogous to the well-known "Schwarzschild singularity" of the gravitational field.

Since the "classical" solution (2.6) inside a spherical region of constant source density p satisfies the new field equation (3.1), it may be taken over bodily into the proposed geometrized theory. An exterior solution of the form (3.31) may then be fitted on to represent the field outside the source; but now, because of the non-linearity of the new field equations, the total strength P of the equivalent point-source will no longer be equal to the spatial integral of the distributed strength p.

Glancing back at the 2-dimensional thermal field discussed in Section 1 above, we see from equation (1.31) that a corresponding geometrization could be obtained by replacing the physical law (1.1) by the requirement that the Gaussian curvature

$$\overline{K} = -\frac{h}{k} p. \tag{3.4}$$

But then in source-free regions $K=0$, and hence the surface is there developable. On making the change of variable.

$$1 + hT = e^{hU} \tag{3.41}$$

* Cf. A. S. Eddington's The Mathematical Theory of Relativity, p. 93 (Cambridge 1923).

the linear element (1.2) becomes

$$d\sigma^2 = e^{-2hU}(dx^2 + dy^2) \tag{3.42}$$

and the fundamental field equation (3.4)

$$k\nabla^2 U + e^{-2hU}p = 0. \tag{3.43}$$

In source-free regions, where $\nabla^2 U = 0$, it follows that the developable surface is obtained by a conformal transformation $x, y \rightarrow \xi, \eta$, and that the geodesics of the measured geometry are those curves which map into the straights of the ξ, η plane; conformal mapping by elementary functions of a complex variable produces here many curious and amusing examples of physical geometry. But we here confine ourselves to the remark that, as may be seen from equation (3.42), the measured geometry is equivalent to that which would result from the "classical" conduction theory on postulating that the expansion δl of a rod of length l due to an increase δT in temperature is given by $\delta l/l = h\delta T$, instead of by $\delta l/l_0$ as in Sections 1–2 above.

Returning to the 3-dimensional case, we see that we have succeeded in geometrizing the theory of heat conduction, in the sense that the content of the fundamental equation (1.1), somewhat modified, is contained in the purely geometrical assertion (3.1) that the mean curvature $\overline{K}$ of the thermal medium is proportional to the source-density p. But the geometry thus obtained is strongly dependent upon the measuring instrument, for the constant of proportionality has in it the ratio h/k of the coefficient of expansion h of the material of which the measuring rod is made, as well as the coefficient of heat conduction k of the thermal medium; thus were we to employ a brass rod, instead of a steel one, we would find that the curvature is trebled. The proposed theory of heat conduction is in consequence a poor example of a geometrico-physical theory, for the geometry obtained lacks the desirable quality of universality. We are therefore better advised to consider the geometry of the thermal field as Euclidean, and to attribute measured deviations from flatness to thermal stresses, differing from material to material, set up in the measuring rods.

This geometrical theory, in fine, would have far more appeal if the ratio h/k were a universal constant, for then the geometry implied by equation (2.32)or its modification (3.1) would be independent of the choice of measuring rod. And the analogue of this is precisely what happens in the general theory of relativity, to a caricature of which we now turn our attention.

5. The gravitational field. To consider the geometry of the gravitational field, it is desirable to proceed immediately to a 4-dimensional formulation, for even though we propose to deal with static fields the most appropriate method of exploration will be dynamical—namely, by examining the motion of test particles set loose in the field. In order to avoid the full mathematical complexity of the general theory of relativity we here set up a theory of gravitation which can be incorporated into the special theory of relativity, and which bears a striking resemblance to the theory of heat conduction presented above.

The kinematical background of this theory is given by the Minkowski linear element

$$ds^2 = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2), \tag{4.1}$$

and the field equation to be geometrized is taken to be the fairly immediate special relativistic generalization

$$\Box^2 V = -4\pi G c^2 \rho \tag{4.11}$$

of Poisson's equation for the gravitational potential V due to a mass distribution of density ρ; here $\Box$ is the 4-dimensional gradient operator whose square

$$\Box^2 = \frac{\partial^2}{\partial t^2} - c^2\nabla^2.$$

A test particle of gravitational mass m_G will be acted upon by a force determined by the contravariant vector

$$m_G f^i = m_G\left(\frac{1}{c^2}\frac{\partial V}{\partial t}, -\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right), \tag{4.12}$$

but since its proper inertial mass m_I is to be constant along the trajectory, only the component of this vector orthogonal to the world-line is effective. This leads to the unique equations of motion

$$m_I \frac{d^2x^i}{ds^2} = m_G\left(f^i - \sum_j f_j \frac{dx^j}{ds}\frac{dx^i}{ds}\right).$$

It is now easy to show that the trajectories (4.13) are in fact the geodesics of the 4-dimensional space-time defined by the linear element

$$d\sigma^2 \equiv \exp\left(\frac{2m_G V}{m_I c^2}\right) ds^2; \tag{4.2}$$

we have thereby accomplished a thorough geometrization of the dynamical trajectories. It would at first sight appear that this geometry depends upon the test particle as well as upon the field, for the ratio m_G/m_I of its gravitational and inertial mass enters explicitly into the linear element. But this is only apparent, for it has been known empirically since Newton that these two masses are directly proportional; on measuring them in the same unit the ratio becomes unity, and the linear element (4.2) may henceforth be taken in the form

$$d\sigma^2 = e^{2V/c^2} ds^2 \tag{4.21}$$

from which all trace of the exploring particle has been removed.*

* The variation principle $\delta\int d\sigma = 0$ defining the geodesics of (4.21) goes over, on allowing $c \to \infty$, into Hamilton's principle $\delta\int (T - V)dt = 0$ of classical mechanics.

This form suggests, in analogy with the development in Sections 1–2 above, that we might base this linear element on the postulate that a measuring rod of length l suffers a fractional change of length $\delta l/l = -\delta V/c^2$ on transportation to a position at which the potential energy V is increased by δV; our time standards must be supposed to suffer the same relative change, in order that the velocity of light c be unaltered. For then the length l of the rod at a position of potential V would be $l = l_0 e^{-V/c^2}$, where l_0 is its length at infinity ($V=0$), and the measurement of ds with such a standard would yield the value $d\sigma$ given by Equation (4.21). On introducing the new dependent variable $U = e^{V/c^2}$ suggested thereby, the linear element becomes

$$d\sigma^2 = U^2\left[dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)\right]. \tag{4.22}$$

In order to carry through the geometrization of the theory, we next examine the geometrical consequences of the given field equation (4.11). Proceeding as in the previous sections, we find that the mean curvature $\overline{K}$ of the 4-dimensional space-time (4.21), defined in fairly straight-forward generalization of the considerations given above for a 3-dimensional space, is

$$\overline{K} = -\frac{1}{2c^2} e^{-2V/c^2}\left[\Box^2 V + \frac{1}{c^2}(\Box V)^2\right]. \tag{4.23}$$

On taking the field equation (4.11) and the definition of force (4.12) into account,

$$\overline{K} = e^{-2V/c^2}[2\pi G\rho - \tfrac{1}{2}f^2]. \tag{4.24}$$

We propose now to replace the above theory by a geometrical one in which the geometry of space-time is related directly to the material content. The choice of field equation which naturally suggests itself from the form of equation (4.24) is

$$\overline{K} = 2\pi G\rho; \tag{4.3}$$

this equation differs from (4.11) only by terms of order $2V/c^2$, and leads in the dynamical applications to deviations of the order $(v/c)^2$ where v is the velocity of the test particle. We have then here a geometrical theory of gravitation which differs from the special relativistic theory set up above, and therefore from the classical Newtonian theory, only by terms of relative order $(v/c)^2$.

This theory is in principle very similar to the general relativity theory of gravitation, only far simpler, and, in the last analysis, far less satisfactory! Here we have a single scalar field equation (4.3) which expresses the mean curvature $\overline{K}$ of space-time directly in terms of the density of its material content; in the general theory the tensor field equations have 10 components, which express certain partial curvatures (the 10 components R_{ij} of the Ricci tensor)

in terms of the density, pressure and state of motion of the material content.* In the latter it is clear from the development that the density which enters directly into the geometry is the density of *inertial* mass; following this hint we postulate that it is this inertial density ρ which appears in equation (4.3).

On taking the linear element in the form (4.22) we find

$$\overline{K} = -\tfrac{1}{2}U^{-3}\Box^2 U = 2\pi G\rho; \tag{4.31}$$

this field equation reduces for the case of a static field to

$$U^{-3}\nabla^2 U = \frac{4\pi G}{c^2}\rho. \tag{4.32}$$

The solution of this equation for a point-mass m is

$$U = e^{V/c^2} = 1 - \frac{\mu}{r}, \tag{4.33}$$

where $\mu = Gm/c^2$ is the length-equivalent of the mass m. Taking the motion of a test particle as along a geodesic of the space-time (4.22), we readily find that in the classical approximation $(v/c)^2 \ll 1$ the motion is Newtonian. It turns out that in the next approximation the perihelion of an elliptic orbit suffers a retardation, the magnitude of which is 1/6 that of the Schwarzschild advance, and that a ray of light is not bent on passing a massive body. As may be seen from the measuring procedure suggested above, light going from a position of lower to one of higher gravitational potential suffers the red-shift $\Delta\lambda/\lambda = G\Delta v/c^2$, as in the Einstein theory. But such refinements are beyond our present interest, for our simple theory is on other, more weighty, counts quite inferior to the general theory of relativity.†

What we have shown here is that the gravitational field, as opposed to the thermal, can be given a quite satisfactory geometrization. The reason for this is that in the gravitational case the geometry arrived at is universal in virtue of the equality of the inertial and gravitational masses of the test particle employed in its exploration, whereas in the thermal the geometry is dependent on the physical properties of the measuring instrument. With this we have completed our announced task—a study of the physical geometry of the gravitational and of the thermal fields, and of the reasons for the success of the former, as opposed to the latter, as a basis for a tenable physical theory.

* In the general theory the "mean curvature" here employed (which is in fact $-1/12$ the contracted Ricci tensor R_i^i is determined by an equation of exactly the same form as (4.3) but with a numerical coefficient $-2\pi/3$ instead of $+2\pi$.

† Thus in the general theory the material-energetic conservation laws and the mechanical equations of motion are direct consequences of the field equations, whereas here they must be postulated independently.

MATHEMATICAL NOTES

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EXTENSION OF A TOPOLOGICAL PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

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1. Introduction. In the August–September (1949) issue of this MONTHLY, a topological proof of the fundamental theorem of algebra was given by B. H. Arnold [1]. We here extend this proof to the case of an equation in quaternions, a result established earlier by Eilenberg and Niven [2]. It is of interest to note that of all the proofs of the fundamental theorem of algebra, only those using topological notions appear to be extensible to the case of quaternions. The well-known analytic proof [3] employing the Liouville theorem on bounded analytic functions does not seem to carry over to the more general case of quaternions. The theory of functions over non-commutative algebras [4] is, of course, not as well developed as that over complex numbers. Similarly the proof (Gauss' second) as given by MacDuffee [5] and van der Waerden [6] involves an algebraic technique which also does not appear to admit generalization.

2. Result. Let $f(z)$ be a polynomial of degree n of the type

$$f(z) = za_1za_2z \cdots za_n + \phi(z), \tag{1}$$

where z, a_1, a_2, $\cdots$, a_n are real quaternions with $a_i \neq 0$ for $i = 1, 2, \cdots, n$, and where $\phi(z)$ is a sum of a finite number of similar monomials each of degree $< n$. We prove that the equation $f(z) = 0$ has at least one quarternion solution. The hypothesis that there is only one term of highest degree in $f(z)$ is necessary, as can be seen from such an example as $zj - iz - 1 = 0$. There is no loss of generality in assuming that the term of degree n in $f(z)$ has left coefficient unity, as indicated in (1); in fact we can also assume that

$$|a_1a_2 \cdots a_n| = 1. \tag{2}$$

3. Proof. Any non-zero quaternion $u = u_1 + u_2i + u_3j + u_4k$ can be written in the form

$$u = |u|(\cos\theta + \epsilon \sin\theta), \tag{3}$$

where $|u| = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$,

$$\cos\theta = \frac{u_1}{|u|}, \qquad \sin\theta = \frac{\sqrt{u_2^2 + u_3^2 + u_4^2}}{|u|}, \tag{4}$$

and ϵ is the unit vector,

$$\epsilon = \frac{u_2 i + u_3 j + u_4 k}{\sqrt{u_2^2 + u_3^2 + u_4^2}}. \tag{5}$$

If u is real, then ϵ is arbitrary, say $\epsilon = 1$, so that θ and $\epsilon \sin\theta$ are continuous functions of u, although ϵ itself is not a continuous function of u.

Let there be r terms in $\phi(z)$ in (1), say $\phi(z) = \phi_1(z) + \phi_2(z) + \cdots + \phi_r(z)$, and write $|\phi_m(1)| = b_m$ for $m = 1, 2, 3, \cdots, r$. Thus

$$|\phi_m(z)| \leqq b_m \qquad \text{for } |z| \leqq 1. \tag{6}$$

Then we define R and u by

$$R = 2 + \sum_{m=1}^{r} b_m, \qquad u = a_1 z a_2 z \cdots z a_n.$$

The mapping which we use is given by

$$g(z) = z - f(z)u^{-1}R^{-1}, \qquad \text{for } |z| \geqq 1; \tag{7}$$

$$g(z) = z - f(z)[\cos(|z|\theta) + \epsilon \sin(|z|\theta)]^{-1}R^{-1}, \qquad \text{for } 0 < |z| \leqq 1; \tag{8}$$

$$g(0) = -f(0)R^{-1}. \tag{9}$$

Now θ and ϵ are given by (3), (4), (5) so that u, θ and $\epsilon \sin\theta$ are continuous functions of z, except at $z = 0$. But $g(z)$ is a continuous function for all values of z, for the following reasons. First, (7) and (8) are continuous in the range indicated. Second, (7) and (8) are identical for $|z| = 1$. Third, the limit of $g(z)$ in (8) as $z \to 0$ exists and the limit is indicated in (9).

The proof that $|g(z)| \leqq R$ for $|z| \leqq R$ is now almost identical with Arnold's. For $|z| \leqq 1$ we have, by use of (2) and (6),

$$|g(z)| \leqq |z| + |f(z)| \cdot R^{-1} \leqq 1 + (1 + b_1 + b_2 + \cdots + b_r)R^{-1} \leqq R.$$

For $1 \leqq |z| \leqq R$, we have

$$|g(z)| = \left|z - \frac{z}{R} - \frac{\phi(z)u^{-1}}{R}\right| \leqq \left|z - \frac{z}{R}\right| + \frac{b_1 + b_2 + \cdots + b_r}{R}$$

$$\leqq R - 1 + \frac{R-2}{R} \leqq R.$$

Thus the mapping $z \to g(z)$ is a continuous transformation which carries the 4-dimensional hypersphere $|z| \leqq R$ into itself. By Brouwer's fixed point theorem [7] there exists a value z_0 such that $g(z_0) = z_0$, and by (7), (8), and (9) we have $f(z_0) = 0$.

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SOLVABILITY OF QUARTICS BY MEANS OF SQUARE ROOTS

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1. Introduction. A geometric configuration A is constructible with ruler and compasses from a geometric configuration B if and only if the numbers defining A are obtainable from the numbers defining B by rational operations and extractions of real square roots.[1]

If $a_1, a_2, \cdots, a_m$ are given complex numbers, a number b is said to be obtainable from the a's by rational operations and extractions of square roots (not necessarily real) if there exists a (finite) sequence $b_1, b_2, \cdots, b_t$ such that $b_t = b$ and each b_i is either

(a) one of the a's, or

(b) the sum, difference, product, or quotient of b_j and b_k, where j and k are less than i (j and k not necessarily distinct), or

(c) a square root of some b_j, where $j < i$.

For brevity we shall then say that b is a *multiple square root* in the a's.

If b is a multiple square root in the a's, there may be many sequences of the kind described. In any one sequence there may be some values of i for which b_i is of more than one of the three types (a), (b), (c); but there will be a certain number of i's, say $p \geqq 0$, such that b_i is of type (c) and not of either of the types (a) or (b). If the smallest value of the p's corresponding to the different sequences is n, we shall say that b is a multiple square root in the a's of order n.

If $n = 0$, then b is said to be *rational* in the a's. In this case b is obtainable by a finite number of rational operations performed upon the a's and upon numbers obtainable from the a's by additions, subtractions, multiplications, and divisions.

It often happens that each of the numbers defining a geometric configuration A is a root of an algebraic equation whose coefficients are rational in the numbers defining configuration B. It is desirable, therefore, to have a criterion for determining when such an equation has roots which are multiple square roots in the numbers of B.

For a linear or quadratic equation there is no problem. For a cubic equation we have the following result.

THEOREM 1. *If p, q, r are rational in $a_1, a_2, \cdots, a_m$, then the equation $x^3 + px^2 + qx + r = 0$ has a root which is a multiple square root in the a's if and only if it has a root which is rational in the a's.*[2]

[1] L. E. Dickson, *First Course in the Theory of Equations*, 1922, p. 30.

[2] The proof is exactly like the proof of the theorem in Dickson, pp. 32–34.

It follows immediately from this theorem (or by direct means) that, with the same hypothesis, if one root of the cubic is a multiple square root in the a's, then all the roots are.

It is easily shown by means of this theorem that the trisection of certain angles, the duplication of the cube, and some other geometric constructions, are not possible with ruler and compasses.[3]

In this note we call attention to a simple criterion, similar to Theorem 1, for quartic equations.

2. Results. Let $f(x)=x^4+px^3+qx^2+rx+s$, where p, q, r, s are rational in $a_1, a_2, \cdots, a_m$. Let $g(y)=y^3-qy^2+(pr-4s)y+4qs-p^2s-r^2$ be the resolvent cubic of $f(x)$. We shall establish the following results.

THEOREM 2. *If $f(x)=0$ has a root which is a multiple square root in the a's, but which is not rational in the a's, then all its roots are multiple square roots in the a's.*

THEOREM 3. *All the roots of $f(x)=0$ are multiple square roots in the a's if and only if all the roots of $g(y)=0$ are multiple square roots in the a's.*

THEOREM 4. *The equation $f(x)=0$ has a root which is a multiple square root in the a's if and only if at least one of the two equations $f(x)=0$ and $g(y)=0$ has a root which is rational in the a's.*

As a simple application of Theorem 4, consider the problem of inscribing in a circle of radius 1 an isosceles triangle of area 1/4. If x is the square of one of the equal sides, we are led to the equation $x^4-4x^3+1=0$. The resolvent cubic equation is $y^3-4y-16=0$. Since neither of these equations has a rational root, the proposed construction is not possible with ruler and compasses.

Consider also the possibility of constructing 1/5 of an acute angle whose cosine is 61/64. If $x=\cos\theta$, where θ is the desired angle, then $\cos 5\theta=16x^5-20x^3+5x=61/64$. One root is 1/4, which is obviously not the desired solution. The reduced quartic equation is

$$x^4+\frac{1}{4}x^3-\frac{19}{16}x^2-\frac{19}{64}x+\frac{61}{256}=0,$$

which has

$$y^3+\frac{19}{16}y^2-\frac{263}{256}y-\frac{2529}{2048}=0$$

as its resolvent cubic. Neither of the last two equations has a rational root. Hence, the proposed construction is not possible with ruler and compasses.

3. Proof of Theorem 2. Let x_1 be a root of $f(x)=0$ which is a multiple square root in the a's of order $n>0$. Let $b_1, b_2, \cdots, b_t$ be a sequence of the kind

[3] Dickson, pp. 34–36.

described in §1 such that $b_t=x_1$, and such that for exactly n values of i, b_i is of type (c) and not of either of the types (a) or (b), and let b_l be the last such b_i. It follows easily that $x_1=\alpha+\beta\sqrt{k}$, where $\sqrt{k}$ is a square root of one of b_1, b_2, $\cdots$, b_{l-1}, and α and β are multiple square roots in the a's of order $n-1$ at most.

Proceeding as in Dickson *loc. cit.*, pp. 32–34, we find that $x_2=\alpha-\beta\sqrt{k}$ is also a root of $f(x)=0$, distinct from x_1. If x_3, x_4 are the other roots, then by eliminating x_3 from the equations

$$x_1+x_2+x_3+x_4=-p, \qquad x_1x_2+(x_1+x_2)(x_3+x_4)+x_3x_4=q,$$

we obtain

$$x_4^2+(p+x_1+x_2)x_4+q+(x_1+x_2)^2+p(x_1+x_2)-x_1x_2=0.$$

Since each of the coefficients of the powers of x_4 in this quadratic equation is a multiple square root in the a's, x_4 is also. It follows that the same is true of x_3.

4. Proof of Theorem 3. We recall[4] the following results.

(1) If y_i is a root of the resolvent cubic, then

$$f(x)=(x^2+\tfrac{1}{2}px+\tfrac{1}{2}y_i+\lambda_ix+\mu_i)(x^2+\tfrac{1}{2}px+\tfrac{1}{2}y_i-\lambda_ix-\mu_i),$$

where $\lambda_i^2=y_i+\frac{1}{4}p^2-q$, $2\lambda_i\mu_i=\frac{1}{2}py_i-r$, $\mu_i^2=\frac{1}{4}y_i^2-s$.

(2) *If* x_1, x_2, x_3, x_4 are the roots of the quartic, then

$$y_1=x_1x_2+x_3x_4, \qquad y_2=x_1x_3+x_2x_4, \qquad y_3=x_1x_4+x_2x_3$$

are the roots of the resolvent cubic.

Suppose x_1, x_2, x_3, x_4 are multiple square roots in the a's. From the expressions in (2) we see that y_1, y_2, y_3 are also.

Suppose, conversely, that one or more of y_1, y_2, y_3 is a multiple square root in the a's. From the expressions in (1) we see that the corresponding λ_i and μ_i are also. Since x_1, x_2, x_3, x_4 are the roots of the quadratic factors of $f(x)$, they also are multiple square roots in the a's.

5. Proof of Theorem 4. Suppose $f(x)=0$ has a root which is a multiple square root in the a's. If this root is rational in the a's, there is nothing more to be proved. If not, then by Theorem 2 all the roots of the quartic are multiple square roots in the a's. By Theorem 3 the same is true of the roots of $g(y)=0$. By Theorem 1, the cubic has a root which is rational in the a's.

Conversely, suppose $f(x)=0$ or $g(y)=0$ has a root which is rational in the a's. If $f(x)=0$ has such a root, there is nothing more to be proved. If $g(y)=0$ has such a root, then the last part of the proof of Theorem 3 applies and shows that all the roots of $f(x)=0$ are multiple square roots in the a's.

[4] Dickson, *loc. cit.*, pp. 50–51.

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FROM SPHERICAL TRIGONOMETRY TO PLANE TRIGONOMETRY

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In the study of spherical trigonometry one is struck with the likeness of some of the spherical trigonometry formulas to certain plane trigonometry formulas. As for example, the law of sines in both trigonometries, the law of tangents for both, some of the right triangle formulas, and others show similarities. But in some cases there seems to be little or no similarity, as for example, the two laws of cosines. Also, there is a law of cosines for angles in spherical trigonometry while of course there is none like this or similar to it in plane trigonometry. And again, in what respect is the formula for area of a spherical triangle similar to or related to any one of the formulas for area of a plane triangle?

In this paper it is shown how, starting with the formulas in spherical trigonometry, to establish a correspondence between these and the formulas of plane trigonometry.

Consider the spherical triangle $\mathcal{ABC}$, Figure 1, and the plane triangle ABC, Figure 2. Letter the triangles in the usual manner and in Figure 1 let the center

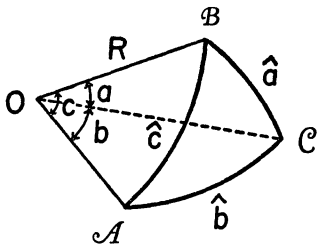


Fig. 1

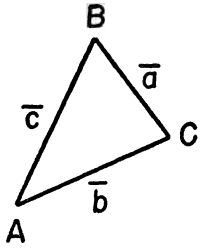


Fig. 2

of the sphere be O, and let the radius of the sphere be $R = O\mathcal{A} = O\mathcal{B} = O\mathcal{C}$. Also, let

a = angle $\mathcal{BOC}$ = arc $\mathcal{BC}$ in circular measure, Figure 1.

$\hat{a}$ = arc $\mathcal{BC}$ in linear measure = Ra, Figure 1;

$\bar{a}$ = side BC in linear measure, Figure 2;

and similarly for the b's and c's.

All angles will be measured in radians.

Next, hold the vertices of the triangle in Figure 1 fixed in space and let the center O of the sphere recede to infinity keeping $O\mathcal{A} = O\mathcal{B} = O\mathcal{C}$. Then $R \to \infty$, $\sphericalangle \mathcal{BOC} = a \to 0$, $\hat{a} \to \bar{a}$, $\sphericalangle \mathcal{A} \to \sphericalangle A$, and similarly for the remaining sides and angles. Also, area $\mathcal{K}$ of $\Delta\mathcal{ABC} \to$ area K of ΔABC.

We shall make use of such simple limits as the limit of a sum, the limit of

a product, the limit of a quotient,

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1, \qquad \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1,$$

and

$$\lim_{\theta\to 0}\cos\theta=1.$$

The equations will be *put in limit form* by replacing $\sin a$ by $Ra\cdot \sin a/a = \hat{a}\sin a/a$, the factor R being introduced either by multiplication of the equation by R or by multiplying the numerator and denominator of the term by R. Similarly for other sines and tangents of sides.

The formulas for the right triangles, plane and spherical, as given by Chauvenet, *A Treatise on Plane and Spherical Trigonometry*, tenth edition, 1908, page 108, for comparison are tabulated below, only here the order is changed and numbers are assigned.

	Spherical Right Triangle		Plane Right Triangle	
	a	b	c	d
1	$\sin\mathcal{A}=\dfrac{\sin a}{\sin c}$	$\sin\mathcal{B}=\dfrac{\sin b}{\sin c}$	$\sin A=\dfrac{\bar{a}}{\bar{c}}$	$\sin B=\dfrac{\bar{b}}{\bar{c}}$
2	$\cos\mathcal{A}=\dfrac{\tan b}{\tan c}$	$\cos\mathcal{B}=\dfrac{\tan a}{\tan c}$	$\cos A=\dfrac{\bar{b}}{\bar{c}}$	$\cos B=\dfrac{\bar{a}}{\bar{c}}$
3	$\tan\mathcal{A}=\dfrac{\tan a}{\sin b}$	$\tan\mathcal{B}=\dfrac{\tan b}{\sin a}$	$\tan A=\dfrac{\bar{a}}{\bar{b}}$	$\tan B=\dfrac{\bar{b}}{\bar{a}}$
4	$\sin\mathcal{A}=\dfrac{\cos\mathcal{B}}{\cos b}$	$\sin\mathcal{B}=\dfrac{\cos\mathcal{A}}{\cos a}$	$\sin A=\cos B$	$\sin B=\cos A$
5	$\cos c=\cos a\cdot\cos b$		$\bar{c}^2=\bar{a}^2+\bar{b}^2$	
6	$\cos c=\cot\mathcal{A}\cdot\cot\mathcal{B}$		$1=\cot A\cdot\cot B$	

To show that 1a reduces to 1c put 1a in the limit form:

$$\sin \mathcal{A} = \frac{\sin a}{\sin c} = \frac{Ra \cdot \dfrac{\sin a}{a}}{Rc \cdot \dfrac{\sin c}{c}} = \frac{\hat{a} \cdot \dfrac{\sin a}{a}}{\hat{c} \cdot \dfrac{\sin c}{c}}.$$

On passing to limits we have 1c, that is,

$$\sin A = \frac{\bar{a}}{\bar{c}}.$$

In like manner formulas 1b, 2a, 2b, 3a, 3b transform respectively into 1d, 2c, 2d, 3c, 3d.

Formulas 4a, 4b, 6a give 4c, 4d, 6c, respectively, since $\cos a$, $\cos b$, and $\cos c$ become 1 in the limit. These formulas, 4c, 4d, and 6c imply that $A+B=\pi/2$, or $A+B+C=\pi$.

Formulas 5a and 5c merit special attention. On the surface it is difficult to see how they "correspond" except that they both involve the three sides. To establish this correspondence, square both sides of 5a, replace $\cos^2 a$ by $1-\sin^2 a$, and do likewise for $\cos^2 b$ and $\cos^2 c$. Then simplify and get

$$\sin^2 c = \sin^2 a + \sin^2 b - \sin^2 a \cdot \sin^2 b.$$

Put in limit form and get

$$\left(Rc \cdot \frac{\sin c}{c}\right)^2 = \left(Ra \cdot \frac{\sin a}{a}\right)^2 + \left(Rb \cdot \frac{\sin b}{b}\right)^2 - \left(Ra \cdot \frac{\sin a}{a}\right)^2 \cdot \sin^2 b,$$

or,

$$\hat{c}^2 \cdot \left(\frac{\sin c}{c}\right)^2 = \hat{a}^2 \cdot \left(\frac{\sin a}{a}\right)^2 + \hat{b}^2 \cdot \left(\frac{\sin b}{b}\right)^2 - \hat{a}^2 \cdot \left(\frac{\sin a}{a}\right)^2 \cdot \sin^2 b.$$

Passing to limits, we find 5c, that is

$$\bar{c}^2 = \bar{a}^2 + \bar{b}^2.$$

Formulas for the general triangle are treated in a similar manner, some transforming into the ordinary formulas of plane trigonometry and others into $A+B+C=\pi$. Three others will be exhibited.

The formula

$$\sin c \cdot \cos \mathcal{A} = \sin b \cdot \cos a - \cos b \cdot \sin a \cdot \cos \mathcal{C}, \tag{7}$$

see Chauvenet, (6), page 154, transforms into

$$\bar{b} = \bar{a} \cdot \cos c + \bar{c} \cdot \cos A,$$

that is, side $\bar{b}$ of the plane triangle equals the sum of the projections of sides $\bar{a}$ and $\bar{c}$ on $\bar{b}$.

Now consider the formula

$$\mathcal{K} = R^2E, \tag{8}$$

for the area of the spherical triangle $\mathcal{ABC}$, where $E=\mathcal{A}+\mathcal{B}+\mathcal{C}-\pi$ is the spherical excess.

If we use Lhuillier's formula (Chauvenet (228), page 230) for the spherical excess we can write (8) as follows:

$$\begin{aligned}\mathcal{K} &= 4R^2\cdot\frac{\frac{1}{4}E}{\tan\frac{1}{4}E}\cdot\left[\tan\tfrac{1}{2}s\cdot\tan\tfrac{1}{2}(s-a)\cdot\tan\tfrac{1}{2}(s-b)\cdot\tan\tfrac{1}{2}(s-c)\right]^{1/2}\\ &= \frac{\frac{1}{4}E}{\tan\frac{1}{4}E}\cdot\left[\left(Rs\cdot\frac{\tan\frac{1}{2}s}{\frac{1}{2}s}\right)\cdot\left(R(s-a)\cdot\frac{\tan\frac{1}{2}(s-a)}{\frac{1}{2}(s-a)}\right)\cdot\ \cdots\ \right]^{1/2}\\ &= \frac{\frac{1}{4}E}{\tan\frac{1}{4}E}\cdot\left[\hat{s}\cdot\frac{\tan\frac{1}{2}s}{\frac{1}{2}s}\cdot(\hat{s}-\hat{a})\cdot\frac{\tan\frac{1}{2}(s-a)}{\frac{1}{2}(s-a)}\cdot\ \cdots\ \right]^{1/2},\end{aligned}$$

where $s=\frac{1}{2}(a+b+c)$. On passing to limits, we find the formula for the area of the plane triangle ABC in terms of the sides of the triangle, with $\bar{s}=\frac{1}{2}(\bar{a}+\bar{b}+\bar{c})$.

If instead of Lhuillier's formula for the spherical excess, we use formula (230), page 231, of Chauvenet's trigonometry, the area can be written

$$\begin{aligned}\mathcal{K} = R^2E &= R^2\cdot\frac{E}{\sin\frac{1}{2}E}\cdot\frac{\sin\frac{1}{2}a\cdot\sin\frac{1}{2}b}{\cos\frac{1}{2}c}\cdot\sin\mathcal{C}\\ &= 2\,\frac{\frac{1}{2}E}{\sin\frac{1}{2}E}\cdot\frac{Ra}{2}\cdot\frac{\sin\frac{1}{2}a}{\frac{1}{2}a}\cdot\frac{Rb}{2}\cdot\frac{\sin\frac{1}{2}b}{\frac{1}{2}b}\cdot\frac{\sin\mathcal{C}}{\cos\frac{1}{2}c}\\ &= \frac{1}{2}\cdot\frac{\frac{1}{2}E}{\sin\frac{1}{2}E}\cdot\hat{a}\hat{b}\cdot\frac{\sin\frac{1}{2}a}{\frac{1}{2}a}\cdot\frac{\sin\frac{1}{2}b}{\frac{1}{2}b}\cdot\frac{\sin\mathcal{C}}{\cos\frac{1}{2}c}.\end{aligned}$$

On passing to limits, we get

$$K = \tfrac{1}{2}\bar{a}\bar{b}\cdot\sin C,$$

the formula giving the area of the plane triangle ABC in terms of two sides and the included angle.

SEMINAR PROGRAM IN MATHEMATICS AT REGIS COLLEGE

SISTER MARY LEONARDA, Regis College

The seminar program was inaugurated at Regis College in the fall of 1947 in an effort to provide for each concentrator a testing ground where she would meet the problems of her field in a setting reproducing as nearly as possible the demands upon her knowledge and training and use of newly acquired knowledge which she would encounter after graduation. Although still in the experimental stage, the program has been undeniably effective. In the present paper, the writer gives some aspects of the seminar system as it has been developed thus far in the Department of Mathematics. The topics for the current year are listed

as typical of the content of the discussions but do not indicate permanency. Classroom experiences and current demands determine the details and contribute significantly to the success of a seminar program by keeping it from becoming too highly systematized and nonadjustable.

Our junior concentrators meet once each week for a two-hour seminar which treats of geometry. The junior seminar is a reading seminar which is made possible because the student comes to it reasonably prepared with knowledge and skills gained from her lower level courses in mathematics. It is directed to stimulate critical and systematic reading of monographs, source material, books of reference, biography, popular writings. Occasionally a text is reviewed. In general, texts are not included since their content is already a digest of some greater finding, and therefore provides no experience of the type we seek. Since "understanding of mathematics cannot be transmitted by painless entertainment any more than education in music can be brought by the most brilliant journalism to those who have never listened intensively,"* an important supplement to our reading requirement is the actual performance of a fair number of proposed problems and "actual contact with the content of living mathematics."* The author quoted points out that such contact is rendered possible at this level if advanced technicalities and profound methods can be laid aside for the purpose of disclosing the goal.

The following topics indicate the trend of our interests in the current reading seminar:

First Quarter: Fundamental Constructions; Constructible Numbers; Three Greek Problems; I Methods of Construction—Inversion; II Methods of Construction—Mascheroni Constructions; III Methods of Construction—Straightedge, Carpenter's Square; Peaucellier Cells; Problem of Apollonius; Reflection in a System of Three Circles.

Second Quarter: Pascal's Theorem; Brianchon's Configuration; Polar Coordinates; Cylindrical Coordinates; Spherical Coordinates; Homogeneous Coordinates; Transformations; The Straight Line.

Third Quarter: Conic Sections Generated by Projective Pencils; Linkages; Cycloids, Tractrix, Catenary; Surfaces of Revolution; Hyperboloid of One Sheet and Hyperbolic Paraboloid; The Pseudosphere; Riemann.

Fourth Quarter: Klein's Non-Euclidean Model; Poincaré's Non-Euclidean Model; Euler's Formula for Polyhedra; The Four-Color Problem; Map Projection; One-sided Surfaces; What is Mathematics?

The required reading pertinent to this outline is taken from the fields of foundation and development of mathematics, algebra, and geometry. It includes the works of such writers as Young, Hardy, Heath, Smith, Baker, Frost, Salmon, Klein, Hudson, Kasner, Thomas, Albert; source material on the topics treated briefly in D. E. Smith's *Source Book in Mathematics*,† and others such as Peacock's *Treatise on Algebra*,‡ and studies on the Rhind papyrus, are expected to arouse further investigation into primary source material.

* R. Courant and H. Robbins: *What is Mathematics?*, Oxford University Press, New York, 1943, p. v.

† McGraw-Hill Book Company, New York, 1929.

‡ Scripta Mathematica Press, New York, 1940.

Not the least important function of the junior seminar is the pursuit of a chosen subject which, for want of a better name, we call a hobby. This hobby is acknowledged and encouraged in our seminar system. The studies pursued are varied: architecture, clocks, market research, inversion, bridges, airplanes, Mascheroni constructions, singular solutions of differential equations, the history of the methods used in assigning the number of representatives in Congress for a given area. Some of the studies have been written formally and bound and placed in our departmental library. A part of the study on airplanes was published, and we are hoping to have the study of bridges appear. With few exceptions the experience which has been gained in the pursuit of the hobby remains only the joy of personal achievement.

Our senior concentrators participate in a coordinating seminar on analysis. The seminar on geometry provides a magnificent foundation upon which to build the coordinating seminar. The following discussions constitute our program for the present coordinating seminar:

First Quarter: The Rhind Papyrus; Pythagorean Arithmetic; Irrational Numbers; Complex Numbers; DeMoivre's Formula and the Roots of Unity; The Number Pi; Euler's Number *e*.

Second Quarter: Calculating Devices; The Sand-Reckoner and His Successors; The Fundamental Theorem of Algebra and Algebraic Equations; Trigonometric and Transcendental Equations; Algebra of Sets—Application to Mathematical Logic.

Third Quarter: Variables, Functions, Graphs; Limits by Continuous Approach; Abel on the Continuity of Functions Defined by Power Series; Theorems of Bolzano and Weierstrass; Heron's Theorem; Extremum Property of Light Rays; Tangent Properties of Ellipse and Hyperbola; Corresponding Extremum Properties; Schwartz' Triangle Problem; The Cycloid.

Fourth Quarter: Maxima and Minima of Functions of Several Variables; Saddle Points; Soap Film Experiments; Plateau's Problem; Pre-Calculus Integration as a Limiting Process; Integral Calculus; Fermat, Newton, Leibniz and the Derivative; Geometric Meanings of the First and Second Derivatives; Differential Equation of the Exponential Function; Radioactive Disintegration; Law of Growth; Compound Interest.

The works of Archimedes, Cauchy, Abel, Dedekind, Hardy, Courant, Ferrar, Young, Whittaker and Watson, Birkhoff and Maclane, N. Bourbaki, Scarborough, and many others which are of significance in the investigations of the students, should provide dependable and interesting reading matter with which to develop the proposed topics. *What is Mathematics?* by R. Courant and H. Robbins contains an excellent directive reading on each topic.

Each seminar course puts the instructor on his mettle as well as the student. The method is an attempt to vitalize instruction and to give a complete view of the field such as cannot be obtained in isolated courses. It puts the burden of his own education on the student where it belongs and is adaptable to his powers. Neither student nor instructor can avoid work. Yet many students and instructors alike do not know how we taught without it, or even how we managed with the lesser forms of seminar which were the forerunners of our present system.

FINDING THE PARTICULAR INTEGRAL IN A SPECIAL CASE

F. A. FICKEN, University of Tennessee and New York University

In a first course in differential equations, considerable attention is usually devoted to the linear equation $f(D)y=g(x)$, where $f(D)$ is a polynomial in D with constant coefficients. If the equation is non-homogeneous, one can find a particular integral in any case by variation of parameters, but special methods are often used to take advantage of the specific character of the right member of the equation. In certain cases, including some of practical interest, the right member is of the form $P(x)e^{\mu x}$, where $P(x)$ is a polynomial and μ is a constant. The object of this note is to emphasize the value of a particular combination of standard processes for handling this specific problem.

We assume throughout that x is a real variable and that the right member $g(x)$ is a linear combination of terms of the form $x^k e^{ax}\cos bx$ and $x^k e^{ax}\sin bx$ with k a non-negative integer and a and b real. The complementary function is not used.

The steps are as follows:

1. Express sines and cosines as imaginary exponentials. Then $g(x)$ is a linear combination of terms of the form $x^k e^{\mu x}$ with μ complex. Next, collect terms with the same μ. Then $g(x)$ is a sum of terms of the form $P(x)e^{\mu x}$, $P(x)$ being a polynomial. For each such term separately, use steps 2–5 to obtain a partial particular integral y_μ, and finally add the y_μ to obtain the whole particular integral.

Note that, if the coefficients of f are real, then y_μ for the complex conjugate of $P(x)e^{\mu x}$ will be the complex conjugate of y_μ for $P(x)e^{\mu x}$, and y_μ for the real (imaginary) part of $P(x)e^{\mu x}$ will be the real (imaginary) part of y_μ for $P(x)e^{\mu x}$. If, as often happens, the coefficients of $P(x)$ are real, one finds y_μ and uses its real or imaginary part according as $e^{\mu x}$ arose from a cosine or a sine. This common technique is illustrated in the example below.

2. Let $y=ze^{\mu x}$ and cancel $e^{\mu x}$, getting $f(D+\mu)z\equiv Q(D)z=P(x)$, where $Q(D)$ is a polynomial whose coefficients are generally complex if μ is.

3. Let $D^r z=w$, choosing r so that the constant coefficient of the polynomial operating on w in the left member is not zero.

4. The equation then has the form $Q_n(D)w=P_m(x)$, where $P_m(x)$ and $Q_n(D)$ are polynomials of degree indicated by the subscripts, and the "constant term" of $Q_n(D)$ *does not vanish.* Using undetermined coefficients, find a polynomial $w_m(x)$ *of degree m* that satisfies this equation. (See lemma at end of note.)

5. Obtain $z_{m+r}(x)$ by integrating $w_m(x)$ r times, letting constants of integration vanish. The required partial particular integral is then $y_\mu=z_{m+r}(x)e^{\mu x}$ (verification trivial).

The process may be illustrated by the following example:

$$f(D)y \equiv (D^3 - 5D^2 + 17D - 13)y \equiv (D - 1)(D^2 - 4D + 13)y$$

$$= \left(\frac{1}{10} - 6x\right) e^{2x}\cos 3x.$$

1. $(D^3-5D^2+17D-13)Y=(1/10-6x)e^{(2+3i)x}$. Since the coefficients of f and P are real, y is the real part of Y.

2. Let $Y=ze^{(2+3i)x}$, getting $f(D+2+3i)z\equiv[D^3+(1+9i)D^2-6(3-i)D]z=1/10-6x$.

3. With $w=Dz$, $[D^2+(1+9i)D-6(3-i)]w=1/10-6x$.

4. Substitute $w=\alpha+\beta x$ and equate coefficients, arriving at

$$w=-\frac{7(7-11i)}{6\cdot 10^2}+\frac{3+i}{10}x=Dz.$$

5. $$z=\int wdx=-\frac{7(7-11i)}{6\cdot 10^2}x+\frac{3+i}{2\cdot 10}x^2=Ye^{-(2+3i)x}$$

$$y=Rl(Y)=\frac{1}{6\cdot 10^2}[(-49+90x)\cos 3x-(77+30x)\sin 3x]xe^{2x}.$$

The factual information needed by the student is as follows: Step 1 and a possible final step (as from Y to y in the example) assume familiarity with the algebra of real and complex numbers and with the basic fact that $e^{it}=\cos t+i\sin t$. Step 2 merely "eliminates the exponential"; this is tactically advantageous in many similar situations. Step 3 introduces a new dependent variable so that the differential equation shall be of lowest order consistent with its form and our ignorance of the complementary function; one does this anyhow. Step 4 makes as economical a use as possible of undetermined coefficients. The particular choice of a trial function is governed by a simple *general* rule requiring no operations on the right members and no knowledge of whether or not or to what order $f(\mu)$ may vanish. Step 5 merely compensates for steps 3 and 2.

This factual equipment is quite modest. It is general and basic rather than specific and *ad hoc*. Most of it is used in earlier parts of the course, and the entire process may be illustrated—perhaps informally—in connection with the *first* order linear equation with constant coefficients. Students with this previous experience accept quite readily the combination of several simple, natural, familiar steps into a method or process, and it quickly becomes automatic. They welcome the absence of formulas and the economy and effectiveness of the process. The algebraic work seems often to be less massive than that required by other processes.

The effectiveness of step 4 is guaranteed by the following well known lemma whose proof is easy.

LEMMA: *Let m and n be non-negative integers. Let $q_0, \cdots, q_n$ be complex constants with $q_0\neq 0$ and $q_n\neq 0$, and let $Q_n(D)\equiv\sum_{j=0}^{n}q_jD^j$, with $D\equiv d/dx$. Let $p_0, \cdots, p_m$ be complex constants with $p_m\neq 0$, and let $P_m(x)\equiv\sum_{k=0}^{m}p_kx^k$.*

Then unique complex constants $a_0, \cdots, a_m$ exist, with $a_m\neq 0$, such that $y_m(x)=\sum_{k=0}^{m}a_kx^k$ satisfy the differential equation $Q_n(D)y=P_m(x)$. The a_k are real if all the p_k and q_j are real.

Students quickly learn, if $n>m$, to discard those q_j for which $j>m$. For example, if $m=0$, $Q_n(D)y=p_0$ has the solution $a_0=p_0/q_0$.

It may be mentioned that a very limited search has revealed a similar discussion of the special problem treated here only in Goursat's *Cours d'Analyse* (vol. II, pp. 458 ff. in fifth edition).

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 911. *Proposed by G. B. Huff, University of Georgia*

The following item appeared in the April 1, 1913, issue of the Erewhon Daily Howler: "Science lovers of Guayazuela will be disappointed to learn that the antarctic expedition of Professor Euclide Paracèlso Bombasto Umbugio, which was planned to start the first of next month and last through November, has been temporarily postponed. In making his characteristically thorough preparations, the eminent numerologist discovered that June 13 of this year falls on Friday and has decided that he will postpone the trip until a year when no Friday the thirteenth occurs in the seven months of the expedition. The Professor, an authority on the doctrine of probability, assured the press that the delay will be of short duration. He explained that the probability of an auspicious month is 6/7 and that hence the probability of seven consecutive auspicious months is $(6/7)^7$ or a little more than 2/5. From this the genius infers that the probability of postponing the trip next year is less than 3/5 and the probability that it will be postponed for nine consecutive years is $(3/5)^9$ or less than 1/100."

Of course, we now know that the Professor has never made his trip, though four times nine years have passed. Is the Professor having a tremendous run of bad luck?

E 912. *Proposed by H. E. G. P.*

Show that the infinite product

$$\prod_{n=1}^{\infty} (n+1)^3/n(n+x)(n+2x)$$

represents, in all points of its set of convergence, the rational function $2x$. (By permission of Prof. E. P. B. Umbugio, April 1, 1950.)

E 913. *Proposed by C. S. Ogilvy, Columbia University*

An isosceles triangle ABC has vertex angle $C=20°$. Points M and N are taken on AC and BC such that angle $ABM=60°$ and angle $BAN=50°$. Prove that angle $BMN=30°$. (Attention Prof. E. P. B. Umbugio.)

E 914. *Proposed by C. W. Trigg, Los Angeles City College*

It is desired to arrange congruent equilateral triangles, with at least one side of each triangle coinciding with a side of another triangle, in a plane configuration which can be folded along the common sides into a tetrahedron with no open edges. (An open edge is one through which there is straight line access to the centroid of the solid figure.)

(1) What is the smallest number of triangles necessary?

(2) How many distinct configurations containing this smallest number may be folded into a closed tetrahedron?

E 915. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The circle passing through the feet L, M, N of the cevians AL, BM, CN which divide the sides $BC=a$, $CA=b$, $AB=c$ of a triangle in the ratios

$$BL/LC = c^n/b^n, \qquad CM/MA = a^n/c^n, \qquad AN/NB = b^n/a^n,$$

intercepts on the lines BC, CA, AB signed segments of lengths x, y, z satisfying the relation

$$x/a^{n-1} + y/b^{n-1} + z/c^{n-1} = 0.$$

SOLUTIONS

Three Arithmetic Progressions

E 877 [1949, 473]. *Proposed by L. J. Burton, Bryn Mawr College*

Each of three arithmetic progressions continues indefinitely in both directions, and each has a difference which is an integer. Prove that if there is a term common to each pair of progressions then there is a term common to all three progressions.

Solution by J. H. Butchart, Arizona State College at Flagstaff. Without loss of generality we may increase or decrease all numbers so that the common term of progressions A and B is zero, and we may change the unit so that the common differences a, b, c will have no divisor greater than 1 common to all three. Thus A consists of multiples of a and B of multiples of b. Let jb be the term common to B and C and let $jb+kc$ be the term common to C and A, or otherwise ia, showing that k contains the greatest common divisor of b and a. Consider the term $jb+kc+pac$ common to A and C. It will be a term of B if

$k+pa=qb$, where p and q are to be determined. By Euclid's algorithm p and q may be found so that $qb-pa=k$, thus completing the proof.

Also solve by Max Beberman, D. H. Browne, Aaron Buchman, W. F. Cahill, M. S. Klamkin, Roger Lessard, T. F. Mulcrone, S. T. Parker, Azriel Rosenfeld, R. W. Shoemaker, and the proposer.

Relations in a Right Triangle

E 878 [1949, 473]. *Proposed by Kaidy Tan, Anglo-Chinese College, Amoy, China*

Let S be the incenter of the right triangle ABC, and X the point of contact of the hypotenuse BC with the incircle. With center X and radius XS describe the circle cutting BS, CS at M and N respectively. Let AD be the altitude on BC. Show that M, N are the incenters of the right triangles ABD and ACD respectively.

Solution by C. S. Ogilvy, Columbia University. Describe the circle with radius XS and center at Z, the point of contact of AB with the original incircle. It is clear that this circle also cuts SB at M. Let it cut ZB at Q. Then $\sphericalangle BAD = 90° - \sphericalangle ABD = 90° - 2\sphericalangle ABM = 90° - (90° - \overset{\frown}{MQ}) = \overset{\frown}{MQ} = 2\sphericalangle QAM$. That is, M lies on the bisector of $\sphericalangle BAD$, and is therefore the incenter of triangle BAD. Similarly it can be shown that N is the incenter of triangle CAD.

Also solved by Leslie Booth, Aaron Buchman, W. E. Buker, J. H. Butchart, P. W. M. John, L. M. Kelly, Sam Kravitz, Joseph Langr, Roger Lessard, F. L. Miksa, M. C. Stapp, C. W. Trigg, Roscoe Woods, and the proposer.

Trigg gave three solutions and remarked that other properties of the figure of the problem can be found in the article, *Relating to some relations in a right-angled triangle*, by Albert Babbitt, this MONTHLY [1918, 347–8].

Minimal Path Connecting n Points

E 880 [1949, 474]. *Proposed by Peter Ungar, University College, London, England*

Let n points be given in the plane, not all on a straight line. The shortest closed route connecting them is a simple polygon.

Solution by O. D. Smith, Oregon State College. Connect the n points by any closed path. Obviously a shorter path is obtained by connecting the points in the same order by a polygonal line with the given points as the only vertices. Suppose that segments P_iP_{i+1} and P_kP_{k+1} of this polygonal path intersect in a point P, so that the path followed is $P_iPP_{i+1} \cdots P_kPP_{k+1} \cdots P_i$. If P is not a point of the given set, then $P_iP_k \cdots P_{i+1}P_{k+1} \cdots P_i$ is a shorter polygonal path and does not contain the intersection P. If P is a point of the given set, then $P_iPP_k \cdots P_{i+1}P_{k+1} \cdots P_i$ is a shorter polygonal path and does not contain P as an intersection. This proves the theorem.

Also solved by L. M. Kelly, D. W. Matlack, E. A. Nordhaus, and the proposer

Factorial Series

E 881 [1949, 552]. *Proposed by C. M. Sandwick, Easton High School, Easton, Pa.*

(a) Show that any positive integer can be represented uniquely in the form $\sum_{n=1}^{\infty} a_n n!$, $0 \leqq a_n \leqq n$.

(b) Show that any positive rational number less than unity can be represented uniquely in the form $\sum_{n=1}^{\infty} a_n/(n+1)!$, $0 \leqq a_n \leqq n$.

Solution by Aaron Buchman, Buffalo, N.Y. (a) Let k be any positive integer and choose m such that $m! \leqq k < (m+1)!$. Then we have

$$k = a_m m! + r_m, \qquad 0 \leqq a_m \leqq m, \qquad 0 \leqq r_m < m!. \tag{1}$$

Again.

$$r_m = a_{m-1}(m-1)! + r_{m-1}, \qquad 0 \leqq a_{m-1} \leqq m-1, \qquad 0 \leqq r_{m-1} < (m-1)!.$$

Proceeding in this manner it is evident that

$$k = \sum_{n=1}^{\infty} a_n n!, \qquad 0 \leqq a_n \leqq n, \tag{2}$$

and representation of k in the required form is always possible. It is evident that $a_n = 0$ when $n > m$.

Since $h! = (h-1)(h-1)! + (h-1)!$, it follows that

$$h! = 1 + \sum_{n=2}^{h-1} (n-1)(n-1)!. \tag{3}$$

Now if, in (1), a_n should be decreased, the new remainder would no longer be less than $m!$. Then, by (3), it would be impossible to complete the summation and keep the rest of the a_i within the limits $0 \leqq a_i \leqq i$. Thus a_m may not be decreased, and of course it cannot be increased. Similarly $a_{m-1}, a_{m-2}, \cdots$ may not be altered, and the representation (2) is unique.

(b) Let u/v be any positive rational number, in lowest terms, with $u < v$. Then we have

$$u/v = a_1/2! + r_1, \qquad 0 \leqq a_1 \leqq 1, \qquad 0 \leqq r_1 < 1/2!. \tag{4}$$

Again,

$$r_1 = a_2/3! + r_2, \qquad 0 \leqq a_2 \leqq 2, \qquad 0 \leqq r_2 < 1/3!.$$

Proceeding in this manner it is evident that

$$u/v = \sum_{n=1}^{\infty} a_n/(n+1)!, \qquad 0 \leqq a_n \leqq n, \tag{5}$$

and representation of u/v in the required form is always possible. Let $(m+1)!$

be the least factorial which is divisible by v. Then, since

$$r_m = u/v - \sum_{n=1}^{m} a_n/(n+1)!, \qquad 0 \leqq r_m < 1/(m+1)!,$$

we see that $r_m = 0$, and consequently $a_n = 0$ when $n \geqq m+1$.

Since $1/h! = h/(h+1)! + 1/(h+1)!$, it follows that

$$1/h! = \sum_{n=h}^{\infty} n/(n+1)!. \tag{6}$$

Now if, in (4), $r_1 \neq 0$ and a_1 should be decreased, the new remainder would exceed $1/2!$. Then, by (6), it would be impossible to complete the summation and keep the rest of the a_i within the limits $0 \leqq a_i \leqq i$. Thus, if $r_1 \neq 0$, a_1 may not be decreased, and of course it cannot be increased. Similarly $a_2, a_3, \cdots, a_{m-1}$ may not be altered. It then follows, from (5) and (6), that only a_m may be decreased by one, and the following zero numerators replaced by $m+1, m+2, \cdots$. Thus there are *two* ways of representing u/v in the required form; one is a unique finite development, and the other is a unique infinite development.

Also solved by M. S. Klamkin, N. D. Lane, Roger Lessard, G. M. Merriman, C. S. Ogilvy, and the proposer. Some of these solutions only showed the possibility of representation in the required form but failed to deal adequately with the question of uniqueness.

Merriman pointed out that part (b) appears, in slightly sharper form, as ex. 12, p. 49, Birkhoff-MacLane, *A Survey of Modern Algebra*, and as ex. 2, p. 31, Hardy, *Pure Mathematics*, 6th ed. Arthur Rosenthal mentioned that although there are exactly two ways of representing a positive rational number less than unity in the required form, any irrational number less than unity has a unique representation. He said that these results were obtained explicitly by M. Stéphanos, *Bull. Soc. Math. France*, vol. 7 (1879), pp. 81–83, but are contained in even older and more general discussions of G. Cantor, *Zeitschrift für Math. u. Physik*, vol. 14 (1869), pp. 121–128. Series of this sort were later called *Cantor series*. See, *e.g.*, O. Perron, *Irrationalzahlen*, Berlin (1st ed., 1921; 2nd ed., 1939; 3rd ed., 1947), sec. 33. For the situation where the series has periodic coefficients see V. E. Dietrich and A. Rosenthal, *Bull. Amer. Math. Soc.*, vol 55 (1949), pp. 594–596.

The proposer stated the following generalization of part (a): *If $f(n)$ is a positive integer when n is a non-negative integer, and if $f(0) = 1$, any non-negative integer can be represented uniquely in the form*

$$\sum_{n=1}^{\infty} a_n \prod_{m=0}^{n-1} f(m), \qquad 0 \leqq a_n < f(n).$$

Thus, when $f(n) = n+1$, we have $N = \sum a_n n!$, $0 \leqq a_n < n+1$; when $f(n) = (n+1)^k$, we have $N = \sum a_n (n!)^k$, $0 \leqq a_n < (n+1)^k$; when $f(n) = k^n$ we have $N = \sum a_n k^{n(n-1)/2}$,

$0 \leqq a_n < k^n$; *etc.* Representation of a number in any scale of notation is also a special case. Let $f(n) = b$ when $n > 0$ and let $f(0) = 1$, then $N = \sum a_n b^{n-1}$, $0 \leqq a_n < b$.

Five Tetrahedra Forming a Decahedron

E 882 [1949, 552]. *Proposed by C. W. Trigg, Los Angeles City College*

Five regular tetrahedra arranged around a common edge just fail to fill space around that edge. (a) What is the closest approximation to a regular tetrahedron such that five such tetrahedra will fill the space around a common edge to form a decahedron having equilateral faces? (b) Show that the edge of the decahedron is $\sqrt{5}$ times the radius of the sphere touching those edges which radiate from the axis of the decahedron.

Solution by C. M. Sandwick, Easton High School, Easton, Pa. (a) Let *P-QAB* be one of the tetrahedra, forming with the other tetrahedra the decahedron *P-ABCDE-Q*. The plane of the regular pentagon *ABCDE* bisects *PQ* perpendicularly at *O*. Draw *OA*. If the edge *AB* is taken as unity, $OA = \sqrt{(5+\sqrt{5})/10}$. But $PA = 1$. Therefore $PO = \sqrt{(5-\sqrt{5})/10}$. Then $PQ = 2PO \doteq 1.05$. The distortion from regularity consists of an increase of slightly more than 5% in the length of *PQ*.

(b) Draw *OR* perpendicular to *PA*. Then $OR/OA = PO/PA$, or $OR = 1/\sqrt{5}$. Hence $PA = OR\sqrt{5}$.

Also solved by Roger Lessard and the proposer.

***N*-ics Having *n*-1 Roots in a Given Interval**

E 884 [1949, 552]. *Proposed by E. P. Starke, Rutgers University*

Show that there exist infinitely many rational integral equations with integral coefficients and leading coefficient unity, and of degree n, such that $n-1$ of the roots occur within a specified interval, however small.

Solution by Paul Erdös, Syracuse University. Let $x_1, x_2, \cdots, x_{n-1}$ be $n-1$ arbitrary distinct rational numbers in the specified interval and consider

$$x^n \pm c_n \prod_{i=1}^{n-1} (x - x_i) = 0, \tag{1}$$

where c_n is a large integer so chosen that all coefficients of (1) are integers. Now if c_n is sufficiently large then equation (1) has $n-1$ roots as close as we please to $x_1, x_2, \cdots, x_{n-1}$, and all conditions are met.

Also solved by C. S. Ogilvy and the proposer.

It is another matter to find the n-ic with smallest coefficients. The proposer gave $x^5+22x^4-47x^3+34x^2-10x-1$ as a quintic with four distinct roots between 0 and 1, and $x^4+12x^3-22x^2+12x-2$ as a quartic with three distinct roots between 0 and 1.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTIONS

4390. *Proposed by Roshan Lal Gupta, Government College, Ludhiana, India*

If $[x]$ denotes the greatest integer function, prove that

$$\sum_{q\geq 1} [(x/q^m)^{1/n}] = \sum_{q\geq 1} [(x/q^n)^{1/m}].$$

Is there any relation between the terms of the two sums?

4391. *Proposed by P. T. Bateman, Institute for Advanced Study*

Given a fixed positive integer k and a complex-valued function $f(n)$ defined on the positive integers and such that $f(n_1)=f(n_2)$ for $n_1\equiv n_2 \pmod{k}$, $|f(n)|\leq 1$ for all n, $f(n)=0$ for $(n, k)>1$, and $\sum_{n=1}^{k} f(n)=0$. Show that

$$\left|\sum_{k=1}^{\infty} \frac{f(n)}{n}\right| < \log k.$$

4392. *Proposed by Paul Erdös, University of Illinois*

Let $f(z)$ be analytic for $|z|\leq 1$. Let z_0 be the point ($|z_0|=1$) where $f(z)$ assumes its maximum on the unit circle. Prove that $f'(z_0)\neq 0$.

4393. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that: (1) The center ω of an equilateral hyperbola ($\mathfrak{H}$) inscribed in a triangle ABC, the orthocenter H of this triangle, and the center N of the nine-point circle (N) of the triangle of contact $\alpha\beta\gamma$ of ($\mathfrak{H}$) with the sides of ABC are collinear. (2) The polar circle of triangle ABC is tangent to the circle (N). (3) If BC, CA, AB; $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ respectively intersect one of the asymptotes of ($\mathfrak{H}$) in A', B', C'; α', β', γ', we have

$$\omega N/\omega H = \omega\alpha'\cdot\omega\beta'\cdot\omega\gamma'/\omega A'\cdot\omega B'\cdot\omega C'.$$

4394. *Proposed by H. F. Sandham, Dublin, Ireland*

Evaluate

$$\int_0^{\infty} \frac{\log x}{e^x+1}\,dx.$$

SOLUTIONS

Mutually Inscribed Tetrahedra

4279 [1948, 34]. Corrected. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Prove that a tetrahedron $ABCD$ whose vertices A, B, C are the inverses, with respect to a sphere with center D, of a right-angled triangle, and the tangential tetrahedron $A_1B_1C_1D_1$ of that tetrahedron are each inscribed in the other.

Solution by the Proposer.* 1. Preliminaries. If the vertices of a triangle ABC, of sides a, b, c, are transformed by an inversion of center D, with D exterior to the plane ABC, and power k, so that the points A', B', C' thus obtained are the vertices of a right triangle $A'B'C'$, with $\overline{B'C'}^2=\overline{C'A'}^2+\overline{A'B'}^2$, we have $B'C'=ka/DB\cdot DC$, $C'A'=kb/DC\cdot DA$, $A'B'=kc/DA\cdot DB$, whence we obtain

$$-a^2\cdot\overline{DA}^2+b^2\overline{DB}^2+c^2\overline{DC}^2=0. \tag{1}$$

The locus of points D satisfying (1) is a sphere whose center is the pole A_1 of BC with respect to the circumcircle of triangle ABC. The barycentric coordinates of A_1 with respect to the reference triangle ABC are $-a^2$, b^2, c^2.

2. By (1) the edges $BC=a$, $AD=a'$, $CA=b$, $DB=b'$, $AB=c$, $DC=c'$, of the tetrahedron $T\equiv ABCD$ satisfy

$$(aa')^2=(bb')^2+(cc')^2.$$

LEMMA. *The antiparallel sections in each of the trihedral angles of the tetrahedron T are right triangles for which the vertices of the right angles are on the edges BC and DA.* The antiparallel sections are similar to the associated triangle of tetrahedron T, since their sides are proportional to aa', bb', cc'. The vertices of the right angles of antiparallel sections in the trihedral angles (B) and (C) are on the edge BC and those of antiparallel sections in the trihedral angles (D) and (A) are on the edge DA.

THEOREM. *The faces CDA and DAB are conjugate with respect to the circumsphere of T. The same statement holds for the faces ABC and BCD.* The plane (π) tangent at A to the circumsphere (O) of T cuts the planes of the faces CDA and DAB in two perpendicular lines Δ_a and Δ_a' (Lemma). The line B_1A which joins A to the pole B_1 of face CDA with respect to (O) is in plane (π) where it is perpendicular to the tangent at A of the circumcircle of triangle CDA: as this tangent line is in the plane (π) it must coincide with the line Δ_a. Hence the line B_1A is the line Δ_a'. Thus the pole B_1 of plane CDA is in plane DAB and the pole C_1 of the latter is in plane CDA. Likewise the pole D_1 of plane ABC is in plane BCD and the pole A_1 of the latter is in plane ABC.

* Translated and checked by W. E. Byrne, Virginia Military Institute.

COROLLARY. *The tetrahedron T and its tangential tetrahedron $T_1 \equiv A_1B_1C_1D_1$ are inscribed one in the other.*

Tetrahedron and Inscribed Sphere

4298 [1948, 320]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $ABCD$, the incenter of which is I, the perpendiculars at A to the faces ACD, ADB, ABC respectively cut the planes ICD, IDB, IBC in A_1, A_2, A_3. (1) The perpendicular through A to the plane $A_1A_2A_3$ passes through the point of contact of the inscribed sphere with the opposite face BCD. (2) The analogous property is true for a triangle ABC.

*Solution by Robert Bouvaist, Nantes, France.** Let (x_1, y_1, z_1, t_1) be the normal coördinates of an arbitrary point M with respect to $ABCD$ taken as the reference tetrahedron. The perpendiculars at A to the faces ACD, ADB, ABC meet the planes MCD, MDB, MBC, at A_1, A_2, A_3. Plane $A_1A_2A_3$ has as its equation

$$\begin{vmatrix} x & y & z & t \\ x_1 & y_1 & -y_1\cos a' & -y_1\cos b \\ x_1 & -z_1\cos a' & z_1 & -z_1\cos c \\ x_1 & -t_1\cos b & -t_1\cos c & t_1 \end{vmatrix} = 0,$$

where $a, \cdots$ designate the dihedral angles of edges $BC, \cdots$. In the above equation we may replace (x_1, y_1, z_1, t_1) by $(1, 1, 1, 1)$ since we are interested in the case where M is the incenter I. The line through A perpendicular to $A_1A_2A_3$ meets the plane BCD at $(0, 1+\cos c', 1+\cos b', 1+\cos a)$. These are precisely the coördinates of the foot of the perpendicular dropped from I on the plane BCD.

The analogous property for a triangle may be established by the same type of calculations or by elementary geometry.

Part (2) also solved by B. R. Leeds.

A Summation Problem

4305 [1948, 431]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$1+\left(\frac{1+\frac{1}{2}}{2}\right)^2+\left(\frac{1+\frac{1}{2}+\frac{1}{3}}{3}\right)^2+\left(\frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{4}\right)^2+\cdots=\frac{17\pi^4}{360}.$$

Solution by Martin Kneser, Tübingen, Germany. Noting that

$$\frac{1}{m}=\left(\frac{1}{m}-\frac{1}{m+n}\right)+\left(\frac{1}{m+n}-\frac{1}{m+2n}\right)+\left(\frac{1}{m+2n}-\frac{1}{m+3n}\right)+\cdots=\sum_{j=0}^{\infty}\frac{n}{(m+jn)(m+jn+n)},$$

* Translated by W. E. Byrne, Virginia Military Institute.

we get

$$\frac{1}{n}\sum_{m=1}^{n}\frac{1}{m}=\sum_{m=1}^{n}\sum_{j=0}^{\infty}\frac{1}{(m+jn)(m+jn+n)}=\sum_{k=1}^{\infty}\frac{1}{k(k+n)},$$

because $k=m+jn$ takes all values 1, 2, 3, $\cdots$ if j takes the values 0, 1, 2, $\cdots$ and m the values 1, 2, $\cdots$, n.

In the following all summations run from 1 to infinity unless indicated differently. If we denote the sum in question by S, then we have

$$\begin{aligned}
S&=\sum_{n}\left\{\frac{1}{n}\sum_{m=1}^{n}\frac{1}{m}\right\}^2=\sum_{n}\left\{\sum_{k}\frac{1}{k(k+n)}\right\}^2=\sum_{k,j,n}\frac{1}{jk(k+n)(j+n)}\\
&=\sum_{k<j}\frac{1}{jk(k+n)(j+n)}+\sum_{k=j}\frac{1}{jk(k+n)(j+n)}+\sum_{k>j}\frac{1}{jk(k+n)(j+n)}\\
&=\sum\frac{1}{k^2(k+n)^2}+2\sum_{k<j}\frac{1}{jk(k+n)(j+n)}\\
&=\sum_{k<j}\frac{1}{k^2j^2}+2\sum\frac{1}{k(k+m)(k+n)(k+m+n)}\\
&=\frac{1}{2}\left\{\sum\frac{1}{k^2j^2}-\sum_{k=j}\frac{1}{k^2j^2}\right\}+2\sum\left\{\frac{1}{kmn(k+m+n)}-\frac{1}{mn(k+m)(k+n)}\right\}\\
&=\frac{1}{2}\left\{\left(\sum\frac{1}{k^2}\right)^2-\sum\frac{1}{k^4}\right\}+2\sum\frac{1}{kmn(k+m+n)}-2S
\end{aligned}$$

according to the third expression for S. Since

$$\sum\frac{1}{k^2}=\frac{\pi^2}{6}\quad\text{and}\quad\sum\frac{1}{k^4}=\frac{\pi^4}{90}$$

we obtain

$$\begin{aligned}
\frac{1}{2}\left(3S-\frac{\pi^4}{120}\right)&=\sum\frac{1}{kmn(k+m+n)}=\sum\int_0^1\frac{x^{k+m+n-1}}{kmn}\,dx\\
&=\int_0^1\left(\sum\frac{x^k}{k}\right)^3\frac{dx}{x}=-\int_0^1\{\log(1-x)\}^3\frac{dx}{x}\\
&=-\int_0^1\{\log x\}^3\frac{dx}{1-x}=-\sum\int_0^1 x^{n-1}(\log x)^3dx\\
&=\sum\frac{6}{n^4}=\frac{\pi^4}{15},
\end{aligned}$$

whence $S=17\pi^4/360$, as required.

Also solved by the Proposer.

Power Residues Modulo p

4307 [1948, 432]. *Proposed by Emma Lehmer, Berkeley, California*

Let p be a prime number of the form $3\cdot 2^{\alpha}\cdot k+1$, and let n be any number not divisible by 3 or 2^{α}. Show that there are numbers a, b, independent of n, such that a, b, $a+1$, $b+1$ are nth power residues modulo p.

Solution by Martin Kneser, Tübingen, Germany. Euler's criterium states that the congruence $x^n \equiv c \bmod p$ is solvable if $c^{(p-1)/d} \equiv 1 \bmod p$, where $d=(n, p-1)$, and then it has d distinct solutions. We therefore must prove the existence of numbers a, b, such that a, b, $a+1$, $b+1$ are zeros of $x^{(p-1)/d}-1 \bmod p$ for all n.

It is sufficient to prove that they are zeros of $x^6-1 \bmod p$, because n is not divisible by 3 or 2, and hence 6 is a divisor of $(p-1)/d$. Now we have

$$\begin{aligned} x^6 - 1 &= (x^3+1)(x^3-1) = g(x)(x^2+x+1), \\ (x+1)^6 - 1 &= \{(x+1)^3 - 1\}\{(x+1)^3 + 1\} \\ &= h(x)\{(x+1)^2 - (x+1) + 1\} = h(x)(x^2+x+1). \end{aligned}$$

So, if a and b are zeros of $x^2+x+1 \bmod p$, then a, b, $a+1$, $b+1$ are zeros of $x^6-1 \bmod p$. Now $x^2+x+1=(x^3-1)/(x-1)$. According to Euler's criterium, $x^3 \equiv 1 \bmod p$ has three distinct solutions and therefore $x^2+x+1 \equiv 0 \bmod p$ has two distinct solutions, a and b.

Remark. Instead of $p=3\cdot 2^{\alpha}\cdot k+1$, we can assume $p=2^{\alpha}\cdot 3^{\beta}\cdot k+1$, if we suppose n not divisible by 2^{α} or 3^{β}.

Also solved by B. A. Hausmann and the Proposer.

4308 [1948, 432]. *Proposed by H. S. Wall, University of Texas*

Let $f(z)$ be a function of the complex variable z, such that

$$f(-w^2) = -\frac{F(w)}{w},$$

where $F(w)$ has the following properties: (a) $F(w)$ is an odd function of w, (b) $F(w)$ is analytic and $I[F(w)] \leqq 0$ for $I(w)>0$. Prove that $I[f(z)] \leqq 0$ for $I(z)>0$.

Solution by M. S. Robertson, Rutgers University. Put

$$w = i(1+t)/(1-t), \qquad F(w) = g(t).$$

Then $R[ig(t)] \geqq 0$ for $|t|<1$. The Poisson-Stieltjes representation in the Hergotz form* gives

$$ig(t) = \beta_0 i + \frac{\alpha_0}{2\pi}\int_0^{2\pi} \frac{1+te^{i\theta}}{1-te^{i\theta}}\, d\alpha(\theta), \qquad |t|<1, \tag{1}$$

* G. Herglotz, Leipziger Berichte, 1911, pp. 501–511.

where $ig(0)=\alpha_0+i\beta_0$, $\alpha_0 \geqq 0$, and $\alpha(\theta)$ is a non-decreasing bounded function in $(0, 2\pi)$. In terms of w, and using $F(-w)=-F(w)$, we obtain

$$F(w)=\frac{\alpha_0}{2\pi}\int_0^{2\pi}\frac{w\,d\alpha(\theta)}{w^2\sin^2\frac{\theta}{2}-\cos^2\frac{\theta}{2}}, \qquad I(w)>0 \tag{2}$$

with $\alpha_0=-I[F(i)]\geqq 0$. Setting $z=-w^2$, $|\arg z|<\pi$, we have

$$f(z)=f(-w^2)=\frac{-F(w)}{w}=\frac{\alpha_0}{2\pi}\int_0^{2\pi}\frac{d\alpha(\theta)}{1+(z-1)\sin^2\frac{\theta}{2}}. \tag{3}$$

Since $\alpha_0 \geqq 0$ it is seen from (3) that $I[F(z)]\leqq 0$ for $I(z)>0$.

It is of some interest to note that $f(1-z)$ is univalent for $|z|<1$ when $f(z)$ is not a constant. If we write

$$\frac{f(1-z)}{\alpha_0}=1+\sum_{n=1}^{\infty}c_n z^n, \qquad \alpha_0 \neq 0, \qquad |z|<1,$$

$$c_n=\frac{1}{2\pi}\int_0^{2\pi}\sin^{2n}\frac{\theta}{2}\,d\alpha(\theta),$$

we see that the sequence $\{c_n\}$ is totally monotonic, and thus by Fejér's test $f(1-z)$ is univalent for $|z|<1$.

The Tritangent Circles

4309 [1948, 432]. *Proposed by R. Goormaghtigh, Bruges, Belgium.*

Let ABC be a triangle, D, E, F the contact points of one of the tritangent circles with BC, CA, AB, respectively, and let A', B', C' and D', E', F' be the projections of a point M of that circle on BC, CA, AB and EF, FD, DE. Show that the lines joining the projections of M on $A'E'$ and $A'F'$, $B'F'$ and $B'D'$, $C'D'$ and $C'E'$ are concurrent.

Solution by the Proposer. The proposed property is a consequence of the following known property of the Simson line: If $JDEF$ is a quadrilateral inscribed in a circle, the projections of a point M of that circle on the Simson lines of M as to the four triangles formed by the vertices of the quadrilateral are on a straight line.

When J and D coincide, the Simson lines of M as to the triangles formed by D counted twice and E or by D counted twice and F are $A'F'$ and $A'E'$; hence the line joining the projections of M on $A'F'$ and $A'E'$ passes through the projection of M on the Simson line of M as to DEF.

This last point is the same when J coincides with E or F, and the theorem follows immediately.

Absolute Value of Polynomial and Derivative

4310 [1948, 504]. *Proposed by Paul Erdös, Syracuse University.*

Let $f(z)=z^n+\cdots$ be a polynomial of degree n. Denote by A_f the closed region (not necessarily connected) where $|f(z)|\leqq 1$. Prove that there always exists a z_0 in A_f with $|f'(z_0)|\geqq n$. Equality occurs only for z^n.

Solution by Robert Breusch, Amherst College, Massachusetts. Call C the boundary of A_f; thus $|f(z)|=1$ along C. Call $\overline{A}_f$ the (infinite) complement of A_f. $\overline{A}_f$ is connected and its boundary is C. Form

$$H(z)\equiv\frac{\{f'(z)\}^n}{\{f(z)\}^{n-1}}\equiv\frac{n^nz^{n(n-1)}+\cdots}{z^{n(n-1)}+\cdots}.$$

$H(z)$ is regular in $\overline{A}_f$; $H(\infty)=n^n$.

(1) Assume $|f'(z)|<n$ everywhere in A_f. Then $|H(z)|<n^n$ on C, and therefore $|H(z)|<n^n$ everywhere in $\overline{A}_f$, in contradiction to $H(\infty)=n^n$.

(2) Assume $|f'(z)|\leqq n$ everywhere in A_f. Then $|H(z)|\leqq n^n$ on C, and therefore (because of $H(\infty)=n^n$) $H(z)\equiv n^n$ in the whole plane. This means

$$\frac{1}{n}\{f(z)\}^{(1-n)/n}f'(z)\equiv 1,$$

whence

$$\{f(z)\}^{1/n}\equiv z+a,\qquad f(z)\equiv(z+a)^n.$$

Also solved by J. H. Curtiss, J. L. Ullman, and the Proposer who remarks that this is an easy corollary of no. 4229 [1948, 171].

Euler's Totient, Iteration

4311 [1948, 504]. *Proposed by V. L. Klee, Jr., University of Virginia*

If k and x are positive integers, let $f_k^1(x)=k\phi(x)$, where $\phi(x)$ is Euler's totient. For $j=2, 3, \cdots$, let

$$f_k^j(x)=f_k^{j-1}[f_k^1(x)].$$

Show that for $k\leqq 3$, the sequence, $f_k^1(x)$, $f_k^2(x)$, $\cdots$, is eventually constant, while for $k\geqq 4$, the sequence is eventually monotonically increasing.

Solution by Leo Moser, University of Manitoba. Let $f_k^j(x)=x_j$. If x_i contains any prime factor greater than k, then the operation $k\phi$ performed on x_i reduces the exponent of its largest prime factor by 1. Hence if k is fixed, there exists a j such that for all $i>j$, all the prime factors of x_i are less than or equal to k.

For $k=1$, for $i>j$, $x_i=1$.

For $k=2$, for $i>j$, $x_i=2^a$ and $2\phi(2^a)=2^a$.

For $k=3$, for $i>j$, $x_i=2^a3^b$ and $3\phi(2^a3^b)=2^a3^b$.

For $k\geqq 4$, for $i>j$,

$$k\phi(x_i) = kx_i \prod_{p \leqq k} (1 - 1/p) > kx_i \prod_{r=2}^{k} (1 - 1/r) = x_i,$$

whence $x_{i+1} > x_i$, which completes the proof.

Also solved by Paul Erdös, Free Jamison, and J. H. Wahab.

The Clairaut Differential Equation

4315 [1948, 586]. *Proposed by Albert Wilansky, Lehigh University*

Consider the Clairaut differential equation $y = px + f(p)$, where $p = y'$ and f has a derivative. Prove that if f' is monotone the singular solution has exactly one point in common with any particular solution.

Solution by S. H. Gould, Purdue University. The singular solution consists of the set of points (x, y) represented parametrically by

$$px - y + f(p) = 0, \qquad x + f'(p) = 0,$$

where $x = x(p)$, $y = y(p)$. A particular solution is represented by

$$cx - y + f(c) = 0,$$

with fixed c. These three equations have in common the point $x = -f'(c)$, $y = -cf'(c) + f(c)$, obtained by setting $p = c$. We now set $p = k \neq c$. If the equations are still consistent, their determinant vanishes:

$$\begin{vmatrix} k & -1 & f(k) \\ 1 & 0 & f'(k) \\ c & -1 & f(c) \end{vmatrix} = f(c) - f(k) + f'(k) \cdot (k - c) = 0.$$

Thus

$$\int_c^k f'(p)dp = f(k) - f(c) = f'(k) \cdot (k - c) = \int_c^k f'(k)dp,$$

which means, since $f'(p)$ is monotone, that $f'(p)$ is constant in $[c, k]$, so that $f'(k) = f'(c)$ and $f(k) = f(c) + f'(c) \cdot (k - c)$. But then

$$x(k) = -f'(k) = x(c) \quad \text{and} \quad y(k) = -kf'(k) + f(k) = y(c),$$

so that the common point $x(c)$, $y(c)$ is unique.

Also solved by E. S. Keeping, and Ingo Maddaus, Jr., and, assuming that also f' is strictly monotone, by Ta Li, Norman Miller, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Fundamental Science. By A. S. Eddington. Cambridge University Press, 1946. 8+292 pages. $6.00.

At the time of his death in November 1944, Sir Arthur Eddington left a manuscript which forms the basis of the present treatise "Fundamental Science." Although Eddington, had he lived, would have added an introduction to the book and also several more chapters one can consider the present work as complete in itself. Sir E. T. Whittaker, in the preface, points out that this book practically replaces all of Eddington's previous writings on his theory of the constants of Nature. The treatment of the theory in the first five chapters is essentially that used by Eddington in his Dublin lectures of 1943. The next three chapters, on sedenion analysis, contain work given in his book Relativity Theory of Protons and Electrons of 1936. In the last four chapters Eddington treats several new problems, including radiation, and the work of these chapters is essentially an addition to his previous published work on his new theory.

The aim and scope of the book is to provide a theory for the explanation of physical phenomena which range from the microscopic to the macroscopic. With such an aim in mind the reviewer feels that it would have been better to have written the book from the point of view of a deductive science. By this we mean that fundamental assumptions and definitions are clearly stated and listed and the main theory derived as consequence of these assumptions and definitions. Many times through the book Eddington seems to borrow results and concepts from quantum theory and one is never quite sure how these tie in with his new ideas. As a simple example we quote from the text:

"The term 'particle' survives in modern physics, but very little of its classical meaning remains. A particle can now best be defined as the conceptual carrier of a set of variates. We shall frequently use the term 'carrier' as an alternative to particle." In view of the fact that Eddington deems it necessary to define the term particle it is hard to see how it can be justified in using old concepts such as mass of a particle, energy of a particle, without a word of explanation of how these classical concepts are related to his new definition of a particle. This constant intermingling of new ideas and concepts and results obtained from modern quantum theory often obscures the justification for many of the statements and derivations given in the text.

How successful Eddington is in completing the aim and scope of his new theory the reviewer does not feel competent to judge. Certainly his theoretical calculations on the constants of Nature are in startling agreement with

observational results. In addition one feels that no work by a man of Eddington's stature in the scientific world could ever be lightly dismissed.

M. WYMAN

Advances in Applied Mechanics. Edited by R. von Mises and T. von Karman. (Vol. I). New York, Academic Press Inc. 8+293 pages. $6.80.

This book consists of six survey articles covering recent advances in applied mechanics and as such, is useful to both the expert and the beginner in any of the diversified fields included. Briefly, the subject matter is concerned with hydro-dynamics including boundary layer phenomena, turbulence, shock waves, compressible flows, elasticity, and problems in non-linear mechanics.

I. The first article by H. L. Dryden of the National Bureau of Standards is concerned with the mechanics of boundary layer flow and is largely non-mathematical and expository in character. While the author notes recent contributions to the mathematical concept of the boundary layer, including work on special flows, and the effects of compressibility on the laminar layer, the main part of the article reviews developments in the problem of stability of laminar flows—that is, the problem of the turbulence in the boundary layer. The recent experimental work (1940) of Schubauer and Skramstad of the National Bureau of Standards have confirmed and unified the theory of Heisenberg-Tolmien-Schlichting of small oscillations causing unstable waves and the theory of G. I. Taylor that free stream turbulence is the main factor in causing transition. s a result of the wealth of experimental data which is reproduced here and the completeness of the discussion this article is highly recommended as providing the reader with an over-all picture of the present state of boundary layer phenomena.

II. The second article by N. Minorsky of Stanford University is concerned with periodic solutions to non-linear differential equations of second order. It is a survey of the material contained in his book which has been published by Edwards Brothers, Ann Arbor, Michigan. Here, necessarily, the treatment is sketchy. But in spite of this, it is an extremely readable account of most of the important ideas in this field. These include the concept and classification of singular points to the differential equation, the concepts of limit cycles, indices and the well known theorems of the Benedixson concerning them, and Liapounoff stability. The analytical methods of solution which are discussed include those of Poincaré, Van der Pol and the Kryloff and Bogoliuboff adaption of the astronomical method of Lindstedt. The article concludes with a discussion of aspects of the problems of non-linear resonance including subharmonic resonance, entrainment of frequence, and parametric excitation. For those unacquainted with this field, this article should be most useful in enabling them to quickly grasp many of the essential ideas. On the other hand, with the publication of the Minorsky book mentioned above and the recently issued translation of "The Theory of Oscillations" by Andronow and Chaikin (Princeton

Univ. Press) most of the same material is available in English in a much more detailed fashion for the worker in this field.

III. The third article of this text is "Elasticity Papers Published in Holland, 1940–1946" by C. B. Biezeno. This article contains short reviews of a number of papers in theoretical and experimental elasticity. The theoretical papers are divided into the following groups: stability and buckling of shells and thin-walled cylinders; stresses in perforated plates; circular plates supported at a number of points regularly distributed along the boundary and rotatory-symmetrically loaded; miscellaneous papers dealing with the bending of beams, torsion of prismatic beams, the convergence of a special iterative process in structural analysis. Much attention is devoted to non-linear problems and their numerical solution which, for the most part, were designed to make the results available for all time. As such, someone contemplating work in this field would do well to scan this article before initiating any large scale calculations. A brief description of numerous experiments is furnished. Finally, two texts, "Technische Dynamik" by C. B. Biezeno and R. Grammel and "Plasticiteitsleer" by F. K. Th. van Iterson are discussed. Unlike the first two articles from which some ideas can be gleaned without too much previous experience in their fields, this article presupposes a thorough knowledge of the theory of elasticity.

IV. In the article "Theory of Turbulence" by J. M. Burgers, the author discusses a system of partial differential equations which he uses as a mathematical model for the theory of turbulence. In the author's terms, the difficulties encountered in the theory of turbulence are of a dual nature, the geometrical complexities introduced by the vectorial nature of the velocity and the complexities introduced by the non-linear terms. Independent variables are time t, and distance y; the dependent variables are U, the velocity of the primary motion, P the "analogue of the exterior force acting upon the primary motion," v the velocity of turbulence. First, the author shows that his system admits a first integral which is interpreted as consisting of various types of energies. Then, for the finite domain laminar solutions ($v=0$), and stationary turbulent solutions ($\partial v/\partial t=0$, $U=$constant) of the author's system are found. The latter involve the replacement of the original system by appropriate approximation equations. In particular, the author studies the spectrum of the solution for v and calculates the dissipation per component, *etc.* Finally, non-stationary solutions ($\partial v/\partial t\neq 0$, $U=$constant) are discussed for finite and infinite domains. Other developments include: a discussion of the correlation function; similarity conditions; systems with two components (u, w) in the secondary motion. The latter part of the article is devoted to a case in which the analogue of turbulent motion is allowed in two dimensions. The article is very readable and should prove of value to the novice in the field since it enables one to become acquainted with some of the important features of turbulent motion.

V. The article "Numerical Methods in Wave Interaction Problems" by H. Geiringer compares the author's and J. von Neumann's unpublished methods

of solving wave interaction problems. The calculations are carried out in detail for a specific simple boundary value problem in which Cauchy data are given at $t=0$ and side conditions are given along two straight walls. In the author's method, a single pressure-volume relation is assumed (weak shocks). The (x, t) plane is then divided into regions. Due to the simple boundary conditions, explicit formulas (due to Riemann) for the solution can be obtained in some of them. However, the principle difficulty lies in determining the solution in the zones beyond the unknown points where the shock line and rarefaction zone meet. The problem, here, is one of solving the partial differential equations of fluid dynamics when the boundary data satisfy certain relations (the shock conditions) along an unknown curve (the continuation of the straight shock line). To avoid this difficulty, the author makes use of a numerical method of essential successive approximations based on difference equations combined with interpolation. A slight modification of this scheme enables one to treat the case of strong shocks. Following this, von Neumann's mechanical model of shock motion and non-linear wave motion is described. Here it is imagined that the individual particles are connected by "springs" which are subject to non-linear laws of attraction and repulsion. Pictorial evidence is shown of the occurrence of oscillations in the solution where the shock line was previously determined by the author's method of calculation. All in all, this article is clearly written and provides an excellent résumé of many interesting results which came from the war effort in this direction.

VI. The sixth and final article by v. Mises and Schiffer provides a very readable account of Bergman's method of integration of Chaplygin's hodograph equation for the stream function of a steady irrotational flow. Chaplygin's basic idea was to make use of the properties of known incompressible flows in order (by analogy considerations) to deduce information about compressible flows. While this is possible in simple cases, there are many flow functions which are neither regular nor single-valued even for the incompressible case when they are expressed in the hodograph plane. Bergman's procedure is to first transform Chaplygin's equation into a form in which the stream function may be expanded in terms of an infinite series of products of harmonic functions and functions of a single variable. Recurrence relations are determined for the successive terms of this expansion, the series explicitly determined. It is shown to converge uniformly in a sector of the complex plane. However, the calculations are very long and tedious. In the second part of the paper, the authors (following Bergman) modify the basic partial differential equation, in order to obtain an easily computed expansion for the stream function. It is shown, in the remainder of the paper, that this implies use of a modified pressure-volume law. Graphical comparisons of the effects of this modified pressure-volume relation, the Karman-Tsien pressure-volume relation, and the pressure-volume relation for a real gas are furnished.

Because of the wide range of topics, the book requires the knowledge of diverse and specialized mathematical techniques on the part of the reader. More-

over the articles range from the first, which is purely expository, to the last, which is almost purely mathematical. Consequently, this book will serve different purposes for different readers.

C. L. DOLPH
N. COBURN

NEW BOOKS RECEIVED

The Theory of Probability, 2d Edition. By H. Reichenbach. Translated by E. H. Hutten and M. Reichenbach. Berkeley, University of California Press, 1949. 16+492 pp. $12.50.

Principles of a New Energy Mechanics. By Jakob Mandelker. Philosophical Library, New York, 1949. 73 pp. $3.75.

Tables of the Confluent Hypergeometric Function $F(n/2, 1/2; x)$ and Related Functions. (Applied Mathematics Series, No. 3.) Prepared by the Computation Laboratory of the National Applied Mathematics Laboratories, National Bureau of Standards. Washington, U. S. Government Printing Office, 1949. 22+73 pp. $.35.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS 1948–1949

Kappa Mu Epsilon, Central Missouri State College

The *Missouri Beta* chapter of *Kappa Mu Epsilon* held regular monthly meetings during the academic year 1948–49 and the summer session of 1949. The following talks were presented by members of the fraternity:

Geography and mathematics, by James Schmer
Pythagorean number triples, by Harold Woods
Calendars of the past and present, by Donna Lee Chitty
Calendrical computations, by Vol Russ
History and construction of magic squares, by Robert Boothe
Computational short-cuts and checks, by Quentin Smith and Keith Stumpff
Squaring the circle, by James Green
Trisecting the angle, by Myron Fitterling
Mathematical induction, by Edward Lowry
Boolean algebras, by Norton Jones
Inversion, by Marvin Goodman.

At the annual banquet in the Spring, Dr. E. J. Clark of the Education Division spoke on *The use of mathematics by the school counselor.* Nine new members were initiated into the fraternity. Eight members were previously initiated during a regular meeting in the Fall.

The members of our chapter journeyed to Central College, Fayette, Missouri in May to install the *Missouri Epsilon* chapter of *Kappa Mu Epsilon.* Dr. Claude H. Brown was the installing officer, and he was assisted by Miss Vonda Leah Lankford, Miss Donna Lee Chitty, Mr. Quentin Smith, and Mr. Robert Boothe, all of *Missouri Beta.*

Officers for 1949–50 are: President, James Green; Vice-President, Robert Boothe; Secretary, Donna Lee Chitty; Treasurer, Quentin Smith; Sponsor and Corresponding Secretary, Prof. L. W. Akers.

Pi Mu Epsilon, Oregon State College

At the five regular meetings of the *Oregon Beta* chapter of *Pi Mu Epsilon* the following programs were given during 1948–49:

The Laplace transformation, by Newton Smith
Paradoxes in infinite series, by O. Dale Smith
A graphometer, by Dr. Howard Eves
Happy mathematics, by James Price
Linkages, by J. Ericksen
Some solved and unsolved problems in geometry of sets of points, by A. S. Besicovitch of Cambridge University, England.

Early in the school year, *Pi Mu Epsilon* sponsored a joint picnic with the mathematics faculty. Another social event of the year's activities was the initiation of fifty-one new members in May. At the banquet held after the initiation, Dr. R. A. Rosenbaum of Reed College, Portland, Oregon, gave a talk on *Some characteristic properties of a circle.*

The officers for the academic year 1949–50 are: Director, R. D. Oeder; Vice-Director, O. Dale Smith; Secretary, R. I. Daniels; Treasurer, Prof. G. A. Williams.

Delta X, University of Toledo

The following papers were presented during the twentieth annual program of *Delta X*, Mathematics Club of the University of Toledo:

Air force research, by Dr. Carroll Amos
Mathematical puzzles and paradoxes, by Dr. Howard Swift
The trisection of the angle, by Richard Castanias
A problem in Diophantine equations, by Douglas Lang
Graphical analysis, by Roger Woodruff
History and computation of pi, by John Suprock
Relativity, by Jack Scarlett
Nomograms, by Richard Reisbach.

Each of the regular meetings concluded with a social period at which refreshments were served. *Delta X* was host to the Maumee High School *Mathematics Club* at one meeting and entertained the *Icosahedron Club* at a Christmas party. The social activities also included an annual banquet and a picnic.

The officers for 1949–50 are: President, Jack Scarlett; Vice-President, George Halgas; Secretary-Treasurer, Anna Tom.

Mathematics Club, Regis College

The annual record of activities of the Regis College *Mathematics Club* for 1948–49 include the following papers:

Shadows, by Anna McFarlane
Approximate construction for e, by Phyllis Moran
Lewis Carroll, mathematician, by Katherine Healey
Moebius strip, by Ann McDonnell
Mathematics in music, by Florence Seaver
Mathematical recreations, by Claire Sweeney
Astronomy, by Ann McDonnell
The abacus, by Ann Downey
Poets' numbers, by Katherine Healey
Peaucellier cells, by Margaret Linney
Architectural wonders of the world, by Barbara Lane
Was man made by Earth or was Earth made for man?, by Louise Kelley
Mathematical designs, by Mary White
Opportunities for mathematically trained college graduates
Use of mathematics in aircraft industry, by Maryann Boyce.

The program included a Christmas party, a supper party at which members of the class of '49 were guests, and a tea at which the Mathematics Alumnae announced the winner of a $25.00 award for the graduate from the department who had earned the highest average in mathematics over a period of four years.

The entire club attended the meeting of the *Greater Boston Mathematics Club* which was held at Harvard College. There was also an attendance of delegates at meetings of the Association of Teachers of Mathematics of New England.

Our projects included *From Near and Far*, a pin map displaying the geographic location from which Regis College students come and to which Regis Alumnae go. A second graph in the College Display Case depicts the survey of the positions which our Alumnae hold and discloses a summary of the vocations in which the Regis College Alumnae are working. Ingenious constructions in balsa have been added to the model collection including two views of the tesseract and simplex.

Officers for 1948–49 were: President, Barbara Lane; Vice-President, Katherine Healy; Secretary, Maryann Boyce; Treasurer, Virginia Lee.

Mathematics Club, Kansas State College

During the academic year 1948–49, the following papers were presented at meetings of the *Mathematics Club* of Kansas State College:

A new look at an old formula, by Prof. F. Sloat
A history of the exponent, by V. Bly
Graphic solution of equations, by K. Loewen
Matrices, by Miss Doris Lloyd
Mathematics and the imagination, by A. Lloyd
Celestial navigation, by L. Trapp
A history of the calculus of variations, by Prof. R. Sanger
Symmetries of the square. by W. Cowell.

Mr. Bly's paper was subsequently published in the February 1949 issue of the *Bulletin of the Kansas Association of Teachers of Mathematics.*

Officers for 1948–49 were: President, W. Cowell; Vice-President, V. Bly; Secretary-Treasurer, A. Lloyd; Faculty Advisor, Prof. P. Young.

Pi Mu Epsilon, Marquette University

The following papers were presented to the members of the *Wisconsin Alpha* chapter of *Pi Mu Epsilon* during 1948–49:

How to draw a straight line, by Dr. H. Pettit
Symmetrical coordinates, by Dr. S. Canterbury
A physicist looks at mathematics, by Dr. G. Barkow
Astronomy, by Mr. Edward Halbach
Mathematical education in central Europe, by Dr. J. Talacko
Sir Isaac Newton, by Dr. G. Parkinson.

A Christmas party and an annual banquet were held during the year.

Officers elected for 1949–50 are: Director, William Golomski; Vice-Director, William Weideman; Recording Secretary, Kathleen Murphy; Corresponding Secretary, Virginia Higgins; Treasurer, Robert Johnson; Librarian, Harry Hesse.

Kappa Mu Epsilon, Southwest Missouri State College

The *Missouri Alpha* chapter of *Kappa Mu Epsilon* at Southwest Missouri State College has an enrollment of 26 members and participated in five meetings during the academic year of 1948–49. The following papers were presented:

A finite geometry of twenty-five points, by Edward Rykowski and Carl Gabriel
A photographic method of portraying cycloidal loci, by L. Shiflett
Schwartz's problem, by Julius Komarmy
The sundial, by Dr. V. Whitney

The first paper listed above also was presented by Mr. Rykowski at the

national biennial convention of *Kappa Mu Epsilon* at Washburn University, Topeka, Kansas, in April.

Officers for 1949–50 are: President, Don Barbarick; Vice-President, Kenneth Mustain; Secretary, Mary Jane Robinette; Treasurer, James Jakobsen; Corresponding Secretary, Harold Skelton; Sponsor, Prof. P. Pummill.

Kappa Mu Epsilon, Coker College

The *South Carolina Alpha* chapter of *Kappa Mu Epsilon* held regular monthly meetings during the school year 1948–49. The programs were composed entirely of discussions conducted by various students on current mathematical problems and puzzles. A number of problems proposed in the PENTAGON and the MONTHLY were reviewed, and some of the members turned in solutions to these problems. For next year a more varied program of papers is being planned.

The officers for 1949–50 are: President, Shirley Jenkins; Secretary, Betty Rose Reeves; Faculty Advisor, Frank W. Saunders.

Mathematics Club, Upsala College

The *Mathematics Club* of Upsala College held a Christmas party and a mid-year festival jointly with *Kappa Mu Epsilon*, and a buffet supper at the home of one of the members.

The Club had bimonthly meetings. The plan was to study a famous personality by investigating his mathematical environment and some of his contributions. The following papers were presented:

The life of Blaise Pascal, by James Gill

The Pascal triangle, by Martin Monroe

The life of Gauss, by John Armstrong

Gauss' law of quadratic reciprocity, by Robert Reed

Development of Gauss' theorem pertaining to the electric intensity about any closed surface, by Robert Warnken

The life and contributions of Zeno, by Seymour Hersh, William Manco, and Edward Pulchlopek

The life of Napier, by Lloyd Thornton

Napier's rods, by Frances Rischmuller

Evariste Galois, by Hannah Magill

The theory of fields and groups, by Jerome Drexler

The life and works of Pierre de Fermat, by Dorothy Fischbeck and William Stachel

Infinite series, by Ethel Larson

Mathematical truths, by James Christakos

Mathematical fallacies, by Jack Presser

The invention of analytical geometry, by Clair Willhelm

The squaring of a circle, by Robert Schenk and Bernt Benson.

The newly elected officers for the year 1949–50 are: President, James Christakos; Vice-President, William Stachel; Secretary, Dorothy Fischbeck; Treasurer, Richard Lawrence.

Mathematics Club, University of Wisconsin

Following is a list of the talks given at the various *Mathematics Club* meetings at the University of Wisconsin:

On the approximation of a function by a power series, by Prof. Max Dehn

Convex functionals, by Prof. W. E. Eberlein

Affine invariants of a pair of hypersurfaces, by Dr. C. C. Hsiung

Elastic stability of plywood plates and shells, by Prof. H. W. March

Multiple Fourier series, by Prof. Lamberto Cesari of the University of Bologna

On certain classes of algebraic integers, by Dr. J. B. Kelly

The topology of function spaces, by Prof. Casimir Kuratowski of the University of Warsaw and the Institute for Advanced Study

Generalized curves, by Prof. L. C. Young

Asymptotic solutions of a second order differential equation, by Dr. E. D. Cashwell

Undecidable sentences in set theory and real numbers arithmetic, by Prof. Andrzej Mostowski of the University of Warsaw and the Institute for Advanced Study

Development of the axiomatic definition of set and Skolem's paradox, by Dr. Alfons M. F. Borgers of the Belgian National Foundation for Scientific Research

Paracompact spaces, by Dr. C. H. Dowker of Princeton University

The generalization of the Moebius mu-function to a partially ordered set, by Mr. A. F. Strehler

Integral valued entire functions, by Dr. R. C. Buck of Brown University.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1950:

Columbia University, Teachers College. July 10 to August 8: Professor Clark, teaching arithmetic in the elementary school; Professor Fehr, current problems in teaching secondary school mathematics, professionalized subject matter in advanced secondary school mathematics; Professor Shuster, modern business arithmetic, teaching algebra in secondary schools; Professor Snader, teaching

geometry in secondary schools, history of mathematics; Professor Yates, field work in mathematics, applications of mathematics.

Duke University. June 14 to July 1: Professor Dressel, solid analytic geometry. July 6 to August 12: Professor Carlitz, finite differences, thesis seminar; Professor Gergen, mathematical statistics, theory of functions. August 14 to August 31: Professor Roberts, theory of functions.

Ohio State University. June 20 to September 1: Professor Reichelderfer, advanced calculus, dimension theory; Professor Miller, solid analytical geometry, fundamental ideas in algebra and geometry; Professor Rechard, advanced calculus, differential equations; Professor Cashwell, integral equations.

State University of Iowa. June 12 to August 9: Professor Knowler, elements of statistics, actuarial aspects of casualty insurance; Professor Wylie, astronomy; Professor Smiley, synthetic projective geometry, introduction to advanced mathematics; Professor Woods, study in secondary geometry, pure geometry; Professor Conkwright, differential equations; Professor Oberg, elementary theoretical mechanics, Fourier and allied series; Professor Price, the supervision of mathematics; Professor Muhly, matrices and determinants, elementary theory of numbers; Professor Cosby, advanced calculus, calculus of variations.

Syracuse University. July 5 to August 11: infinite series; intermediate course in algebra; introduction to modern mathematics; history of mathematics; aids in the teaching of high school mathematics.

University of Buffalo. July 5 to August 12: Professor Gehman, infinite series; Professor Montague, algebras and their arithmetics, geometrical constructions; Professor Schneckenburger, elementary finite groups.

University of California at Los Angeles. June 19 to August 11: Professor Bell, workshop in secondary school mathematics; Professor Daus, geometrical constructions; Professor Hedlund, differential geometry in the large; Professor Horn, advanced calculus; Professor Sokolnikoff, partial differential equations; Professor Taylor, functions of a complex variable; Dr. Wing, vector analysis; Professor Zelinsky, theory of fields.

University of Colorado. June 19 to July 21 and July 24 to August 25: Professor Kurt Mahler, University of Manchester, England, geometry of numbers; Mr. Charlesworth, advanced course in teaching of mathematics (first term); Professor Farnell, higher algebra (second term), calculus of variations; Mr. Hunt, elementary differential equations; Professor Hutchinson, Fourier series and boundary value problems; Professor Jones, higher algebra (first term), elementary topology (first term); Professor Kendall, selected topics in geometry (second term).

University of Detroit. June 26 to August 4: Professor Smith, theory of equations, differential geometry; Professor McCarthy, advanced calculus, Fourier series and harmonic analysis; Professor Mehlenbacher, intermediate differential equations.

University of Illinois. June 20 to August 19: Professor Bourgin, theory of rings; Professor Day, functions of real variables; Professor Ketchum, theory of

functions of a complex variable; Professor Landin, projective geometry; Professor Munroe, measure and integration; Staff, fundamental concepts of mathematics; introduction to higher algebra; introduction to higher geometry; introduction to higher analysis, complex variables; mathematical methods in engineering and science; introduction to higher analysis, real variables.

University of Michigan. June 26 to August 19. Professor Eckmann, general spaces, combinatorial methods; Professor Hildebrandt, theory of differential equations; Professor Myers, differential geometry; Professor Carver, theory of statistics II, finite differences; Professor Craig, multivariate analysis; Professor Dwyer, analysis of variance, statistical and computational methods with punched cards; Professor Dushnik, foundations of mathematics; Professor Fischer, intermediate mathematics of life insurance; Professor Hay, advanced mechanics, vector analysis; Professor Rothe, application of functions of a complex variable, topics in mathematical physics; Professor Thrall, higher algebra, theory of matrices; Professor Bartels, Fourier series; Professor Coburn, dynamics; Professor Dolph, operational mathematics; Professor Jones, teaching of algebra, history of geometry; Professor Reade, functions of a complex variable, harmonic and subharmonic functions; Professor Young, advanced calculus; Dr. Darling, theory of probability, theory of statistics I; Dr. Leisenring, synthetic projective geometry.

University of Nebraska. June 6 to July 29: Professor Basoco, differential equations, solid analytic geometry; Professor Leavitt, advanced calculus; Professor Halfar, fundamental concepts of mathematics; Professor Camp, analysis.

University of North Carolina. June 12 to July 20: Professor Cameron, elementary algebra from an advanced viewpoint; Professor Garner, history of mathematics; Professor Hill, elementary mathematical statistics; Dr. Mann, advanced calculus; Professor Brauer, advanced number theory of rational integers; Professor Brown, algebraic invariants. July 21 to August 29: Professor Mackie, theory of equations; Professor Hoyle, differential equations; Professor Whyburn, advanced calculus II; Professor Hershner, advanced theory of functions of a complex variable; Professor Lasley, differential geometry.

University of Oregon. June 19 to August 11: Professors Civin and Niven, algebra for teachers; Professor Ghent, geometry for teachers, foundations of mathematics for teachers; Professor Moursund, selected topics (pure mathematics); Professor Civin, transformations of Fourier and Laplace.

University of Wisconsin. June 26 to August 18: Professor Bing, advanced analytic geometry, higher mathematics for engineers; Professor Mayor, topics in geometry; Professor Sokolnikoff, higher mathematics for engineers; Professor Owens, higher analysis; Professor Young, advanced calculus, divergent series; Professor Mark, advanced calculus; Professor Arnold, mathematics of elementary statistics; Professor MacDuffee, survey of the foundations of arithmetic, tensor analysis; Professor Colvin, mathematical applications; Professor Langer, harmonic analysis; Staff, determinants and matrices, higher algebra.

University of Wyoming. June 19 to July 21: Professor Barr, numerical analysis; Professor Neubauer, history of mathematics; Professor Schwid, advanced calculus, theory of functions of a complex variable; Professor Smith, ordinary differential equations; Professor Varineau, theory of equations, abstract algebra. July 24 to August 25: Professor Barr, vector analysis; Professor Schwid, seminar in analysis, college geometry, theory of functions of a complex variable; Professor Smith, partial differential equations; Professor Varineau, higher algebra, fundamental concepts of mathematics, abstract algebra.

PERSONAL ITEMS

Dr. Ilse L. Novak of Wellesley College has been awarded one of the Frank B. Jewett postdoctoral fellowships for 1950–51 by the Bell Telephone Laboratories.

The Duodecimal Society of America announces that the following officers have been elected for 1950: Chairman of the Board, F. E. Andrews; President, H. C. Robert, Jr.; Vice-Presidents, P. E. Friedemann and Dr. Nathan Lazar; Editor, G. S. Terry; Secretary, R. H. Beard; Treasurer, H. K. Humphrey.

Harvard University reports: Professor Georges de Rham of the University of Lausanne and the University of Geneva was Visiting Professor for the fall term, 1949; Professor W. V. D. Hodge of Cambridge University is Visiting Professor during the spring term, 1950.

Iowa State College announces the appointments of the following instructors: Mr. A. M. Feyerherm, Mr. R. N. Goss and Mr. R. A. Worsing.

At Miami University: Dr. W. C. Sangren, formerly an Atomic Research Fellow at the University of Michigan, has been appointed to an assistant professorship; Instructor Alberta L. Wolfe has been promoted to an assistant professorship; Graduate Assistants Kathryn B. Aldrich and R. H. Johnston have been promoted to instructorships.

Oklahoma Agricultural and Mechanical College announces: Dr. C. J. Clark of Ohio State University has been appointed Instructor and Research Assistant; Dr. Arthur Zeichner, formerly research associate at Harvard University, has been appointed Instructor and Research Assistant; Associate Professor Joseph Barnett, Jr., has retired with the title of Associate Professor Emeritus; Associate Professor C. E. Marshall and Assistant Professor J. V. Robison have returned to the staff.

Pennsylvania State College makes the following announcements: Dr. E. J. Akutowicz, formerly at the Institute for Advanced Study, has been appointed to an assistant professorship; Dr. Nathan Jaspen has been appointed Lecturer; Mr. A. J. Goldstein, Mr. J. D. E. Konhauser and Mr. E. F. Ormsby have been appointed to part-time instructorships; Miss Alice M. Hanna has been appointed graduate assistant; Professor Orrin Frink has been promoted to the position of Head of the Department of Mathematics; Professor F. W. Owens is now Professor Emeritus; Assistant Professor Helen B. Owens has retired.

At Rensselaer Polytechnic Institute: Dr. M. R. Spiegel, previously graduate

assistant at Cornell University, and Mr. J. C. Gibson, graduate assistant at Yale University, have been appointed to instructorships; Assistant Professor R. F. Tessier has retired.

Texas Technological College announces the following: Professor J. M. Michie, formerly head of the Department of Mathematics and Astronomy, and Professor E. L. Thompson retired on February 1, 1950.

Tulane University makes the following announcements: Dr. Sze Tsen Hu of Academia Sinica, Shanghai, China, has been appointed Lecturer; Dr. Herman Cohen of the University of Wisconsin has been appointed to an instructorship; Mr. J. W. Ellis has been appointed Atomic Energy Commission Fellow.

University of Illinois reports: Dr. Herman Chernoff, formerly statistician for the Cowles Commission, University of Chicago, and Instructor Harry Lass of the University of California have been appointed to assistant professorships; Dr. Meyer Jersion has been appointed Research Instructor; Associate Professor M. M. Day has been promoted to a professorship; Instructors W. F. Atchison, M. K. Fort, Jr., F. E. Hohn, and E. J. Scott have been promoted to assistant professorships; Assistant Clarence Phillips has been promoted to an instructorship.

At the University of Kentucky: Miss H. Fay Hays and Mr. Sherman Vanaman have been appointed to graduate assistantships; Mrs. Lydia R. Fischer has retired.

The University of Southern California reports the following appointments: Dr. L. A. Henkin of Princeton University to an assistant professorship; Dr. Harold Shniad, Purdue University, to an instructorship.

University of Washington makes the following announcements: Mr. G. E. Uhrich has been appointed Acting Instructor; Assistant Professor Mary E. Haller has been promoted to an associate professorship; Dr. Fumio Yagi has been promoted to an assistant professorship.

University of Wyoming reports the following appointments: Associate Professor R. L. Calvert of Utah State Agricultural College to an assistant professorship; Mr. J. A. Weiss, formerly research chemist, Glenn L. Martin Company, Baltimore, Ohio, to an instructorship.

Utah State Agricultural College announces the following appointments to instructorships: Mr. J. K. Everton, Mr. J. E. Faulkner, Mr. Karl Koerner, Mr. P. A. Madden, Mr. D. W. Stoddard and Mr. J. M. Ward.

At Virginia State College: Associate Professor H. M. Linette of Virginia Union University has been appointed to an associate professorship; Miss Dorothy M. Cheek has been appointed Instructor in Mathematical Education; Mrs. Bertrice W. King has been promoted to an assistant professorship.

Washburn University announces the following appointments to instructorships: Mr. M. C. Hoover and Mr. H. Q. Rolfs of the University of Kansas.

Wayne University reports: Mr. Milton Sobel has been appointed to an assistant professorship; Assistant Professor R. L. Ackoff has been granted a leave of absence for six months so that he may serve as consultant for the United States Bureau of Census.

At West Virginia University: Miss Elizabeth H. McLaughlin and Mr. Peter Zaphyr have been appointed to instructorships; Assistant Professor M. L. Vest has been promoted to an associate professorship; Assistant Professor I. D. Peters is on leave for the year and is studying at Columbia University.

Yale University makes the following announcements: Dr. Shizuo Kakutani of the Institute for Advanced Study has been appointed to an assistant professorship; Dr. R. R. Benard of the University of Virginia and Dr. D. T. Finkbeiner, California Institute of Technology, have been appointed to instructorships; Associate Professor Nathan Jacobson has been promoted to a professorship; Dr. D. F. Votaw has been promoted to an assistant professorship; Professor W. R. Longley has retired with the title of James E. English Professor Emeritus of Mathematics.

Dr. M. M. Andrew who has been engaged in operational research with the Naval Operations Evaluation Group has joined the staff of the machine development section of the applied mathematics laboratories, National Bureau of Standards.

Miss Betty J. Barnett, graduate assistant at the University of Kentucky, has received an appointment at Sue Bennett College.

Miss Adrian M. Baucom, graduate assistant at the University of Kentucky, is teaching at Kenton High School, Kenton, Tennessee.

Miss Dorothy De Witt has been appointed to an instructorship at Iowa State Teachers College.

Professor Patric DuVal of the University of Istanbul, Turkey, has been appointed Professor of Mathematics at the University of Georgia.

Assistant Professor W. C. G. Fraser of Rensselaer Polytechnic Institute has accepted an assistant professorship at Dartmouth College.

Mr. R. P. Hobbs is now Vice-President of Rinehart & Company, Incorporated.

Dr. Nathan Lazar, part-time instructor at Teachers College, Columbia University, has been appointed Associate Professor of Education at Ohio State University.

Mr. L. B. McBride, formerly instructor at Utah State Agricultural College, is teaching at Shelley High School, Shelley, Idaho.

Mr. Jacques Saint-Pierre of the University of Montreal has been promoted to the position of assistant lecturer.

Mr. F. W. Stallard has been appointed to an instructorship at East Tennessee State College.

Assistant Professor E. C. Stopher, Jr., of Miami University has been appointed to a professorship at Fort Hays Kansas State College.

Mr. J. A. Wilson of Case Institute of Technology has received an appointment as assistant professor at Baldwin-Wallace College.

Professor L. A. H. Warren of the University of Manitoba died on October 6, 1949. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following one hundred and six persons have been elected to membership on applications duly certified.

M. W. AL-DHAHIR, B.A.(Baghdad) Student Columbia University, New York, N. Y.

SHOLOM ARZT, M.S.(N.Y.U.) Instructor, New York University, N. Y.

R. P. BACON, A.B.(Miami U.) Grad. Asst., Miami University, Oxford, Ohio.

BARBARA BAKER, A.B.(Brown) Research Asst., Lehigh University, Bethlehem, Pa.

NATHAN BAROTZ, B.S.I.E.(N.Y.U.) Instructor, Brooklyn College, N. Y.

H. S. BERG, B.S.(North Dakota) Grad. Asst., University of North Dakota, Grand Forks, N. D.

L. G. CAMPBELL, A.M.(New Jersey S.T.C.) Instructor, New Jersey State Teachers College, Upper Montclair, N. J.

J. H. CASE, Student, Alabama Polytechnic Institute, Auburn, Ala.

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REPORT OF THE TREASURER FOR THE YEAR 1949

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1949. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

I. TOTAL FUNDS OF THE ASSOCIATION ON JANUARY 1, 1949

M & T Trust Co., Buffalo.....	$ 7,306.07	Current Fund.....	$ 7,306.07
Securities.....	61,885.56	Carus Fund.....	8,081.26
		Chace Fund.....	9,776.25
		Houck Fund.....	9,536.64
		Chauvenet Fund.....	624.58
		General Fund.....	33,866.83
	$69,191.63		$69,191.63

II. CURRENT FUND

Balance, January 1, 1949	$ 7,306.07	MONTHLY	
Dues	13,998.82	Publication	$11,999.63
Initiation fees	1,090.00	Reprints (net)	179.65
Subscriptions	3,978.16	Editor's office	511.34
Sale of back numbers (net)	1,291.56	Secretary-Treasurer's Office	
Advertisements	1,831.50	Clerical help	5,224.17
Contributions for publication of		Office expenses	2,111.78
the MONTHLY	400.00	Bank fee	59.70
Sales of exchange periodicals	55.50	Auditing fee	25.00
Interest on General Fund	1,143.71	Board of Governors	2,535.63
Income from Hardy Fund	120.00	Meetings	265.09
Transfer from General Fund	593.48	Representatives	45.73
		Subventions	925.00
		Balance, Dec. 31, 1949	7,926.08

III. CARUS FUND

Balance, January 1, 1949	$ 8,081.26	Reprinting 7th Monograph	$ 1,422.20
Sales of Monographs	3,296.94	Balance, January 31, 1949	10,337.74
Interest	270.34		
Increase in value of securities	111.40		

IV. CHACE FUND

Balance, January 1, 1949	$ 9,776.25	Honorarium, 2nd Slaught Paper.	$ 114.00
Sale of Papyrus	300.00	Printing, 2nd Slaught Paper.	2,156.68
Sale of Slaught Papers	727.76	Balance, Dec. 31, 1949	9,003.16
Interest	332.72		
Increase in value of securities	137.11		

V. HOUCK FUND

Balance, January 1, 1949	$ 9,536.64	Expenses of Index	$ 116.00
Interest	311.93	Balance, Dec. 31, 1949	9,861.11
Increase in value of securities	128.54		

VI. CHAUVENET FUND

Balance, January 1, 1949	$ 624.58	Balance, Dec. 31, 1949	$ 653.95
Interest	20.80		
Increase in value of securities	8.57		

VII. GENERAL FUND

Balance, January 1, 1949	$33,866.83	Transfer to Current Fund	$ 593.48
Increase in value of securities	471.32	Balance, Dec. 31, 1949	33,744.67

VIII. TOTAL FUNDS OF THE ASSOCIATION ON DEC. 31, 1949

Current Fund	$ 7,926.08	M & T Trust Company, Buffalo	
Carus Fund	10,337.74	Checking Account	$ 7,926.08
Chace Fund	9,003.16	Securities	63,600.63
Houck Fund	9,861.11		
Chauvenet Fund	653.95		
General Fund	33,744.67		
	$71,526.71		$71,526.71

THE MAY MEETING OF THE INDIANA SECTION and a CONFERENCE ON NONLINEAR PROBLEMS

The Spring meeting of the Indiana Section of the Mathematical Association of America was held May 7, 1949 at the University of Notre Dame, Notre Dame, Indiana, simultaneously with a Conference on Nonlinear Problems sponsored by the Department of Mathematics of the University of Notre Dame.

One hundred and one persons were present at the meeting, including the following forty-seven members of the Association: Rev. H. B. Albiser, Juna L. Beal, W. R. Beck, C. B. Bell, Stanley Bolks, Richard Brauer, H. E. Burns, G. E. Carscallen, H. E. Crull, Rev. H. F. DeBaggis, CSC, M. W. DeJonge, Sister M. Virgilia Dragowski, OSF, R. D. Edwards, Ky Fan, Cleota G. Fry, E. L. Godfrey, Michael Golomb, S. H. Gould, Noel Gottesman, G. R. Grainger, Sister M. Charlotte Holland, RSM, C. T. Hazard, Smith Higgins, Jr., Ralph Hull, H. F. S. Jonah, Wilfred Kaplan, M. W. Keller, E. L. Klinger, Rev. Bonaventure Knaebel, J. P. LaSalle, Florence Long, Sister M. Ferrer McFarland, RSM, P. M. Nastucoff, F. S. Nowlan, C. E. Olsen, R. R. Otter, P. W. Overman, P. M. Pepper, J. C. Polley, Arthur Rosenthal, A. E. Ross, G. X. Saltarelli, A. H. Smith, R. A. Struble, Anna K. Suter, W. D. Wood, L. J. Zimmerman.

Officers elected at the meeting were: Chairman, Ralph Hull, Purdue University; Vice-Chairman, J. C. Polley, Wabash College. P. M. Pepper, University of Notre Dame, continues as Secretary. The annual meeting in 1950 is to be held on Saturday, April 29, at Wabash College, Crawfordsville, Indiana.

Professor Richard Brauer, of the University of Michigan, gave an interesting and stimulating talk, entitled *Some Reflections on the Teaching of Algebra*, describing what he believes should be the aims in the teaching of this subject at the undergraduate level in the colleges. Professor Ralph Hull and Professor George Whaples officially led the animated discussion which followed Professor Brauer's provocative comments. The majority seemed to favor, with Professor Brauer, the placing of emphasis on understanding rather than on development of techniques with scant comprehension. Some persons were concerned that this emphasis might lead to the teaching of insufficient techniques. It was brought out in the subsequent remarks that the most satisfactory procedure would strive to accomplish both these aims, and that an attempt should be made to extend this attitude to the teaching in secondary schools.

Dr. Mary L. Cartwright, of Girton College, Cambridge, England, and

Princeton University, gave an address, *On Nonlinear Differential Equations and Some of Their Applications*. Dr. Cartwright illustrated a method for study of nearly linear resonance. This method (due to J. E. Littlewood and the speaker) was applied to the study of the equation $\ddot{x}-k(1-x^2)\dot{x}+x=pk\lambda \cos(\lambda t+\alpha)$, where k is small and λ is near 1. It was shown that many results can be obtained from the study of a related autonomous system.

Professor P. D. Edwards, Ball State Teachers College, Chairman of the *Committee on High School Mathematical Prerequisites for College Curricula in the State of Indiana*, gave a preliminary report of that committee. Responses have been received from all colleges in the state offering full four year degree courses, and from most others offering work of college grade. Variations existing in the requirements between colleges make the formation of general statements difficult. In most cases the only departments which do not require a knowledge of high school mathematics are English, Foreign Languages, Music, and Art. It is the hope of the Committee that the complete report may be made available to principals and student advisors in each high school in Indiana.

The following papers were presented:

1. *The eigenvalue problem for completely continuous normal operators*, by Professor Michael Golomb, Purdue University.

Most of the classical eigenvalue problems in differential and integral equations may be considered as special cases of the eigenvalue problem for completely continuous linear operators in Hilbert space. An operator is normal if it commutes with its adjoint. Important subclasses of the normal operators are the self-adjoint and the unitary operators. A simple proof is given for the theorem that every completely continuous normal operator has a complete system of eigenfunctions. Similar proofs have been known for completely continuous self-adjoint operators.

2. *The range of synchronization of subharmonic resonance*, by Mr. Hans Schaffner, Department of Electrical Engineering, University of Illinois, introduced by the Secretary.

Mr. Schaffner reported on the range of synchronization when $p\lambda=q\lambda_0$, where p and q are small integers, λ_0 is the resonant frequency, and λ is the frequency of the driving force.

3. *Numerical methods for solution of nonlinear differential equations*, by Professor Wilfred Kaplan, University of Michigan.

Professor Kaplan gave a general method for the analysis of differential equations $dx_i/dt=f_i(x_1, \cdots, x_n)$, $(i=1, \cdots, n)$ in a bounded phase space. The essential idea is to assume the f_i to be only approximately known. This leads to a blurring of the structure of the family of solutions; fine details are lost while only a simple structure remains; this is considered to be more in accordance with reality than the precise solutions of the exact equations. A numerical method for determining the stable states was given.

4. *Van der Pol's equation with forcing term*, by Dr. Mary L. Cartwright, Girton College, Cambridge, England, and Princeton University, introduced by Professor J. P. LaSalle.

It was suggested by van der Pol that for k very large the equation

$$\ddot{x}-k(1-x^2)\dot{x}+x=bk\lambda \cos(\lambda t+\alpha) \tag{1}$$

represents a physical system in which two stable subharmonics of period $(2n\pm1)2\pi/\lambda$ occur. This was investigated by J. E. Littlewood and the speaker. The stable solutions of (1) for k large and $0<b<2/3$ consist of descending waves from near $x=2$ to $x=1$, followed by a rapid descent to $x=-2$ approximately, and then a repetition with $-x$ for x. The waves from $x=2$ to $x=1$ are approximately solutions of $x-x^3/3=b\sin(\lambda t+\alpha)+C$, where C is a constant, and C is reduced by $k^{-1}\int_0^{2\pi/\lambda}x(t, C)dt$ after one complete period. There may be $n\pm(1/2)$ waves above $x=1$, and the same number below $x=-1$, where n is of the order of k, giving solutions with periods $(2n\pm1)2\pi/\lambda$. There are also unstable solutions with more complicated behavior near $x=\pm1$.

5. *Nonlinear circuit problems*, by Mr. Hans Schaffner, University of Illinois.

Mr. Schaffner outlined the nonlinear problems which appear important to the electrical engineers, and discussed their present status.

6. *Relaxation oscillations*, by Professor J. P. LaSalle, University of Notre Dame.

Professor LaSalle pointed out that the problem of proving the existence of periodic solutions of differential equations can be reduced in some cases to finding periodic solutions of suitable differential inequalities. The solutions of the inequalities provide a region of the phase plane which encloses a periodic solution (a closed path). This enclosure approximates the closed path and gives bounds for the amplitude and period of the periodic solution.

PAUL M. PEPPER, *Secretary*

CALENDAR OF FUTURE MEETINGS

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December 30, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Allegheny College, Meadville, May 6, 1950

ILLINOIS, Southern Illinois University, Carbondale, May 12–13, 1950.

INDIANA, Wabash College, Crawfordsville, May 6, 1950.

IOWA, State University of Iowa, Iowa City, April 21–22, 1950.

KANSAS, State Teachers College, Pittsburg, April 22, 1950.

KENTUCKY, University of Kentucky, Lexington, April 29, 1950.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Richmond, May 6, 1950.

METROPOLITAN NEW YORK, City College, April 1, 1950.

MICHIGAN

MINNESOTA, Macalester College, St. Paul, May 6, 1950.

MISSOURI

NEBRASKA, Nebraska Wesleyan University, Lincoln, May 6, 1950.

NORTHERN CALIFORNIA, University of San Francisco, January 27, 1951.

OHIO, Denison University, Granville, April 22, 1950.

OKLAHOMA

PACIFIC NORTHWEST, University of Washington, Seattle, June 16, 1950.

PHILADELPHIA, Lehigh University, Bethlehem, Pennsylvania, November 25, 1950.

ROCKY MOUNTAIN, University of Denver, April 28–29, 1950.

SOUTHEASTERN, University of Florida, Gainesville, April 7–8, 1950.

SOUTHERN CALIFORNIA

SOUTHWESTERN, Flagstaff, Arizona, April 29, 1950.

TEXAS, Abilene, April 14–15, 1950.

UPPER NEW YORK State, Syracuse University, April 22, 1950.

WISCONSIN, Marquette University, Milwaukee, May 13, 1950.

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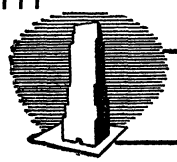
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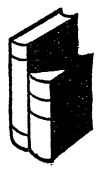
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THE FUTURE OF MATHEMATICS*

ANDRÉ WEIL

"At one time," says Poincaré in his Rome conference on the future of mathematics, "there were prophets of misfortune; they reiterated that all the problems had been solved, that after them there would be nothing but gleanings left. . . . " "But," he added, "the pessimists have always been compelled to retreat . . . so that I believe there are none left to-day."

Our faith in progress, our belief in the future of our civilization are no longer as strong; they have been too rudely shaken by brutal shocks. To us, it hardly seems legitimate to "extrapolate" from the past and present to the future, as Poincaré did not hesitate to do. If the mathematician is asked to express himself as to the future of his science, he has a right to raise the preliminary question: what kind of future is mankind preparing for itself? Are our modes of thought, fruits of the sustained efforts of the last four or five millennia, anything more than a vanishing flash? If, unwilling to stumble into metaphysics, one should prefer to remain on the hardly more solid ground of history, the same questions reappear, although in different guise: are we witnessing the beginning of a new eclipse of civilization? Rather than to abandon ourselves to the selfish joys of creative work, is it not our duty to put the essential elements of our culture in order, for the mere purpose of preserving it, so that at the dawn of a new Renaissance, our descendants may one day find them intact?

These questions are not purely rhetorical; upon each man's answer, or rather (for such questions do not have answers), upon the attitude which he takes in front of them, depends in large measure the trend of his intellectual efforts. It was necessary, before writing about the future of mathematics, to formulate these questions, just as the faithful cleansed themselves before consulting the oracle. Let us now interrogate destiny.

Mathematics, as we know it, appears to us as one of the necessary forms of our thought. The archaeologist and the historian have shown us civilizations from which mathematics were absent. It is indeed doubtful whether they would ever have become more than a technique, at the service of technologies, if it had not been for the Greeks; and it is possible that, under our very eyes, a type of human society is being evolved in which they will be nothing but that. But for us, whose shoulders sag under the weight of the heritage of Greek thought and who walk in the paths traced out by the heroes of the Renaissance, a civilization without mathematics is unthinkable. Like the parallel postulate, the postulate that mathematics will survive has been stripped of its "evidence"; but, while the former is no longer necessary, we would not be able to get on without the latter.

The clinical student of ideas who limits his prognosis to the immediate fu-

* Authorized translation by Arnold Dresden of the article entitled L'avenir des mathématiques in the volume Les grands courants de la pensée mathématique, edited by F. Le Lionnais. Cahiers du Sud, Marseille, 1948.

ture, and does not risk long-range prophecies, certainly observes more than one favorable symptom in contemporary mathematics. To begin with, while some sciences, conferring, as they now do, an almost unlimited power upon a ruthless possessor of their results, tend to become caste monopolies, treasures jealously guarded under a seal of secrecy which must of necessity become fatal to any genuine scientific activity, the real mathematician does not seem to be exposed to the temptations of power nor to the straight-jacket of state secrecy. "Mathematics" said G. H. Hardy in substance in a famous inaugural lecture, "is a useless science. By this I mean that it can contribute directly neither to the exploitation of our fellowmen, nor to their extermination."

It is certain that few men of our times are as completely free as the mathematician in the exercise of their intellectual activity. Even if some State ideologies sometimes attack his person, they have never yet presumed to judge his theorems. Every time that so-called mathematicians, to please the powers that be, have tried to subject their colleagues to the yoke of some orthodoxy, their only reward has been contempt. Let others besiege the offices of the mighty in the hope of getting the expensive apparatus, without which no Nobel prize comes within reach. Pencil and paper is all the mathematician needs; he can even sometimes get along without these. Neither are there Nobel prizes to tempt him away from slowly maturing work, towards a brilliant but ephemeral result. Mathematics is taught the world over, well here, badly there; the exiled mathematician—and who among us can to-day feel free from the danger of exile—can find everywhere the modest livelihood which allows him to pursue his work to some extent. Even in gaol one can do good mathematics if one's courage fail him not.

To these "objective conditions," or rather, as the physician would say, to these external symptoms, must be added others revealed by a more penetrating clinical examination. In recent times mathematics has demonstrated its vitality by passing through one of these periods of growing pains, to which it has been accustomed for a long time, and which are designated by the strange name of "foundation crises." It has come through it, not only without damage, but with great gain. Whenever wide domains have been added to the field of mathematical reasoning, it is necessary to inquire what techniques are allowed in the exploration of the new territory. One wants certain objects to have certain properties, one wants certain modes of reasoning to be admissible and one behaves as if they were. But the pioneer who proceeds in this way knows very well that some day the police will come to put an end to the disorder and to bring everything under the control of the general law. Thus, when the Greeks defined the ratio of two magnitudes for the first time with enough precision to raise the problem of the existence of incommensurable magnitudes, they seem to have believed and to have wanted all ratios to be rational and to have based the first sketch of their geometrical reasonings on this provisional hypothesis; some of the greatest advances in Greek mathematics are connected with the discovery of their initial error at this point. In the same way, at the beginning of the era

of the theory of functions and of the infinitesimal calculus, one wished every analytical expression to define a function, and every function to have a derivative; we know to-day that these requirements were incompatible. The last crisis, which grew out of the sophistries, for which the "naive" theory of sets opened a way in its early stages, has led for us to a no less happy result, which can now be considered as permanently established. We have learned to trace our entire science back to a single source, constituted by a few signs and by a few rules for their use; this is unquestionably an unassailable stronghold, inside which we could scarcely confine ourselves without risk of famine, but to which we are always free to retire in case of uncertainty or of external danger. Only a few backward spirits still maintain the position that the mathematician must forever draw on his "intuition" for new, alogical or "prelogical" elements of reasoning. If certain branches of mathematics have not yet been axiomatized, *i.e.*, reduced to a form of exposition in which all terms are defined, and all axioms made explicit, in terms of the basic notions of set theory, this is simply because there has not yet been the time to do it. It is of course possible that some day our successors will want to introduce into set-theory modes of reasoning which we do not admit. It is even possible that the germ of a contradiction, which we do not perceive to-day, may later be discovered in the modes of reasoning we now use, although the work of the modern logicians makes this very unlikely. A general revision will then become necessary; one can feel certain even now that this will not affect the essential elements of our science.

But, if logic is the hygiene of the mathematician, it is not his source of food; the great problems furnish the daily bread on which he thrives. "A branch of science is full of life," said Hilbert, "as long as it offers an abundance of problems; a lack of problems is a sign of death." They are certainly not lacking in our mathematics; and the present time might not be ill chosen for drawing up a list, as Hilbert did in the famous lecture from which we have just quoted. Even among those of Hilbert, there are still several which stand out as distant, although not inaccessible, goals which will continue to suggest research for perhaps more than a generation; an example is furnished by his fifth problem, on Lie groups. The Riemann hypothesis, after the attempts to prove it by function-theoretic methods had been given up, appears to-day in a new light, which shows it to be closely connected with the conjecture of Artin on the L-functions, thus making these two problems two aspects of the same arithmetico-algebraic question, in which the simultaneous study of all the cyclotomic extensions of a given number field will undoubtedly play a decisive role. Gaussian arithmetic was centered around the law of quadratic reciprocity; we know now that this law is only a first example, we might better say the pattern of, the laws of "class fields," which control the abelian extensions of algebraic number-fields; we know how to formulate these laws so as to make them look like a coherent set. But, pleasant as this facade may be to the eye, we do not know whether it might not hide deeper lying symmetries. The automorphisms induced in the class groups by the automorphisms of the field, the properties of the norm-residues in the

non-cyclic cases, the passage to the limit (inductive or projective) when the base field is replaced by extensions, for example, cyclotomic extensions, of indefinitely increasing degree, all these are questions on which our ignorance is almost complete and in whose study the key to the Riemann hypothesis is perhaps to be found. Closely connected with these questions is the study of Artin's conductor and, in particular, in the local case, the search for the representation, whose trace can be expressed by means of simple characters with coefficients equal to the exponents of their conductors. These are some of the directions which can and must be followed up in order to penetrate the mystery of non-abelian extensions; it is not impossible that we are here close to principles of extraordinary fertility and that, once the first decisive step on this road will have been taken, we shall gain access to vast domains whose existence is hardly suspected. For, however wide our generalizations of Gauss' results may be, we can hardly claim to have as yet really moved beyond them.

Even in the realm of abelian extensions, we have not made any progress towards the generalization of the theorems of "Kronecker's youth dream," the generation of class fields, whose existence is known, by means of values of analytic functions. While it has been possible, without serious difficulties, to complete Kronecker's unfinished work and to obtain the solution of this problem, in the case of imaginary quadratic fields, by means of complex multiplication, the key to the general problem, considered by Hilbert as one of the most important of modern mathematics, still escapes us, in spite of the conjectures of Hilbert himself and the efforts of his pupils. Must we look for it perhaps in the new automorphic functions of Siegel, in his modular functions of several variables? Or can the theory of the endomorphisms of abelian varieties, which has now made considerable progress, be of some help here? It is too early to risk acceptable conjectures on these questions; but their closer examination is bound to produce interesting results, even though they should be negative in character.

The foregoing discussion shows clearly not only the vitality of modern arithmetic, but also the close ties which connect it, to-day as in the days of Euler and the days of Jacobi, with the most deep-lying parts of the theory of groups and of the theory of functions. This essential unity, which appears in so many and in such diverse ways, is also found in many other places. The introduction by Hermite of continuous variables into the theory of numbers has led to the systematic study of discontinuous groups of arithmetical nature by means of the continuous groups in which they can be imbedded, of the symmetric Riemannian spaces associated with these groups, of the differential and topological properties of their fundamental domains (or rather, in modern terminology, of their quotient spaces), and of the automorphic functions which belong to them. The work of Siegel, continuing the great tradition of Dirichlet, of Hermite and of Minkowski, has opened entirely new paths here. On the one hand, we connect with Fermat, Lagrange and Gauss, the representation of numbers by forms, and the genera of quadratic forms. At the same time, we begin to see in outline the fertile principle, according to which the global aspect of an arithmetical problem

can, under certain circumstances, be reconstructed from its local aspects. For instance, in the work of Siegel we see repeatedly that the number of solutions of some arithmetical problem in the field of rational numbers is expressed by means of numbers defined by the corresponding local problems, density of solutions in the real field and in the p-adic fields for all values of the prime p. This is a principle, analogous to Cauchy's theorem for the Riemann surface of an algebraic curve, with which one may connect also the famous "singular series" which appear in the application of the method of Hardy-Littlewood to problems in the analytic theory of numbers. Is it possible to formulate this principle in a general statement, which would allow us to obtain at one stroke all results of this character, just as the discovery of Cauchy's theorem made it possible to calculate by a single method a number of integrals and of series which were formerly treated by special distinct processes? It looks as if this were not yet a problem for the immediate future; so much the more reason to prepare for its solution by the study of well-chosen particular cases. It may be that this same principle will one day reveal the deep reason for the existence of Eulerian products, of which the extreme importance for the theory of numbers and the theory of functions has only become clear through the work of Hecke. Here we deal with the classes of quadratic forms, and not merely, as in the work of Siegel, with their genera; at the same time, we find ourselves at the core of the theory of modular functions, which has been infused with new life by these studies, and of the theory of theta functions. This domain is still so full of mystery, the questions which it raises so numerous and fascinating, that it would be premature to try to arrange them in order of importance.

At the same time, Siegel has taught us to construct discontinuous groups and automorphic functions by arithmetical methods; in this field the theory of functions, by its own efforts, had been unable to move forward since Poincaré. Indeed it is very likely, that, just as in the case of functions of a single variable, the thorough study of special functions of several complex variables will have to prepare the ground for an attack on the general theory. In the work of Siegel, the local and global geometrical study of fundamental domains, in effect of manifolds with a complex-analytic structure, tends to occupy a dominating role. Along this road, connection is made with the immense work accomplished by E. Cartan and its various extensions; at the same time one gets into the center of modern topology, the theory of fibre-spaces; and the invariants of Sitefel-Whitney appear, along with their generalizations. The intimate connections between these two domains had been suspected for some time, but their actual merging was made possible only through the recent discoveries of Chern, stimulated in their turn, at least in part, by considerations of algebraic geometry. Indeed, algebraic varieties, at least varieties without singularities in the complex field, are nothing but a special, and particularly interesting, class of manifolds with a complex-analytic structure; more precisely, they are manifolds on which, at least in all known cases, one can define one of these remarkable Hermitian metrics which were introduced by Kähler in connection with functions

of several complex variables, and of which results of S. Bergmann, not yet fully clarified, furnish other examples. By a systematic, although not explicit use of these metrics, Hodge has recently obtained the first existence theorems for this type of manifold, generalizing the classical results of Riemann. While it may be too much to hope that such methods may one day lead to the uniformization of algebraic varieties (which, contrary to what happens in the case of curves, can not be done in general by means of unramified functions), there is little doubt that they can be extended to integrals of the third kind. The analogous generalization of the methods of Hodge to differential forms with singularities in the real domain raises still more important problems. It appears to be connected, on the one hand, with local properties of the equations of elliptic type which harmonic forms satisfy; on the other hand, it seems to be linked with an extension of de Rham's theory which would make it possible to obtain the homologic torsion of a manifold by means of differential forms with singularities. De Rham's results have, as a matter of fact, definitely clarified a certain aspect of the relation between homology groups and multiple integrals, and this accounts for the fundamental role they play in the work of Hodge and of Chern; but until now they have only made the homology groups with real coefficients accessible to differential methods; moreover, the striking and fertile analogy between chains and differential forms, which is expressed in these results, remains a mere heuristic principle, until we succeed in finding a common basis for these two concepts. To convert this principle into a method of proof has thus far succeeded only in a few special cases, for example, in some of the beautiful papers with which Ahlfors has in recent years given fresh life to the theory of analytic functions.

But, while algebraic geometry, as we have just seen, receives a fresh stimulus from the most recent developments in topology and in differential geometry, this field does not lack purely algebraic problems; and, thanks to the methods of modern algebra, our understanding of them no longer need depend upon flashes of intuition of a few privileged mortals. At present, the theory of surfaces, brilliantly but too rapidly developed by the Italian school, must yield place to a general theory of algebraic varieties, freed from restrictive assumptions as to the nature of the base field and as to the absence of singularities. The structure of the groups of divisor classes, with respect to the different known concepts of equivalence (linear, continuous, numerical), the study of unramified extensions of a field of algebraic functions, both abelian and non-abelian, these are the questions that call for solution first of all. Thanks to the results obtained, or at least made plausible, by the Italian geometers, we can more or less guess the answers; and their solution, perhaps already within our reach, must open the road for important advances. On the other hand, the study of algebraic geometry over various special fields of constants is still gropingly taking its first steps. In view of the fact that algebraic geometry over the complex field, studied for almost a century, has arrived by its own methods (topological and transcendental) at well-known important results, it is probable that other fields,

finite fields, p-adic fields, fields of algebraic numbers, deserve to be studied, each by itself, by methods suited to their purpose. From this point of view, geometry over a finite field appears somewhat like a turntable, from which one may at will direct one's further progress either towards algebraic geometry proper, with the powerful tools already at its disposal, or towards the theory of numbers; it is precisely in that way that we are beginning to get a better insight into the nature of the Zeta-function and into the true nature of the Riemann hypothesis. In the same way, before undertaking the determination of the extensions of a field of algebraic numbers by means of their local properties, it might be indicated to solve the analogous problem, already difficult enough, concerning algebraic functions of one variable over a finite field, *i.e.*, to extend Riemann's existence theorem to such functions. To mention merely a particular case, one might ask whether the modular group, whose structure determines the fields of functions of a complex variable with only three points of ramification, plays the same role, at least with respect to the extensions of degree prime to the characteristic, when the field of constants is finite. It is not impossible that all questions of this kind can be treated by a uniform method, which would make it possble to deduce, from a result established (for instance by topological methods) for characteristic 0, the corresponding result for characteristic p; the discovery of such a principle would constitute an advance of the greatest importance. Of the same character, but still more difficult, are the problems arising in the modern study of finite groups. Is the theory of finite simple groups an analogue of the theory of simple Lie groups? It would probably be premature to make a frontal attack on this question at the present stage; by means of indirect procedures, in particular the study of p-groups, some progress has been made in this direction in recent years. As in many other questions of algebra and of the theory of numbers, so also here a new element has recently been introduced by the definition of the homology groups of an abstract group. The discovery of this concept, which generalizes the fruitful concepts of character and factor-set, is due to Eilenberg and MacLane, who introduced it in connection with H. Hopf's studies in pure combinatorial topology; it will have to be subjected for some time to systematic study, before its scope and possibilities of application can be estimated.

While arithmetic, in the widest sense, is for its devotees always the queen of mathematics, and while for that reason we have allowed ourselves to dwell on it with predilection, this is not to say that other branches of mathematics do not offer as many problems worthy of sustained attention. The work of a Cartan alone contains enough material to keep busy several generations of geometers. His general theory of systems in involution has not been carried to its conclusion by its author, who appears not to have been able to overcome all the difficulties of algebraic character which it involves. Concerning the theory of "infinite Lie groups," undoubtedly very important, but for us very obscure, we know nothing beyond what is found in the memoirs of Cartan, a first exploration into an almost impenetrable jungle; this jungle threatens to overgrow the

paths which already have been marked out, if the indispensable task of clearing is not undertaken very soon. The modern theory of Lie groups proper, studied by a combination of Cartan's methods with those of modern topology, is far from complete; even in the theory of semi-simple groups and in the theory of the symmetric Riemannian spaces associated with them, a good many results are attainable only by *a posteriori* verification, making use of our knowledge (also due to Cartan) of all simple groups. But, as has already been suggested, it is principally in the topological theory of fibre spaces, in the theorems of de Rham and in the notion of homotopy group, that we now find the tools best suited to the global study of the generalized geometries of Cartan. To give but a single example, the classical Gauss-Bonnet formula, until recently the only result in which a topological invariant was expressed by means of the integral of an invariant differential form, appears to us now as but the first term of a whole sequence of formulas, to which Chern's methods give access and whose systematic study has barely been started.

But, even though involutory systems should, in principle, enable us to obtain everything which can be reduced to the local problem of Cauchy-Kowalewski in the theory of partial differential equations, this is merely one aspect of the existence problem for solutions of these equations; and, from several points of view, it is not its most interesting aspect. Beyond this, we find important results concerning equations of very special types, chiefly elliptic and hyperbolic, some of which are of very recent date; but, although the study of these types, to which our predecessors were led by mathematical physics more than a century ago, is far from complete, it will not do to stop indefinitely at this point. The system which is satisfied by the real part of an analytic function of several complex variables does not belong to any of these simple types; however, function-theory has taught us, for example, that the most general singularities which they can have are, in a sense which is still not easy to specify, made up out of elementary singularities which are characteristic varieties; at any rate, one can interpret the theorems of Hartogs and of E. E. Levi in this manner. In this form, they present an obvious analogy to known results concerning hyperbolic equations; it is this analogy which suggests that we look for the germ of a general theory in a fuller development of the concepts of characteristic variety and of elementary solution. On the other hand, in the work of Delsarte, and in that of S. Bergmann and of his pupils, we find the first examples of the transformation of differential equations by means of integral or of integro-differential operators. It looks as if we have here the germ of entirely new developments and of a classification of systems of partial differential equations, which falls entirely outside the framework of classical methods. In particular, as was shown by Delsarte, the series of orthogonal functions to which elliptic problems lead naturally, are found to be transformed into series of much more general types; some isolated examples of these are found in classical analysis, but their general study presents problems of the greatest interest. The mathematician can no longer be satisfied here with Hilbert space, with which he has become as familiar as with

the Taylor series or the Lebesgue integral; must he look for the most appropriate tool in the theory of Banach spaces, or must he have recourse to more general spaces? It must be admitted that Banach spaces, interesting and useful as they have already proved to be, have not yet brought about the revolution in analysis which some people expected of them; but it would be a counsel of despair to abandon their study now before the various possibilities of application have been more fully explored. However it is possible that they are both too general to be suited to as exact a theory as that of Hilbert spaces, and too special to lend themselves to the study of the most significant operators. They do not include, for instance, the space of indefinitely differentiable functions; and it is only in that space that the operators of L. Schwartz can be defined, which represent formally the derivatives of all orders of arbitrary functions. Perhaps the basis for a new calculus, founded on the generalized Stokes theorem, is to be looked for here; and this may give access to the relations between differential operators and integral operators. Ideas of this character have already proved very useful in special problems, for instance in the calculus of variations under the name of Haar's lemma as well as in certain papers of Friedrichs. Similarly, the well-known theorem, which asserts that the mean of a harmonic function on a circle is equal to its value at the center, expresses the fact that a certain operator, defined by a mass distribution in the plane, is, in a certain sense, a linear combination of the values of the Laplacian in the closed domain bounded by the circle. Also connected with these questions is the problem, mentioned above, of the representation of differential forms as sums of chains, which arises from the theory of de Rham. It is possible that we have in these researches the dim outlines of an operational calculus, destined to become in one or two centuries as powerful an instrument as the differential calculus has been for our predecessors and for ourselves.

All of this has to do only with the local or semi-local study of partial differential equations; indeed, apart from the simple cases which can be treated by means of the theory of Hilbert spaces or by the direct methods of the calculus of variations, the study of partial differential equations in the large, for instance on a compact analytic manifold, appears too difficult to justify any hope that it can be attacked for a long time to come. On the other hand, the study in the large of ordinary differential equations raises a large number of interesting problems; they are difficult, but within our reach. It will suffice to mention as an example the recent beautiful proof, by E. Hopf, of the ergodic character of the geodesics on every compact Riemannian manifold of everywhere negative curvature. Related to this subject is also the study of van der Pol's equation and of relaxation oscillations, one of the few interesting problems which contemporary physics has suggested to mathematics; for the study of nature, which was formerly one of the main sources of great mathematical problems, seems in recent years to have borrowed from us more than it has given us.

But, incomplete as the foregoing enumeration can not fail to seem to our colleagues, it has undoubtedly exhausted the attention of more than one reader;

and yet, for lack of space and for lack of necessary competence, we have not spoken of the geometry of numbers, nor of diophantine approximations, of the calculus of variations, of the calculus of probabilities, or of hydrodynamics, neither have we mentioned at all several problems, to-day in the background of interest, which could be reactivated by a new idea and restored to the vital stream of mathematics. As a matter of fact, we neither can nor want to lay out a route for the future development of our science; this would be a futile task, indeed it would be a ridiculous enterprise, for the great mathematicians of the future, like those of the past, will flee from the beaten track. They will solve the great problems which we shall bequeath to them, through unexpected connections, which our imagination will not have succeeded in discovering, and by looking at them in a new light. It was our purpose, in passing some of the principal branches of our mathematics in review, to draw attention to their robust vitality and to their fundamental unity. We believe to have shown, not only that there are large numbers of problems, but also that there are very few really important problems which are not intimately related to others which, at first sight, seem to be far removed from them. When a branch of mathematics ceases to interest any but the specialists, it is very near to its death, or at any rate dangerously close to a paralysis, from which it can be rescued only by being plunged back into the vivifying sources of the science. "Mathematics," said Hilbert at the end of his 1900 lecture (and it would be quite in order to quote the conclusion of this lecture in full), "is an organism for whose vital strength the indissoluble union of the parts is a necessary condition."

Does this mean that mathematics is becoming a science for erudites, and that it will no longer be possible to do creative work in mathematics until one has grown gray in the harness, and exhausted from burning the midnight oil for many years in the company of dusty tomes? This would at the same time be a sign of its decline; for, be it strength or weakness, mathematics is not a science that prospers on details, painstakingly collected in the course of a long career, on patient reading, on observations or on filing cards, amassed one by one so as to form a bundle from which an idea will ultimately come forth. Perhaps it is more true in mathematics than in any other branch of knowledge that the idea comes forth in full armor from the brain of the creator. Moreover, mathematical talent usually shows itself at an early age; and the workers of the second rank play a smaller role in it than elsewhere, the role of a sounding board for sounds in whose production they had no part. There are examples to show that in mathematics an old person can do useful work, even inspired work; but they are rare, and each case fills us with wonder and admiration. Therefore, if mathematics is to continue to exist in the way in which it has manifested itself to its votaries until now, the technical complications with which more than one of its subjects is now studded, must be superficial or of only temporary character; in the future, as in the past, the great ideas must be simplifying ideas, the creator must always be one who clarifies, for himself and for others, the most complicated tissues of formulas and concepts. Hilbert indeed asked himself: "Is it not

going to become impossible for the individual worker to embrace all the branches of our science?" and he justified his negative answer, not only by his example, but by remarking that every important advance in mathematics is related to the simplification of methods, to the disappearance of old procedures which have lost their usefulness, and to the unification of branches which were until then foreign to each other. It is quite likely that the contemporaries of Apollonius for example, or those of Lagrange, were familiar with the same feeling of growing complexity which tends to overwhelm us to-day. It is undoubtedly true that the modern mathematician does not know certain details of the theory of conic sections as well as Apollonius did, or as a candidate for a French competitive examination, but this does not lead any one to think that the theory of conic sections should form an autonomous science. Perhaps the same fate is in store for some of the theories of which we are proudest. The unity of mathematics would not be threatened by such an occurrence.

The danger lies elsewhere. Although it is more contingent in character, it does not strike us as less serious; and it seems to us that we cannot bring our reflections on the future of mathematics to a conclusion without saying something of it. We have already said that our civilization itself seems to be under attack from all sides; but this remark was couched in too general terms. "Ne sutor ultra crepidam": it is as mathematicians that we must look at the contemporary world. Our tradition is healthy; are we assured of transmitting it undamaged? In some European countries, particularly in Germany until the start of the Hitler regime, there existed, still a short time ago, university instruction, based on a solid secondary education, which made sure that the mathematical apprentice acquired specific subject-matter knowledge and also the general culture without which nothing of importance can be accomplished. What do we see to-day? In France, none of the essential parts of modern mathematics is taught in our universities, except by a lucky chance. One looks in vain for a university course which puts the advanced student in contact with any one of the great problems which we have listed.* Even the elements of the science are too frequently taught in such a way that the student has to learn everything over again if he wants to push on; the extreme rigidity of a mandarin-caste founded on obsolete academic institutions is the cause of the fact that every attempt at modernization is doomed to failure, unless it remains restricted to verbal changes. Italy, which had formerly a flourishing mathematical school, seems to have fallen into a state of sclerosis analogous to that with which France is threatened, but which has had there still more immediate and destructive effects. We do not know what principles guide, at present, secondary and higher education in the U.S.S.R. There are in that country a number of first-rate mathematicians, but they seem to be absolutely prohibited from crossing the frontiers; and, if such practice should persist, it can hardly in the long run have

* (Author's footnote) This was written in 1946; the same would not be unqualifiedly true to-day.

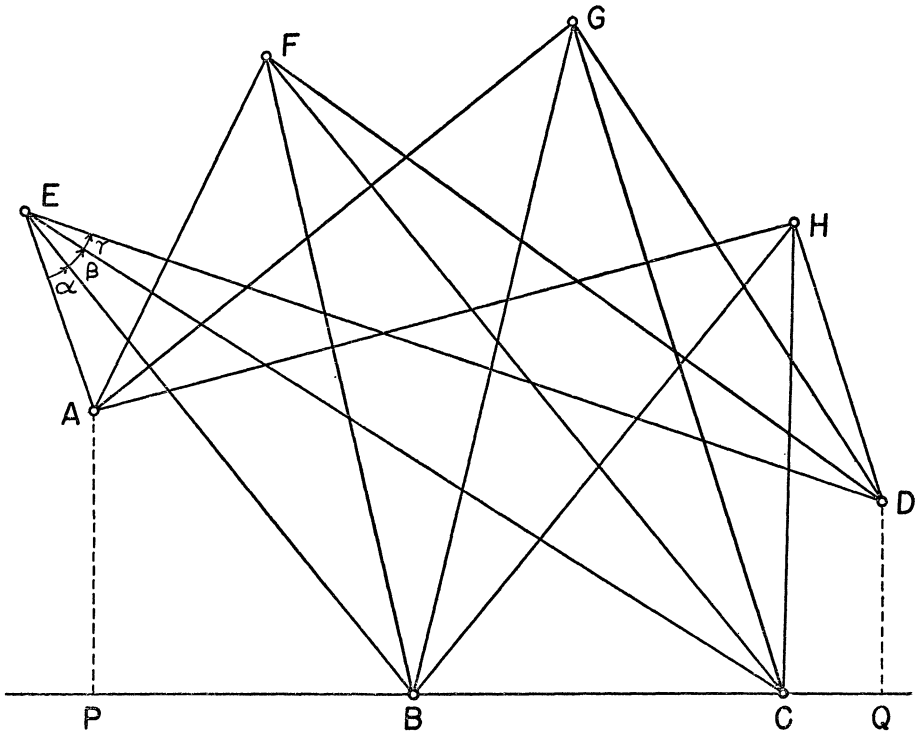
any other result than the slow choking off of all scientific life. The most remote, as well as the most recent history of our science, shows sufficiently to what extent the contacts between one country and another, prolonged sojourns of students and of teachers at foreign universities, not official sessions at which one drinks toasts while waiting for the next airplane, are an indispensable condition for all progress. We believe that more favorable conditions are found in England and in some of those nations of western Europe which are small only in military statistics. As to Germany, only the future can show whether she will find within herself the necessary elements for linking up with the brilliant tradition interrupted by fifteen years of organized stupidity. Beyond the Atlantic finally, we find a large country, which counts its universities by the hundreds, its students by the hundred thousands, and where, in the words of H. Morrison, the great American specialist in educational problems, "one wanted the education of the masses, one has mass production in education." Thorstein Veblen once sketched in a small book, too little read, the plan of higher education in the United States, and he has done it in a masterly manner; let us merely indicate how the future mathematician is formed in this country which produces more "mathematicians" than perhaps all the rest of the world. In the most favorable cases, one sees a student who, towards the end of his stay at the University, has three or four years at his disposal in which to acquire at the same time the knowledge, the method of work and the elementary intellectual apprenticeship, for which nothing that he has experienced before has in the least prepared him. His only way out under these circumstances is to seek his salvation in the most narrow specialization; in this way, he can often, if he is intelligent and has good guidance, do useful work. Beyond this, he runs the great risk of not being able to survive the stupefying effects of the purely mechanical teaching which he will have to inflict on others, in order to earn his living, after having undergone it himself for too long a time. Whether, in other fields, mass production, thus understood, may produce good results, we are not qualified to determine; we hope to have made it clear that this can not be the case in mathematics. If, unfortunately, the plausible doctrine of making education available to all has had such consequences in a country which lacks, it is true, a strong intellectual tradition, do we not have reason to fear the spread of the contagion to a Europe enfeebled by a catastrophe without precedent?

But if, as Panurge, we ask the oracle questions which are too indiscreet, the oracle will answer us as it did Panurge: Trinck! This advice the mathematician follows gladly, pleased as he is to believe that he will be able to slake his thirst at the very sources of knowledge, convinced as he is that they will always continue to pour forth, pure and abundant, while others have to have recourse to the muddy streams of a sordid reality. If he be reproached with the haughtiness of his attitude, if he be summoned to do his part, if he be asked why he persists on the high glaciers whither no one but his own kind can follow him, he will answer, with Jacobi: For the honor of the human spirit.

THE PROBLEM OF EIGHT POINTS

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1. Introduction. In a *Treatise on Surveying* by William M. Gillespie* one finds the following problem with figure and comment:



"*Problem of the Eight Points.* Four points, *A*, *B*, *C*, *D*, are inaccessible, but visible from four other points *E*, *F*, *G*, *H;* it is required to find the relative distances of these eight points; the only data being the observation, from each of the points of the second system, of the angles under which are seen the points of the first system.

This problem can be solved, but the great length and complication of the investigation and resulting formulas render it more a matter of curiosity than of utility. It may be found in Puissant's *Topographie*, page 55; Lefevre's *Trigonometrie*, page 90, and Lefevre's *Arpentage*, No. 387."

Puissant gives† a method for handling the problem which is entirely formal and leaves unanswered all the interesting questions with regard to the existence

* Revised by Cady Staley, 1887, p. 256.

† Traite de Topographie, d'Arpentage, et de Nivellement by Puissant, Paris, 1820, p. 55. The problem was brought to my attention by Mr. F. M. Garnett of Savannah, Georgia, who also furnished me with a copy of Puissant's solution.

and number of solutions. The purpose of this paper is to indicate a method of solution (essentially the same as that given by Puissant), to call attention to two Cremona transformations related to the problem, and to give numerical examples of various special cases in which Puissant's treatment breaks down.

2. Puissant's solution. To make the statement of the problem more definite we assume that the eight points are all distinct and lie in a horizontal plane, and we define α as the angle AEB through which an observer at E and facing A must turn to face B, $-180° < \alpha \leqq 180°$, α positive or negative according as the observer turns counter-clockwise or clockwise. At the point E we measure the three such angles, $\alpha = AEB$, $\beta = BEC$, $\gamma = CED$, and at each of the other points of observation the three similar angles; these twelve angles are the only data of the problem. The figure as given by Gillespie and Puissant shows the simple case in which the eight points are vertices of a convex octagon, but there is no assumption of this sort in the problem. The eight points may be anywhere in the plane, with only the restriction that no two of them may coincide.

We set up a rectangular Cartesian coordinate system with B as the origin and C as the point (1, 0), using BC as the unit of length. Let P and Q respectively be the feet of perpendiculars dropped from A and D to the x-axis BC; and let h, k, r, s be four numbers such that

$$|h| = PB, \qquad |k| = PA, \qquad |r| = CQ, \qquad |s| = QD,$$

with the signs so chosen that $(-h, k)$ and $(r+1, s)$ shall be the coordinates, respectively, of A and D. For the figure shown, h, k, r, s are all positive.

Now let the coordinates of E be (u, v), and let the tangents of the given angles at E be

$$\tan\alpha = \frac{a_1}{a_2}, \qquad \tan\beta = \frac{b_1}{b_2}, \qquad \tan\gamma = \frac{c_1}{c_2},$$

with the understanding that a_2, b_2 or c_2 will be zero when the corresponding angle is $\pm 90°$. It can then be shown readily that u, v must satisfy the equations

$$\begin{aligned} &a_1(u^2+v^2) + (a_1h - a_2k)u - (a_1k + a_2h)v = 0 \\ (1)\quad &b_1(u^2+v^2) - b_1u - b_2v = 0 \\ &c_1(u^2+v^2) + (c_2s - c_1r - 2c_1)u - (c_1s + c_2r)v + (c_1r - c_2s + c_1) = 0. \end{aligned}$$

The elimination of u and v from these three equations gives the equation

$$\begin{aligned} &b_1(a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1 - a_1b_1c_1)(hr - ks) \\ (2)\qquad &+ b_1(a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2 - a_2b_2c_2)(hs + kr) \\ &- a_1c_1(b_1^2 + b_2^2)(h + r + 1) + a_2c_1(b_1^2 + b_2^2)k + a_1c_2(b_1^2 + b_2^2)s = 0. \end{aligned}$$

The angles α, β, γ being given, this is the condition on h, k, r, s that the equations (1) may have a simultaneous solution for u, v. Geometrically, it is a condition on the situation of the four points A, B, C, D such that they may, at some

point of the plane, subtend the given angles α, β, γ.

Using the given angles at each of the four observation points we have four equations like (2), and we want to solve this set of equations for h, k, r, s. If we set

$$hr - ks = z, \qquad hs + kr = w, \qquad h + r + 1 = t,$$

this set of equations takes the form

$$A_{i1}z + A_{i2}w + A_{i3}t + A_{i4}k + A_{i5}s = 0, \qquad i = 1, 2, 3, 4, \tag{3}$$

linear and homogeneous in z, w, t, k, s.

Puissant then proceeds as follows:

Equations (3) are solved for z, w, t, s as multiples of k, giving

$$z = Zk, \qquad w = Wk, \qquad t = Tk, \qquad s = Sk.$$

From $w = Wk$ and $s = Sk$ we have

$$Shk + kr = Wk, \quad \text{or} \quad Sh + r = W,$$

and from $t = Tk$ we have

$$h + r = Tk - 1.$$

Solving these last two equations for h and r, we substitute the results in

$$z = hr - Sk^2 = Zk$$

and obtain the quadratic equation in k,

$$S\{T^2 + (S-1)^2\}k^2 - \{2ST + TW + STW - Z(S-1)^2\}k + (W+1)(W+S) = 0. \tag{4}$$

Substituting the roots, k and k', of equation (4) in the preceding equations, we obtain two solutions, h, k, r, s and h', k', r', s', of equations (3). We then substitute these sets of values in equations (1), using the tangents of the angles at each of the four observation points, and obtain the coordinates (u, v) of each of these points.

Puissant says that the problem will thus have two solutions; but it is obvious that his procedure may break down at several points. The set of equations (3) may not determine z, w, t, s as multiples of k; equation (4) may be linear rather than quadratic (if $S=0$); equations (1) may not determine u and v uniquely; *etc.* But by far the biggest loop-hole in Puissant's treatment of the problem is due to the fact that the method depends only on the tangents of the twelve given angles.* Hence very often after formal solutions have been found, they will have to be discarded because one or more of the angles of the solution will differ from the given angles by 180°. By the modified problem we shall

* Puissant's equations are expressed in terms of the sines and cosines of these angles, but in such a way that the equations are not altered when any angle is changed by 180°. Hence they depend essentially on the tangents of the angles.

mean the problem in which not the twelve angles, but only their tangents are given. It is this modified problem that Puissant really solves.* Any solution of the problem is of course a solution of the modified problem, but not conversely.

In what follows it is assumed that the twelve given angles (or their tangents) are chosen arbitrarily within the prescribed limits. The problem as worded by Puissant and Gillespie says that these angles are "observed," which would seem to imply the existence of at least one solution; but for angles chosen arbitrarily it will often happen that there are no solutions even for the modified problem (see examples 4, 6, 9, 15, 16 below).

3. Related Cremona transformations. In attempting to make simple numerical examples one discovers two interesting Cremona transformations connected with the problem. Suppose one starts with a set of object points A, B, C, D with rational coordinates, that is, with a rational set of values for h, k, r, s. Rational values are obtained for z, w, t, and then (if $k \neq 0$) for Z, W, T, S, and hence one has the quadratic equation (4) with rational coefficients. One root of this equation is the original k, and hence the other root k' must be rational; and rational values for h', r', s' are then obtained. This means that the set h', k', r', s' is related to the set h, k, r, s by a rational transformation. This transformation is found to be

$$
\begin{aligned}
h' &= \frac{(hs + kr + k)\{r(h + r + 1) - s(k - s)\}}{s\{(h + r + 1)^2 + (k - s)^2\}} \\
k' &= \frac{(hs + kr + k)(hs + kr + s)}{s\{(h + r + 1)^2 + (k - s)^2\}} \\
r' &= \frac{(hs + kr + s)\{h(h + r + 1) + k(k - s)\}}{k\{(h + r + 1)^2 + (k - s)^2\}} \\
s' &= \frac{(hs + kr + s)(hs + kr + k)}{k\{(h + r + 1)^2 + (k - s)^2\}}.
\end{aligned} \tag{5}
$$

This may be regarded as a transformation between points (h, k, r, s) and (h', k', r', s') of a four-dimensional space. It is an involutorial Cremona transformation. Each point in each of the six planes

$$k = s = 0, \quad h = k = 0, \quad r = s = 0, \quad h + 1 = k = 0, \quad r + 1 = s = 0,$$
$$h + r + 1 = k - s = 0,$$

is a fundamental point, and each point of the cubic variety

$$k(r^2 + s^2 + r) - s(h^2 + k^2 + h) = 0$$

is a fixed point.

* Puissant, in his only numerical example, gives "two solutions"; but in one of them the angle γ at E is $-139°$ while the required angle is 41°, so that it is not a solution.

We now have two sets of related object points A, B, C, D and A', B', C', D'. If we next choose any observation point E with rational coefficients, we find rational tangents for the angles α, β, γ at E; and using these tangents with h', k', r', s' in equations (1) we find u', v' as rational coordinates of a related observation point E'. The geometric relationship is that the tangents of the angles α, β, γ subtended by A, B, C, D at E will be the same as the tangents of the angles α', β', γ' subtended by A', B', C', D' at E'. The equations expressing the coordinates (u', v') of E' rationally in terms of the coordinates (u, v) of E are somewhat long and complicated, and it seems hardly worth while to give them here. They represent, however, a Cremona transformation of a well known type,* between points E of a plane π, and points E' of a plane π'. We shall refer to this as transformation (6). The transformation is of the fifth degree with six fundamental points in each plane, A, B, C, D and the two imaginary "circular points" I, J in the plane π, and A', B', C', D' and I', J' in the plane π'. The fundamental curves in each plane are the six conics determined by the fundamental points taken five at a time. This means that the four real fundamental curves in each plane are the four circles determined by A, B, C, D (or A', B', C', D') taken three at a time. Thus if E is taken on the circle ABC, E' will coincide with D', *etc.* (See example 16 below). The properties of the Cremona transformations (5) and (6) correspond to interesting features of the problem of eight points, and enable one to make up rational numerical examples. One chooses A, B, C, D and finds A', B', C', D' by (5); then chooses the observation points E, F, G, H, and for each one (and quite independently of the others) finds the corresponding point E', F', G', or H' by transformation (6); and then finally finds the tangents of the angles α, β, γ at each observation point.

4. **Examples of special cases.** In the table of numerical examples, page 313, the given angles are indicated by a pair of homogeneous numbers in parentheses preceded by a Roman numeral. Thus II(−5:3) indicates the angle in the second quadrant having the tangent $-5/3$. The quadrantal angles $-90°$, $0°$, $90°$, $180°$ will be designated respectively by III(1:0), IV(0:1), I(1:0), II(0:1). With the restriction $-180° < \alpha \neq 180°$ this notation specifies the angle uniquely. One solves the modified problem ignoring the quadrant specification, obtaining solutions in the form of coordinates for the eight points. In each solution of the modified problem one then checks the angles and discards those solutions in which one or more angles differ from the required angles by 180°. It is quite obvious that if the twelve given angles are chosen arbitrarily, the chance of obtaining any solution for the problem is small.

Examples 1, 2. These represent the normal case in the sense that Puissant's procedure goes through without trouble, giving in each example two solutions for the modified problem. In the first example there are no solutions for the problem proper because the angle α at G turns out to be $-45°$ instead of 135°. If we change the problem by specifying the angle IV(−1:1) for α at G, there

* See Cremona Transformations by Hilda P. Hudson, 1927, p. 120.

will be two solutions. In example 2 there is one solution for the problem, but the other fails because α at H is $-135°$ instead of $45°$ as required. No change in the quadrant specification could give us two solutions for this problem, because the angle α at H differs in the two solutions of the modified problem, and if one is right the other must be wrong.

The next few examples indicate ways in which Puissant's handling of the set of equations (3) may break down. In the 4×5 matrix of coefficients of these equations we designate by D_z, D_w, D_t, D_k, D_s respectively the five determinants of fourth order obtained by omitting each column in order. If not all of these determinants vanish, there is homogeneously unique solution for z, w, t, k, s; but we cannot solve for z, w, t, s as multiples of k if $D_k=0$.

Example 3. Here $D_k=0$, $D_s\neq0$. We can solve for z, w, t, k in terms of s, and this leads to a linear equation in s instead of the quadratic equation (4) in k. We find $h=-2$, $k=0$, $r=-2$, $s=1$, and in a strict sense there is only one solution of the modified problem and the problem, namely,

$$A(2, 0),\ B(0, 0),\ C(1, 0),\ D(-1, 1),\ E(0, 1),\ F(1, 2),\ G(-1, -1),\ H(2, -1).$$

But if we make use of transformation (5), keeping h, r, s fixed as above and letting k approach zero, we find that $h'\to-7/5$, $k'\to1/5$, and r' and s' become very large with the ratio $s'/(r'+1)$ approaching $-1/3$. If we then think of D as so far away that the lines of sight from E', F', G', H' to D' all have measurably the same slope, $-1/3$, we have what might be regarded as a second solution

$$A'\left(\frac{7}{5}, \frac{1}{5}\right),\quad B'(0, 0),\quad C'(1, 0),\quad D'\left(\text{slope } -\frac{1}{3}\right),$$

$$E'\left(\frac{2}{5}, \frac{6}{5}\right),\quad F'\left(\frac{3}{2}, \frac{1}{2}\right),\quad G'\left(\frac{7}{5}, -\frac{14}{5}\right),\quad H'\left(\frac{20}{13}, \frac{-4}{13}\right).$$

Examples 4, 5. In example 4, $D_k=D_s=0$, $D_w\neq0$. This gives $k=s=0$, and $w=hs+kr\neq0$, which is impossible, and there is no solution even for the modified problem. In example 5, $D_s=D_k=D_w=0$, $D_z\neq0$. Hence we have $s=k=w=0$, $t=3z$, and it follows that h may be arbitrary and $r=(h+1)/(3h-1)$. We have infinitely many solutions of the modified problem; and we have also infinitely many solutions of the problem proper given by values of h greater than $1/3$. A particularly interesting solution is obtained for $h=1$.

Examples 6, 7. In each of these examples the coefficients of w in equations (3) all vanish, so that we have a set of four homogeneous equations in the four quantities z, t, k, s. In example 6 the determinant does not vanish and the only solution of the set would be $z=t=k=s=0$; but this would make A and D coincide with either B or C and so furnishes no solution of the modified problem. In example 7 the determinant vanishes, and we have

$$z=-\frac{1}{5}k,\qquad t=\frac{8}{5}k,\qquad s=-\frac{1}{5}k,$$

Example		E	F	G	H
1	α	II (−1:2)	IV (−2:1)	II (−1:1)	I (2:3)
	β	III (1:0)	I (1:3)	I (1:1)	IV (−3:11)
	γ	III (2:1)	I (1:2)	IV (−1:2)	I (1:2)
2	α	I (4:3)	I (21:25)	I (1:4)	I (1:1)
	β	I (1:4)	I (1:5)	I (1:13)	I (1:9)
	γ	I (7:6)	I (1:1)	III (3:1)	I (11:24)
3	α	IV (−2:1)	IV (−4:3)	I (1:2)	I (2:1)
	β	I (1:1)	I (1:2)	IV (−1:3)	IV (−1:3)
	γ	III (1:1)	IV (−2:1)	I (2:1)	I (1:5)
4	α	IV (−1:3)	I (1:1)	I (1:2)	IV (−1:1)
	β	I (1:2)	IV (−1:3)	I (1:1)	I (2:1)
	γ	IV (−4:3)	I (1:3)	IV (−14:13)	IV (−12:1)
5	α	I (1:1)	I (1:1)	I (1:3)	I (1:7)
	β	I (1:3)	I (1:1)	I (1:1)	I (1:3)
	γ	I (1:7)	I (1:3)	I (1:1)	I (1:1)
6	α	IV (−1:4)	I (1:2)	I (1:1)	I (8:7)
	β	I (2:1)	I (1:1)	I (5:9)	I (20:39)
	γ	I (6:7)	I (1:3)	I (2:7)	I (1:4)
7	α	I (1:3)	I (7:11)	I (17:31)	I (11:23)
	β	I (1:1)	IV (−1:4)	I (1:7)	I (5:12)
	γ	I (1:2)	III (3:1)	I (4:3)	I (17:19)
8	α	II (−8:1)	I (2:1)	I (8:9)	I (8:19)
	β	I (6:17)	I (1:2)	I (2:5)	I (14:73)
	γ	I (2:1)	I (1:0)	II (−2:1)	II (−2:3)
9	α	I (1:7)	I (1:7)	I (1:3)	I (1:1)
	β	I (3:4)	I (3:4)	I (3:2)	II (−3:2)
	γ	I (1:2)	I (1:2)	I (2:3)	I (1:1)
10	α	I (1:7)	I (1:7)	I (1:3)	I (1:6)
	β	I (3:4)	I (3:4)	I (3:2)	I (6:1)
	γ	I (1:2)	I (1:2)	I (2:3)	II (−2:3)
11	α	II (−1:2)	IV (−2:1)	III (9:1)	IV (−1:2)
	β	III (1:0)	I (1:3)	I (1:1)	I (13:51)
	γ	III (2:1)	I (1:2)	I (1:3)	I (1:60)
12	α	I (1:2)	I (3:4)	IV (−1:4)	I (5:12)
	β	I (2:3)	I (4:3)	IV (−2:3)	I (2:3)
	γ	I (11:16)	I (7:6)	IV (−7:22)	I (3:2)
13	α	I (1:18)	I (1:3)	I (1:6)	IV (−1:1)
	β	III (12:5)	I (3:2)	I (6:1)	IV (−3:4)
	α	I (8:27)	I (2:3)	II (−2:3)	I (13:34)
14	α	I (1:7)	I (1:3)	I (1:6)	IV (−1:1)
	β	I (3:4)	I (3:2)	I (6:1)	IV (−3:4)
	γ	I (1:2)	I (2:3)	II (−2:3)	I (13:34)
15	α	I (173:29)	I (33:1)	I (21:2)	I (33:17)
	β	I (3:1)	I (2:1)	I (1:1)	I (1:0)
	γ	I (1:1)	I (1:1)	I (2:1)	I (1:1)
16	α	IV (−3:4)	I (5:2)	I (1:0)	I (1:1)
	β	I (1:3)	IV (−3:7)	IV (−1:1)	IV (−1:3)
	γ	IV (−1:7)	III (1:1)	II (−1:3)	III (1:3)

which leads to the equation

$$5r^2 - (8k - 5)r - (k^2 + k) = 0.$$

Choosing an arbitrary value for k we get two values for r and h, and hence two solutions for the modified problem. We will therefore have an infinite number of solutions of the modified problem, and in this particular example also an infinite number of solutions of the problem proper. Two fairly simple solutions are given by taking $k=1$ with $r=1$ and $k=5/7$ with $r=4/7$. The vanishing of all the coefficients of w in this example corresponds to the geometric fact that $\alpha+\beta+\gamma=\pm 90°$ at each of the observation points.

Example 8. All the coefficients of z in the equations (3) vanish, giving four equations in w, t, k, s, and the determinant is of rank 2. We may choose both k and s arbitrarily, and then

$$h = -2k + 4, \qquad r = 2s - 5,$$

which means that A and D lie anywhere on the line $x-2y+4=0$. It is then found that the four observation points also lie on this line at fixed positions for all choices of k and s. The vanishing of the coefficients of z means that at each observation point $\alpha+\beta+\gamma=0°$ or 180°, and hence each observation point must lie on the line AD. There are infinitely many solutions for the modified problem; and also, if we take $k<3/2$ and $s>7/2$, infinitely many solutions for the problem.

Examples 9, 10. The angles at E are exactly the same as the angles at F, so that the first two of the four equations (3) are the same, and hence $D_z=D_w=D_t=D_k=D_s=0$. It is easy to show, however, that the angles at two distinct points E and F could be the same only if A, B, C, D, E, F all lie on one circle. Using this fact and the tangents of the angles at E and F (and abandoning Puissant's procedure), we deduce without difficulty that

$$h = \frac{1}{6}, \qquad k = \frac{1}{6}, \qquad r = \frac{1}{3}, \qquad s = \frac{2}{3},$$

and hence

$$z = -\frac{1}{18}, \qquad w = \frac{1}{6}, \qquad t = \frac{3}{2}.$$

These values may or may not satisfy the two remaining equations of set (3). In example 9 the last equation is not satisfied, and hence there are no solutions even for the modified problem. In example 10 the last two equations are satisfied There is only one possible position for the points A,B,C,D (on a circle), but there are infinitely many solutions of the modified problem because E and F may be taken anywhere on the circle. The three equations of set (1) become identical for the points E and F. There are infinitely many solutions of the problem proper with E and F restricted to certain arcs of the circle.

Example 11. For the set of equations (3) we have $D_z = D_w = D_t = D_k = D_s = 0$ without any situation quite so obvious as in examples 7, 8, 9, 10. The matrix is of rank 3, and we can solve for z, w, t in terms of k and s,

$$z = \frac{-8k + 13s}{9}, \qquad w = \frac{2k - 19s}{9}, \qquad t = \frac{-5k + 7s}{9}.$$

We can then give arbitrary values to S in the relation $s = Sk$, and proceed with Puissant's method, obtaining (usually) two solutions of the modified problem for each value assigned to S. There are infinitely many solutions of both the modified problem and the problem. For $S = 2$ we get two fairly simple solutions of the modified problem, both of which are solutions of the problem.

In the remaining examples the determinant D_k for the set of equations (3) does not vanish, and there is a unique solution for z, w, t, s as multiples of k; also we can set up the quadratic equation (4). For this equation the discriminant may be positive, negative, or zero, and Puissant ignores the last two possibilities. The discriminant vanishes under either of the two following geometric conditions: when AD is parallel to BC so that $s = k$ ($S = 1$), and when A, B, C, D all lie on one circle for which the condition is

$$k(r^2 + s^2 + r) - s(h^2 + k^2 + h) = 0. \tag{7}$$

Example 12. We have

$$z = k, \qquad w = 2k, \qquad t = 6k, \qquad s = k,$$

and equation (4) becomes

$$36k^2 - 36k + 9 = 0,$$

giving $\frac{1}{2}$ as the only value for k and for s. But there are two values each for h and r, and we have the two solutions for equations (3),

$$h = \frac{1}{2}, \qquad k = \frac{1}{2}, \qquad r = \frac{3}{2}, \qquad s = \frac{1}{2}$$

$$h' = \frac{3}{2}, \qquad k' = \frac{1}{2}, \qquad r' = \frac{1}{2}, \qquad s' = \frac{1}{2}.$$

This leads to two solutions for both the modified problem and the problem.

Examples 13, 14. In both of these examples we have

$$z = -\frac{1}{3}k, \qquad w = k, \qquad t = 9k, \qquad s = 4k,$$

and equation (4) is

$$360k^2 - 120k + 10 = 0,$$

giving $k = 1/6$ and the one set of values

$$h = \frac{1}{6}, \qquad k = \frac{1}{6}, \qquad r = \frac{1}{3}, \qquad s = \frac{2}{3}.$$

These values satisfy the relation (7), showing that A, B, C, D lie on a circle. In example 13 the observation points are uniquely determined (none of them lying on the circle $ABCD$), and we have one solution for the problem. But in example 14, when we try to find the coördinates of E, the three equations of set (1) are all the same as the equation of the circle $ABCD$, and E is any point on this circle giving infinitely many solutions of the modified problem. There are also infinitely many solutions of the problem proper, with E on the major arc AD of the circle.

Example 15. Here we find

$$z = 24k, \qquad w = 8k, \qquad t = 3k, \qquad s = 2k,$$

and equation (4) is

$$20k^2 - 60k + 90 = 0.$$

The values of k are imaginary, and therefore there are no solutions even for the modified problem.

Example 16. Puissant's method goes through, giving what would be two solutions of the modified problem except for the fact that in each solution an observation point coincides with an object point, F with B in one solution and G' with D' in the other. In a strict sense we have no solutions. But it is possible to take a certain line through F (the line with slope $3/7$) as the line of sight from F to B, and similarly a line through G' (with slope 1) as the line $G'D'$, giving two pseudo-solutions of the modified problem. In terms of the Cremona transformation (6), G is on the fundamental circle ABC in the plane π, and F' is on the fundamental circle $A'C'D'$ in the plane π', so that G' and F coincide respectively with the fundamental points D' and B.

The interested reader will find it not too difficult to make up examples of other cases for which the Puissant procedure will break down, and for which there may be no solutions, one, or infinitely many.

FACTORIZATION OF POLYNOMIALS

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1. Introduction. Let D be an integral domain* and let $D[x]$ be the integral domain of polynomials in x with coefficients in D. Then

THEOREM. *If D is a unique factorization domain (abbreviated UFD), $D[x]$ is also.*

It follows from this theorem, by mathematical induction on n, that when D is a UFD then the domain $D[x_1, x_2, \cdots, x_n]$ of polynomials in $x_1, x_2, \cdots, x_n$ with coefficients in D is also a UFD.

In the usual proof† of the theorem, the desired result is first established when D is a field. The proof of this depends upon:

(a) the division algorithm, which is used to prove
(b) if $f(x)$ and $g(x)$ are non-zero polynomials in $D[x]$, there exists a greatest common divisor (abbreviated GCD) $d(x)$ expressible in the form $a(x)f(x)+b(x)g(x)$, where $a(x)$ and $b(x)$ are elements of $D[x]$, which is used to prove
(c) if a prime polynomial divides the product of two polynomials it divides one of the factors. Using this special case of the theorem and
(d) Gauss' lemma and
(e) the existence of a quotient field for any integral domain, the general case of the theorem is established.

The usual proof of the fact that the integers form a UFD also makes use of results similar to (a), (b) and (c). However, E. Zermelo‡ has given a very simple proof which avoids these considerations. It is the purpose of this note to present a similar proof of the theorem above which does not require the use of (a)–(e).

2. Existence of factorization. Let D be a UFD. We show that every non-zero element $f(x)$ of $D[x]$ which is not a unit is expressible in the form $p_1(x)p_2(x)\cdots p_k(x)$, $k \geqq 1$, where the $p_i(x)$ are primes in $D[x]$.

We shall make use of the following easily demonstrable facts:

(1) The units of $D[x]$ are the units of D.
(2) If a constant divides a polynomial it divides every coefficient of the polynomial.
(3) A constant is prime in $D[x]$ if and only if it is prime in D.
(4) If $a_1, a_2, \cdots, a_k$ $(k \geqq 2)$ are elements of D not all zero, there exists a GCD a in D, and if $a_i = ab_i$ $(i = 1, 2, \cdots, k)$ then the b_i have no common factors in D except units.

Now let $f(x)$ be of degree n. If $n = 0$, $f(x)$ is in D. Since it is not a unit of $D[x]$, it is not a unit of D. Hence it is a product of primes in D which are also primes in $D[x]$.

* For definitions of terms see Birkhoff-MacLane, A Survey of Modern Algebra, Macmillan, 1941.

† Birkhoff-MacLane, pp. 91–100.

‡ Nachr. Ges. Wiss. Göttingen I, N.F. 1, 1934, pp. 43–46. The complete proof will also be found in a few lines in Zentralblatt 10, 1935, p. 293.

Proceeding by induction on n, let $n>0$ and suppose the desired result established for polynomials of degree $0, 1, \cdots, n-1$.

Let

$$\begin{aligned} f(x) &= a_0x^n + a_1x^{n-1} + \cdots + a_n, && n > 0,\ a_0 \neq 0 \\ a &= \text{GCD of } a_0, a_1, \cdots, a_n \\ a_i &= ab_i && (i = 0, 1, \cdots, n) \\ g(x) &= b_0x^n + b_1x^{n-1} + \cdots + b_n. \end{aligned}$$

Then

$$f(x) = ag(x).$$

Since $a \neq 0$ is in D, it is a unit or is factorable into primes.

If $g(x)$ is a prime, the desired result is established (using, if necessary, the fact that the product of a unit and a prime is a prime). Otherwise, there exists a factorization $g(x)=s(x)t(x)$, where $s(x)$ and $t(x)$ are elements of $D[x]$ neither of which is a unit. Neither $s(x)$ nor $t(x)$ can be a constant. For, if either were a constant, it would have to divide each of the coefficients of $g(x)$, and this is impossible since the b_i have no common divisors except units. Thus, each of $s(x)$ and $t(x)$ is of degree less than n. Therefore, by the hypothesis of the induction, each is factorable into primes, and the desired result is established.

3. Uniqueness of factorization. We show that if

$$f(x) = p_1(x)p_2(x) \cdots p_k(x) = q_1(x)q_2(x) \cdots q_l(x),$$

where the $p_i(x)$ and $q_j(x)$ are primes, then $k=l$ and the $p_i(x)$ can be paired with the $q_j(x)$ in such a way that the $p_i(x)$ and $q_j(x)$ in each pair are associates.

If $n=0$ then $f(x)$ and the prime factors are in D. Since D is a UFD, the desired results is immediate. Proceeding by induction on the degree of $f(x)$, let $n>0$, and suppose the desired result established for polynomials of degree $0, 1, \cdots, n-1$.

If $k=l=1$, there is nothing more to be proved.

If $k=1$ and $l>1$, then $q_1(x)[q_2(x) \cdots q_l(x)]$ is a prime. Hence $q_2(x) \cdots q_l(x)$ is a unit, so that $q_2(x)$ is a unit. This is impossible.

Similarly, $k>1$ and $l=1$ is impossible.

Let

$$\begin{aligned} & k > 1, \quad l > 1 \\ f(x) &= a_0x^n + a_1x^{n-1} + \cdots + a_n, && n > 0,\ a_0 \neq 0 \\ p_i(x) &= a_{i0}x^{n_i} + \cdots + a_{in_i},\ a_{i0} \neq 0 && (i = 1, 2, \cdots, k), \\ q_j(x) &= b_{j0}x^{m_j} + \cdots + b_{jm_j},\ b_{j0} \neq 0 && (j = 1, 2, \cdots, l). \end{aligned}$$

Let n_1 be one of the smallest of the n_i, and let m_1 be one of the smallest of the m_j. Either $n_1 \leqq m_1$ or $m_1 \leqq n_1$. Suppose, to be specific, $n_1 \leqq m_1$. Then $n_1 \leqq m_j$ for

$j=1, 2, \cdots, l$.

We show that $p_1(x)$ is an associate of some $q_j(x)$.

If $n_1=0$,* then $p_1(x)=a_{10}$ is a prime in D which is a factor of

$$a_0 = a_{10}a_{20}\cdots a_{k0} = b_{10}b_{20}\cdots b_{l0}.$$

Since D is a UFD, a_{10} is an associate of one of the primes in the factorizations of $b_{10}, b_{20}, \cdots, b_{l0}$. Therefore one of $b_{10}, b_{20}, \cdots, b_{l0}$ is divisible by a_{10}. Let

$$\begin{aligned} b_{u0} &= a_{10}c \\ P(x) &= q_1(x)\cdots q_{u-1}(x)p_1(x)q_{u+1}(x)\cdots q_l(x) \\ &= (b_{10}\cdots b_{u-1,0}a_{10}b_{u+1,0}\cdots b_{l0})x^m + \text{lower powers of } x, \text{ if any,} \end{aligned}$$

where

$$\begin{aligned} m &= m_1 + \cdots + m_{u-1} + n_1 + m_{u+1} + \cdots + m_l \\ &\leqq m_1 + \cdots + m_{u-1} + m_u + m_{u+1} + \cdots + m_l = n. \end{aligned}$$

Let $R(x)=f(x)-cx^{n-m}P(x)$. Then either $R(x)=0$, or the degree of $R(x)$ is less than n.

If $n_1>0$, let u be any integer between 1 and l inclusive and define $P(x)$ as above. Let

$$R(x) = a_{10}f(x) - b_{u0}x^{n-m}P(x).$$

Again either $R(x)=0$ or the degree of $R(x)$ is less than n. Combining both cases we have

$$R(x) = \alpha f(x) - Q(x)P(x)$$

where $Q(x)$ is in $D[x]$, $\alpha\neq 0$ is in D, and $\alpha=1$ if $p_1(x)$ is a constant.

We consider two cases:

(a) $R(x)=0$. Then

$$\begin{aligned} \alpha f(x) &= \alpha q_1(x)q_2(x)\cdots q_l(x) \\ &= Q(x)q_1(x)\cdots q_{u-1}(x)p_1(x)q_{u+1}(x)\cdots q_l(x), \end{aligned}$$

so that

$$\alpha q_u(x) = q(x)p_1(x).$$

If $n_1=0$ then $\alpha=1$ and $p_1(x)$, $q_u(x)$ are associates.

If $n_1>0$ then every $m_j\geqq n_1>0$. Since $l\geqq 2$, therefore every $m_j<n$. Hence $m_u<n$. Therefore, by the hypothesis of the induction, $\alpha q_u(x)$ is uniquely factorable into primes. By factoring α (unless it is a unit) we see that one of the primes is $q_u(x)$ or an associate and the others are constants. By factoring $Q(x)$ (unless it is a unit) we see that one of the primes is $p_1(x)$ or an associate. Hence $p_1(x)$ is an associate of $q_u(x)$ or of a constant. Since $n_1>0$, $p_1(x)$ is not a constant. Therefore it is an associate of $q_u(x)$.

* This cannot happen if D is a field.

(b) $R(x) \neq 0$. We have

$$R(x) = \alpha q_1(x)q_2(x) \cdots q_l(x) - Q(x)q_1(x) \cdots q_{u-1}(x)p_1(x)q_{u+1}(x) \cdots q_l(x)$$
$$= q_1(x) \cdots q_{u-1}(x)q_{u+1}(x) \cdots q_l(x)[\alpha q_u(x) - Q(x)p_1(x)].$$

Also

$$R(x) = \alpha p_1(x)p_2(x) \cdots p_k(x) - Q(x)q_1(x) \cdots q_{u-1}(x)p_1(x)q_{u+1}(x) \cdots q_l(x)$$
$$= p_1(x)S(x).$$

$R(x)$ is not a unit, since $p_1(x)$ is a factor of $R(x)$. Therefore, by the hypothesis of the induction, $R(x)$ is uniquely factorable into primes. Its prime factors are $q_1(x), \cdots, q_{u-1}(x), q_{u+1}(x), \cdots, q_l(x)$ and the prime factors of $\alpha q_u(x) - Q(x)p_1(x)$. Also, its prime factors are $p_1(x)$ and the prime factors of $S(x)$. Therefore, $p_1(x)$ is an associate of one of $q_1(x), \cdots, q_{u-1}(x), q_{u+1}(x), \cdots, q_l(x)$ or of one of the prime factors of $\alpha q_u(x) - Q(x)p_1(x)$. In the latter case $p_1(x)$ is a factor of $\alpha q_u(x) - Q(x)p_1(x)$, so that

$$\alpha q_u(x) - Q(x)p_1(x) = T(x)p_1(x)$$
$$\alpha q_u(x) = [Q(x) + T(x)]p_1(x).$$

It now follows as in case (a) that $p_1(x)$ is an associate of $q_u(x)$.
Thus, in all cases, $p_1(x)$ is an associate of $q_u(x)$.

If $p_1(x) = \epsilon q_v(x)$ where ϵ is a unit, then $p_1(x)p_2(x) \cdots p_k(x) = q_1(x)q_2(x) \cdots q_l(x)$ yields

$$[\epsilon p_2(x)]p_3(x) \cdots p_k(x) = q_1(x) \cdots q_{v-1}(x)q_{v+1}(x) \cdots q_l(x)$$

where $\epsilon p_2(x)$ is a prime.

If the degree of $\epsilon p_2(x)p_3(x) \cdots p_k(x)$ is less than n, the desired result follows by the hypothesis of the induction. If not, then we may again proceed as above.

Since the process reduces the number of primes, we must, after applying it a sufficient number of times, arrive at a stage where either the hypothesis of the induction applies or else there is only one prime on one side of the equality. In either case the desired result follows.

MATHEMATICAL NOTES

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ON THE LATENT ROOTS OF QUASI-COMMUTATIVE MATRICES

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We consider matrices A, B of order n such that

$$AB = \omega BA,$$

where ω is a primitive qth root of unity, $\omega^q = 1$. Thus B is a commutant[1] of A and ωA, so that, if $B \neq 0$, A and ωA must have a common latent root[1]; that is, for at least one pair α, β of latent roots of A we have $\beta = \omega\alpha$. If B is non-singular, since $B^{-1}AB = \omega A$, the latent roots of A are clearly of the form α_1, $\omega\alpha_1$, $\omega^2\alpha_1$, $\cdots$, $\omega^{q-1}\alpha_1$, α_2, $\omega\alpha_2$, $\cdots$. It is simple to prove the following result.

THEOREM 1. *If α is a latent root of A, x a corresponding latent vector such that $Bx \neq 0$, then $\omega\alpha$ is also a latent root of A.*

Proof. Put $y = Bx$; then we have

$$Ay = ABx = \omega BAx = \omega\alpha Bx = \omega\alpha y,$$

since $Ax = \alpha x$. Also $y \neq 0$; hence $\omega\alpha$ is a latent root of A.

On the other hand we have the example

$$A = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -\alpha \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Here $AB = -BA$, α and $-\alpha$ are latent roots of A, and yet $Bx = 0$ for each latent vector x of A.

THEOREM 2. *The latent roots, $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n$, of A, B may be taken in such an order that the latent roots of*

(i) $$A + B \text{ are } (\alpha_r^q + \beta_r^q)^{1/q} \qquad (r = 1, \cdots, n),$$

and of

(ii) $$AB \text{ are } \omega^{\frac{1}{2}(q-1)}\alpha_r\beta_r \qquad (r = 1, \cdots, n).$$

[1] Cf. H. W. Turnbull, and A. C. Aitken, *An Introduction to the Theory of Canonical Matrices*, London, 1932, p. 148.

Proof. Let

$$(A + B)^k = \sum_{r=0}^{k} c_r^{(k)} B^r A^{k-r},$$

so that $c_0^{(k)} = c_k^{(k)} = 1$. Since $(A+B)^{k+1} = (A+B)(A+B)^k = (A+B)^k(A+B)$, we have

$$\sum_{r=0}^{k+1} c_r^{(k+1)} B^r A^{k+1-r} = \sum_{r=0}^{k} \omega^r c_r^{(k)} B^r A^{k+1-r} + \sum_{r=0}^{k} c_r^{(k)} B^{r+1} A^{k-r}$$

$$= \sum_{r=0}^{k} c_r^{(k)} B^r A^{k+1-r} + \sum_{r=0}^{k} \omega^{k-r} c_r^{(k)} B^{r+1} A^{k-r};$$

for $A^r B^s = \omega^{rs} B^s A^r$. Thus

$$c_r^{(k+1)} = \omega^r c_r^{(k)} + c_{r-1}^{(k)} = c_r^{(k)} + \omega^{k+1-r} c_{r-1}^{(k)},$$

and we deduce that this generalized binomial coefficient $c_r^{(k)}$ has the value

$$c_r^{(k)} = \frac{\phi_k}{\phi_r \phi_{k-r}}, \quad \text{where} \quad \phi_r = \prod_{s=1}^{r} (\omega^s - 1).$$

In particular, if ω is a primitive qth root of unity, we have $c_r^{(q)} = 0$, for $r = 1, 2, \cdots, q-1$, and

$$(A + B)^q = A^q + B^q.$$

By induction, $(AB)^k = \omega^{k(1-k)/2} A^k B^k$. As $A^q B^q = B^q A^q$, we have by Frobenius' theorem[2] that, if $f(x, y)$ is any rational function, then the latent roots of $f(A^q, B^q)$ are $f(\alpha_r^q, \beta_r^q)$ $(r = 1, \cdots, n)$ for some pairing of the roots α_r, β_r, and this is the same for every function $f(x, y)$. Thus the latent roots of $(A+B)^q$, $(AB)^q$ are

$$\alpha_r^q + \beta_r^q, \qquad \omega^{\frac{1}{2}q(q-1)} \alpha_r^q \beta_r^q \qquad (r = 1, \cdots, n).$$

Hence the latent roots of $A+B$ are $(\alpha_r^q + \beta_r^q)^{1/q} (r = 1, \cdots, n)$, and those of AB are $\omega^{\frac{1}{2}(q-1)} \alpha_r \beta_r$ $(r = 1, \cdots, n)$.

A CRITERION FOR A COMMON ROOT OF k ALGEBRAIC EQUATIONS

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1. Introduction. A necessary and sufficient condition that two algebraic equations $f(x) = 0$ and $g(x) = 0$ have a common root is the vanishing of Sylvester's determinant $R(f, g)$. But there seems to exist no simple criterion that more than two equations have a common root. In this paper such a criterion will be obtained under the assumption that at least one of the equations has only real roots.

The origin of this paper was Problem 2 of the Mathematical Series of the

[2] Frobenius. Berliner Sitzungsberichte, I (1896), pp. 601–614.

Rand Corporation in Santa Monica; the statement of this problem follows:

Given $n+1$ polynomials of degree n, every n of which are linearly independent, and each of which is non-negative for $0 \leqq x \leqq 1$ and has n roots (counting multiplicities) in that interval. Is it true that these polynomials have no root in common if, and only if, all $n+1$ are linearly independent? If not true, what additional conditions are necessary for the $n+1$ polynomials to have a common root?

2. Solution of problem 2. Let $f_1, f_2, \cdots, f_{n+1}$ be $n+1$ arbitrary polynomials of degree n. If these polynomials have a common root, then it is necessary that they are linearly dependent. This follows at once from the fact that they can be considered as linear forms in $1, x, x^2, \cdots, x^n$.

The converse is not correct even if the polynomials are non-negative in the interval $0 \leqq x \leqq 1$ and have there all their roots. This is shown by the example:

$$f_1(x) = (x - 1/2)^2(x - 1/4)^2, \qquad f_2(x) = (x - 1/3)^4,$$
$$f_3(x) = x^2(x - a)^2, \qquad f_4(x) = x^2(x - b)^2, \qquad f_5(x) = x^2(x - c)^2,$$

where $0 \leqq a \leqq 1$, $0 \leqq b \leqq 1$, $0 \leqq c \leqq 1$. They are linearly dependent since

$$\begin{vmatrix} & & \cdot & 3 & 1 \\ & & \cdot & -\frac{}{16} & \frac{}{64} \\ & & \cdot & & \\ & & \cdot & 4 & 1 \\ & & \cdot & -\frac{}{27} & \frac{}{81} \\ & & \cdot & & \\ \cdots & \cdots & \cdot & \cdots & \cdots \\ & & \cdot & & \\ & & \cdot & 0 & 0 \\ & & \cdot & & \\ & \Delta & \cdot & 0 & 0 \\ & & \cdot & & \\ & & \cdot & 0 & 0 \end{vmatrix} = \frac{\Delta}{16 \cdot 27} \begin{vmatrix} 3 & \frac{1}{4} \\ 4 & \frac{1}{3} \end{vmatrix} = 0.$$

The following condition is sufficient and necessary. Let f be one of the $n+1$ polynomials and F the sum of the n other polynomials. The polynomials have a common root if and only if f and F have a common root; that is, if and only if Sylvester's determinant $R(f, F)$ satisfies $R(f, F) = 0$.

Proof: If the polynomials have a common root, then this root is a common root of f and F. Conversely, assume that f and F have a common root. Then this root is a root of f and therefore real and lies in the interval $0 \leqq x \leqq 1$. On the other hand, F is non-negative in this interval and vanishes there only if all the terms of F have a common root. Hence all the polynomials have a common root.

Sometimes the computation of the resultant $R(f, F)$ can be simplified by using $c_\nu f_\nu(x)$ with positive c_ν instead of $f_\nu(x)$.

3. Generalization. This result can be generalized as follows.

THEOREM. *Let $f_1, f_2, \cdots, f_k$ be polynomials of arbitrary degree with real or complex coefficients. Set*

$$f_\kappa(x) = g_\kappa(x) + ih_\kappa(x) \qquad (\kappa = 1, 2, \cdots, k), \tag{1}$$

where $g_\kappa(x)$ and $h_\kappa(x)$ have real coefficients. Moreover, set

$$\bar{f}_\kappa(x) = g_\kappa(x) - ih_\kappa(x) \qquad (\kappa = 1, 2, \cdots, k).$$

Assume that one of these polynomials, say f_1, has only real roots. Set

$$F(x) = \sum_{\kappa=2}^{k} f_\kappa(x)\bar{f}_\kappa(x) = \sum_{\kappa=2}^{k} g_\kappa^2(x) + h_\kappa^2(x). \tag{2}$$

A necessary and sufficient condition that the polynomials (1) *have a common root is that f_1 and F have a common root.*

Proof. If the polynomials (1) have a common root ξ, then ξ is a root of $F(x)$ by (2). Conversely, if f_1 and F have a common root ξ, then ξ is real since it is a root of f_1. But F is non-negative for real values of x and vanishes only if all the polynomials $g_\kappa(x)$ and $h_\kappa(x)$ have a common root. Hence

$$g_\kappa(\xi) = h_\kappa(\xi) = 0 \qquad (\kappa = 2, 3, \cdots, k).$$

It follows that ξ is a common root of the polynomials (1).

ON A DETERMINANT OF SYLVESTER*

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1. Introduction. Let $\epsilon = e^{2\pi i/n}$. Then the element in the rth row and cth column of the determinant V of order n which is the subject of this note is defined by $v_{rc} = \epsilon^{(r-1)(c-1)}$. The first known application of this interesting determinant is due to Sylvester [1], who gave a formula for the value of V. In this note we show that V is reducible and obtain the values of the two component determinants; their elements contain only cosines and sines, respectively.

2. Discussion. Consider the transformation

$$x_j = \sum_{k=1}^{n} \epsilon^{(j-1)(k-1)} x_k', \qquad j = 1, \cdots, n, \tag{1}$$

of determinant V. This may be written as

$$x_{2-j} = \sum_{k=1}^{n} \epsilon^{(1-j)(k-1)} x_k', \qquad i = 1, \cdots, n, \tag{2}$$

where it is assumed that n is added to any subscript not in the range $1, \cdots, n$. The coefficients of x_k' in (1) and (2) are the same conjugate complex numbers as the coefficients of x_{2-k}' in (2) and (1). For an odd $n > 1$, we obtain, from (1) and (2),

* Presented to the Southeastern Section of the Association, March 19, 1949.

$$\frac{1}{2}(x_j + x_{2-j}) = x_1' + \sum_{k=2}^{(n+1)/2} 2\cos(j-1)(k-1)\frac{2\pi}{n}\cdot\frac{1}{2}(x_k' + x_{2-k}'), \qquad j = 1, \cdots, \frac{n+1}{2};$$

(3)

$$\frac{1}{2}(x_j - x_{2-j}) = \sum_{k=2}^{(n+1)/2} 2i\sin(j-1)(k-1)\frac{2\pi}{n}\cdot\frac{1}{2}(x_k' - x_{2-k}'), \qquad j = 2, \cdots, \frac{n+1}{2}.$$

Introducing new variables by means of the non-singular transformation,

$$y_j = \frac{1}{2}(x_j + x_{2-j}), \qquad j = 1, \cdots, \frac{n+1}{2},$$

(4)

$$y_{(n-1)/2+j} = \frac{1}{2}(x_j - x_{2-j}), \qquad j = 2, \cdots, \frac{n+1}{2},$$

of determinant T, we obtain

$$y_j = y_1' + \sum_{k=2}^{(n+1)/2} 2\cos(j-1)(k-1)\frac{2\pi}{n}\cdot y_k', \qquad j = 1, \cdots, \frac{n+1}{2},$$

(5)

$$y_{(n-1)/2+j} = \sum_{k=2}^{(n+1)/2} 2i\sin(j-1)(k-1)\frac{2\pi}{n}\cdot y_{(n-1)/2+k}', \qquad j = 2, \cdots, \frac{n+1}{2}.$$

The determinant of the transformation (5) is given by TVT^{-1}.

The elements of V^2 are defined by $n\delta_{2-c}^r$, where $\delta_{2-c}^r = 1$, if $r \equiv 2-c \pmod{n}$ and $\delta_{2-c}^r = 0$, if $r \not\equiv 2-c \pmod{n}$. The only non-zero elements of TV^2T^{-1} are in the diagonal. The first $(n+1)/2$ of these are n and the last $(n-1)/2$ are $-n$.

Let A and B represent the determinants of (5) considered as two transformations in the order in which they are written. Then

$$A^2 = n^{(n+1)/2}, \qquad B^2 = (-n)^{(n-1)/2}.$$

If in A we add the elements of the remaining columns to the elements of the first column, the first element in the new first column is n and the remaining elements are zero. Hence $A_1^2 = n^{(n-3)/2}$, where A_1 represents the minor of the element in the upper left-hand corner of A.

If we make the necessary adjustments for an even $n > 2$, it may be shown that the minor corresponding to A_1 is defined by

$$a_{rc} = 2\cos rc\frac{2\pi}{n}, \qquad r = 1, \cdots, \frac{n}{2},$$

(6)

$$a_{r(n/2)} = (-1)^r, \qquad c = 1, \cdots, \frac{n}{2} - 1.$$

If square roots are extracted, a result due to J. S. Frame [2] may be used to select the proper sign.

3. Results. We have thus established the following theorems.

THEOREM 1. *If* $n=4k+1$, *then we have the following values of determinants*:

$$\left|2\cos rc\frac{2\pi}{n}\right| = (-1)^k n^{(n-3)/4},$$

$$\left|2i\sin rc\frac{2\pi}{n}\right| = n^{(n-1)/4}, \qquad r, c = 1, \cdots, \frac{n-1}{2}.$$

If $n=4k+3$, *then*

$$\left|2\cos rc\frac{2\pi}{n}\right| = (-1)^{k+1} n^{(n-3)/4},$$

$$\left|2i\sin rc\frac{2\pi}{n}\right| = in^{(n-1)/4}, \qquad r, c = 1, \cdots, \frac{n-1}{2}.$$

THEOREM 2. *If* $n=4k$, *the determinant defined* by (6) *equals* $(-1)^k n^{(n-2)/4}$; *also*

$$\left|2i\sin rc\frac{2\pi}{n}\right| = in^{(n-2)/4}, \qquad r, c = 1, \cdots, \frac{n}{2}-1.$$

If $n=4k+2$, *the determinant defined* by (6) *equals* $(-1)^{k+1} n^{(n-2)/4}$; *also*

$$\left|2i\sin rc\frac{2\pi}{n}\right| = n^{(n-2)/4} \qquad r, c = 1, \cdots, \frac{n}{2}-1.$$

References

1. J. J. Sylvester, Mathematical Papers, vol. II, p. 621, Cambridge University Press, 1908.
2. J. S. Frame, this MONTHLY, vol. 46 (1939), p. 304.

ON A PROBLEM OF ERDÖS

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1. Introduction. In what follows, p's denote different primes; p_i denotes the i-th prime, with $p_1=2$; all small letters denote integers $\geqq 0$; m_i denotes an integer in the canonical factorization of which exactly i primes appear to the first power; and M_i denotes the set of all such integers.

Recently, P. Erdös stated and J. Lampek[1] proved that for every k, the equation:

$$\phi(x) = k!, \tag{1}$$

is solvable, ϕ being Euler's totient function.

[1] This MONTHLY, 55, 1948, 103.

The object of this note is to obtain Lampek's solution by an independent method, and to show that the equation has at least one solution in M_1. In fact, if $S_i(k)$ denotes the number of solutions of (1) in M_i, then it is shown that for every k we have

$$S_1(k) \geqq 1, \tag{2}$$

and that, further,

$$S_1(k) \longrightarrow \infty \quad \text{with} \quad k.$$

These results were conjectured by V. L. Klee,[2] but he did not prove them. Klee was, however, able to prove that

$$S_0(x) = 1. \tag{3}$$

2. Lampek's solution. To get Lampek's solution, we divide integers from 1 to k into two classes A and B. We place $r(\leqq k-1)$ in class A if it is either a prime or one less than a prime, and in class B otherwise. The integer k itself is placed in class A if and only if it is an odd prime. A number of the form $(p-1)$ thus appears in class B first and then shifts (for $k \geqq p$) to class A.

For a given k, let B_k denote the product of integers in class B. We can then write

$$B_k = \prod_{i=1}^{j} p_i^{\alpha_i}, \qquad \alpha_i \geqq 0,$$

where p_j is the greatest prime $\leqq k$.

One solution of (1) is then readily seen to be given by

$$\begin{aligned} x &= p_1^{\alpha_1+1} p_2^{\alpha_2+2} p_3^{\alpha_3+2} \cdots p_j^{\alpha_j+2} \\ &= \frac{k!(2\cdot 3\cdot 5\cdot 7\cdot \cdots \cdot p_j)}{1\cdot 2\cdot 4\cdot 6 \cdots (p_j-1)} = \frac{k!}{\prod_1^j \left(1-\frac{1}{p_i}\right)} = \frac{(k!)^2}{\phi(k!)}. \end{aligned}$$

This is Lampek's solution.

If $\alpha_1 \geqq 1$, this value of x is in M_0. If $\alpha_1 = 0$, as is the case when $k = 3$, 5 or 7, then

$$x = \frac{(k!)^2}{2\phi(k!)}$$

is another solution of (1); this is in M_0.

It is thus shown that for every k, the equation has a solution in M_0; this can further be shown to be unique.

3. Klee's Conjectures. For values of $k \leqq 8$, Klee's conjecture (2) can easily be

[2] This MONTHLY, 56, 1949, 21.

verified. We may therefore take $k \geqq 9$ in establishing (2).

LEMMA. *If for some $p > k$ we have*

$$(p-1) \mid B_k, \tag{4}$$

then (1) *has a solution in* M_1.

Let

$$\frac{B_k}{p-1} = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdot p_3^{\beta_3} \cdots p_j^{\beta_j}, \qquad \beta \geqq 0,$$

where p_j is the greatest prime $\leqq k$; then the required solution is given by

$$x = p_1^{\beta_1+1} p_2^{\beta_2+2} p_3^{\beta_3+2} p_4^{\beta_4+2} \cdots p_j^{\beta_j+2} \cdot p, \quad \text{if} \quad \beta_1 \geqq 1,$$

and by

$$x = p_2^{\beta_2+2} p_3^{\beta_3+2} p_4^{\beta_4+2} \cdots p_j^{\beta_j+2} \cdot p, \quad \text{if} \quad \beta_1 = 0.$$

As interesting examples of (4), we may give the following:

$$72 \mid B_9, \quad 1008 \mid B_{14}, \quad 15120 \mid B_{15}, \quad \text{and} \quad 17136 \mid B_{34},$$

and it follows that Klee's conjecture holds for values of $k \leqq 17136$.

To prove Klee's conjecture, we now give the following result.

THEOREM.[3] *If $k \geqq 32$, and p is any prime $> k$ and $\leqq 2k+1$, then the equation* (1) *has a solution of the form $x = pm_0$.*

Proof. First suppose $(p-1)/2$ is not a prime, and let

$$p - 1 = 2^{\gamma_1} \cdot 3^{\gamma_2} \cdot 5^{\gamma_3} \cdot \cdots \cdot p_t^{\gamma_t}, \qquad \gamma_i \geqq 0,$$

where p_t is the greatest prime $\leqq k/3$.

Evidently then

$$\gamma_1 \leqq 1 + \log_2 k.$$

Since 2^n+1 cannot be a prime except when $n = 2^s$, $s \geqq 0$, it follows that 2^3, 2^5, 2^6, 2^7, 2^9, $\cdots$, are all members of class B. For $k \geqq 32$, the greatest power of 2 which divides B_k is

$$\geqq 3 + 5 + 6 + 7 + 9 + \cdots + \log_2 h,$$

where $h = k/2$ or k according as $\log_2 k$ is or is not of the form 2^s, $s > 0$. This is greater than $1 + \log_2 k$. Hence[3]

$$2^{\gamma_1} \mid B_k \quad \text{when} \quad k \geqq 32.$$

[3] This holds for $k \geqq 14$, as can be easily verified.

Again, for $\gamma_i > 1$ and $2 \leqq i \leqq t$, $p_i^{\gamma_i}$ belongs to class B. Hence $p_i^{\gamma_i} \mid B_k$, provided $\gamma_i > 1$.

If $\gamma_i = 1$ for any value of i, then since $3p_i$ is a member of class B, p_i still divides B_k.

Hence $(p-1) \mid B_k$, and the theorem follows.

Now, suppose that $(p-1)/2$ is a prime, say p_l, and that[4] $p_l \nmid B_k$. Then $k \geqq p_l > k/3$. Evidently

$$(p-1) \mid p_l(p_l - 1)B_k.$$

Let

$$p_l(p_l - 1)B_k = 2^{c_1} \cdot 3^{c_2} \cdot 5^{c_3} \cdot \cdots \cdot p_l^{c_l} \cdots p_j^{c_j}, \qquad c_i \geqq 0,$$

where p_j is as before the greatest prime $\geqq k$.

Since p_l is not a factor of B_k, we have $c_l = 1$, and we therefore get the solution:

$$x = 2^{c_1} \cdot 3^{c_2+2} \cdot 5^{c_3+2} \cdot \cdots \cdot p_{l-1}^{c_{l-1}+2} \cdot p_{l+1}^{c_{l+1}+2} \cdot \cdots \cdot p_j^{c_j+2} \cdot p.$$

The theorem is thus completely proved.

Klee's second conjecture follows as a corollary, since the number of primes between k and $2k$ goes to infinity with k (as follows easily from the prime number theorem).

CLASSROOM NOTES

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ARTIFICIAL MATHEMATICS

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We teachers of college mathematics take pride in the fact that our subject matter is based on logic, that our derivations and proofs are obtainable by "reasonable" means and are the "natural" consequences of our hypotheses. Yet at times I wonder just how reasonable and natural some of our teaching methods are.

[4] If $p_l \mid B_k$, then $(p-1) \mid B_k$, provided $k \geqq 8$.

How often are "proofs" merely verifications of unmotivated, dogmatic statements? Any experienced teacher who has examined a number of textbooks has, I think, found instances galore. Here I shall cite just three examples of what is unfortunately only too common and which is unfair to the student, unjustified, and unnecessary.

EXAMPLE 1. Consider the linear differential equation of first order,

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{1}$$

Frequently the method of solving this equation is given in the following form: The function

$$e^{\int P dx} \tag{2}$$

is "seen" to be an integrating factor, for, when we multiply both members of (1) by function (2), we get an exact equation, as may be easily verified.

This *does* produce the result—but where on earth did the function (2) come from? Will not most students ask themselves, if not their instructor, how *they* can be expected to do mathematical thinking if it requires this sort of genius, or at least a crystal ball?

EXAMPLE 2. Consider the linear differential equation of order n with constant coefficients,

$$\phi(D)y = (a_0D^n + a_1D^{n-1} + \cdots + a_n)y = 0. \tag{3}$$

Here a common procedure is to "take" $y = e^{mx}$, so that we get

$$\phi(D)e^{mx} = \phi(m)e^{mx}, \tag{4}$$

which reduces to zero provided that m is a root of the equation $\phi(m) = 0$, and therefore we get a solution. But again the student may well ask: how did we hit upon e^{mx} as a function that turns out to do the job so effectively?

EXAMPLE 3. In order to prove the law of the mean for derivatives, the argument sometimes is made to begin, disconcertingly, with: Consider the function

$$g(x) = (a - b)f(x) + [f(b) - f(a)]x + bf(a) - af(b). \tag{5}$$

Since $g(a) = g(b) = 0$, and $g'(x)$ must vanish for some value of x between a and b, we get the desired result; but, once more, where did function (5) come from?

These examples, and others of the same ilk, provide illustrations of how artificial mathematical techniques can become. With a little thought and care, however, each proof and derivation can be well motivated, natural, and therefore more acceptable to the student. Moreover, naturalness can then produce the desired confidence in the student, so that he will be more ready to attack new problems and situations himself.

A COLOSSAL PRIMITIVE PYTHAGOREAN TRIANGLE

H. S. UHLER, Yale University

In *Scripta Mathematica*, vol. XV, 1949, pp. 247–251 may be found the author's table of non-consecutive positive integral values of 2^n, the highest value of n being 4001. A few of these data enabled him to calculate with great care the lengths of the sides of an enormous primitive right triangle. The numbers of digits in a, b, and c are respectively 1205, 1204, and 1205. The expressions used were $a=2^{4n}+2^{2n+1}$, $b=2^{4n}-2^{4n-2}-2^{2n}-1$ and $c=2^{4n}+2^{4n-2}+2^{2n}+1$, $n=1, 2, 3, \cdots$. (The questioning reader may substitute $u=2^{2n}$ and $v=2^{2n-1}+1$ in equations (8-5) on page 168 of Professor O. Ore's elegant book entitled *Number Theory and its History*.) Putting $n=1000$ we obtain the purely numerical expressions $a=4\cdot 2^{3998}+2^{2001}$, $b=3\cdot 2^{3998}-2^{2000}-1$, and $c=5\cdot 2^{3998}+2^{2000}+1$. The procedure by which the author intentionally adjusted the data so as to cause the first 600-odd figures of a, b, and c to be the same exact multiple of the corresponding elements of the historical triangle 4, 3, 5 should be read clearly from the transformed values

$$a = 2^{3998}(4 + 2^{-1997}) = 2^{3998}[4 + (0.696784785\cdots)10^{-601}],$$

$$b = 2^{3998}(3 - 2^{-1998} - 2^{-3998}) = 2^{3998}[3 - (0.348392392\cdots)10^{-601} - 2^{-3998}],$$

$$c = 2^{3998}(5 + 2^{-1998} + 2^{-3998}) = 2^{3998}[5 + (0.348392392\cdots)10^{-601} + 2^{-3998}].$$

This work is not altogether as fantastic as it may seem to be at first blush. Students just practicing with involution may square the unexpanded numerical values of a, b, c and test their results by the condition $a^2+b^2=c^2$. More proficient pupils may apply their skill to the above functions of n. Obviously many other practice primitive triangles may be found by letting $n=2, 3, 4\cdots$. Later when the student is being introduced to congruences, moduli, residues, etc., he or she may show that the immense constants given below lead to $a\equiv 37084\ (m_5)$, $b\equiv 19983\ (m_5)$, $c\equiv 81659\ (m_5)$, $a^2\equiv 9304\ (m_5)$, $b^2\equiv 16296\ (m_5)$, $a^2+b^2\equiv 25600\ (m_5)$, and $c^2\equiv 25600\ (m_5)$, where m_5 denotes mod. 10^5+1. Then other moduli such as 10^7+1 and $10^{29}+1$ may be tried. The latter is easily applied with a reliable calculating machine and it lends confidence in the present work by giving

$$\begin{aligned}
a &\equiv 515719865\ 6233667385\ 4495330061\ (m_{29})\\
b &\equiv 3686259740\ 4224446729\ 4228735908\ (m_{29})\\
c &\equiv 424331736\ 7521601946\ 0133505593\ (m_{29})\\
a^2 &\equiv 529973252\ 6166485746\ 8892932125\ (m_{29})\\
b^2 &\equiv 648590322\ 1905618795\ 2838387251\ (m_{29})\\
c^2 \equiv a^2 + b^2 &\equiv 178563574\ 8072104542\ 1731319375\ (m_{29}).
\end{aligned}$$

Finally the functioning of an electronic calculating machine may be verified by actually squaring the 1204 and 1205-digit numbers tabulated on page 332.

$a =$ 13182

0409343094	3100103889	7942365913	6318401916	1093272769	0928034502
4175692811	2834455107	9752123172	1220331409	4075648071	6823038446
8176942405	8128173106	2452512184	0385446744	4438688895	6328970642
7719939300	3658655292	4249514488	8321833894	1583237562	0009284922
6089461110	3857875407	7913265440	9185831255	8605043164	7284603636
4908238500	0782681167	2468900210	6891044880	8948534719	2152708820
1197650061	2594485839	7761874669	3012787452	3350479658	6994514054
4352170538	0373270324	0283400815	9261693483	6479947271	6094576894
0072431686	6256888660	3065832486	8306061250	1764335646	9732407252
8745672177	3369482423	6675323341	7556818392	2195469382	0456072249
8800234260	7773169142	5276429749	2719440301	2786312910	8110276523
1791250003	9324450202	0158585930	4304919378	1774876967	1953220419
0542848312	1124562345	8850343351	5183070273	3949122803	8974497727
1904776640	9658550797	9096195582	3614866289	5884470059	3540289002
5614727146	4790864241	1262007050	8297458635	8000281716	8946915944
4567759251	1693734052	7736385060	5142174692	5088960985	2079084482
3799705399	8732549654	4541159162	4129752556	5023893091	9749021654
2178578874	6307774977	0114507233	6119847609	9196512050	4342804980
8328964656	9380080473	2217857144	6992977330	4159099451	1618533227
2629192527	9641335050	2194424741	0970572936	3002894540	9209008128

$b =$ 9886

5307007320	7325077917	3456774435	2238801437	0819954576	8196025876
8131769608	4625841330	9814092379	0915248557	0556736053	7617278835
1132706804	3596129829	6839384138	0289085058	3329016671	7246727982
0789954475	2743991469	3187135866	6241375420	6187428171	5006963691
9567095832	7893406555	8434949080	6889373441	8953782373	5463452727
3681178875	0587010875	4351675158	0168283660	6711401039	4114531615
0898237545	9445864379	8321406001	9759590589	2512859744	0245885540
8264127903	5279952743	0212550611	9446270112	7359960453	7070932670
5054323764	9692666495	2299374365	1229545937	6323251735	2299305439
6559254133	0027111817	7506492506	3167613794	1646602036	5342053900
3773437509	9466771036	0657027891	4579524431	7037517303	0963296708
1779021849	4684079805	0954067868	6963014737	5182041750	7099221952
5109734076	4460136111	1298715850	3423824455	3175514032	9229151347
5222581358	9386681350	9163431074	8170763126	0316261557	6862651650
7552196087	2344008640	2449482926	5482953394	0587827246	6155261126
6392215215	6524185913	1796340083	3658779546	8583026613	7986806570
0184133286	8547379181	7702639968	6400231487	2118170969	8766448686
2137474054	9190867936	9457028201	6242333409	4258644591	9282923637
7244674562	2877414666	0712328899	2094344789	0922719730	3577627632
2577297330	3437142405	4643761448	8957480565	8811624692	9034182655

$c =$ 16477

5511678867	8875129862	2427957392	0398002395	1366590961	3660043128
0219616014	1043068884	9690153965	1525414261	7594560089	6028798058
5221178007	2660216382	8065640230	0481808430	5548361119	5411213303
4649924125	4573319115	5311893111	0402292367	6979046952	5011606153
2611826387	9822344259	7391581801	1482289069	8256303955	9105754545
6135298125	0978351459	0586125263	3613806101	1185668399	0190886025
1497062576	5743107299	7202343336	6265984315	4188099573	3743142568
0440213172	5466587905	0354251019	9077116854	5599934089	5118221117
5090539608	2821110825	3832290608	5382576562	7205419558	7165509066
0932090221	6711853029	5844154177	1946022990	2744336727	5570090140
1304249914	5898597935	6615360534	2923266900	0451560710	5066199243
4500413112	8270008044	6699309465	5621744345	0529126624	1822109506
7500119095	5575731543	3859504191	4200750891	9064606662	8717088990
2657370128	1358849448	8775015110	7794350907	2897332982	0949822192
5023099369	5239096577	0479295396	5927738945	0752921721	4850689431
5489536530	3269498790	1266912098	4309007482	1220984756	4455351527
9150244292	0114467232	6254511311	1143940937	5290017055	5059086535
1025347532	6560740743	7765822707	8641278133	5912396395	5043998167
3009976462	9230513178	2201683055	2850988737	7880911399	1441403161
1649732420	5775353483	4541796662	2150946688	5689290448	4787716097

PROBLEMS ON NATURAL NUMBERS

J. B. ROSSER, Cornell University

In a recent issue of this journal (see *On the System of Natural Numbers*, M. O. Gonzalez and J. D. Mancill, this MONTHLY, vol. 57 (1950), pp. 104–111. All references will be to this paper unless stated otherwise) there appeared a treatment of natural numbers intended to stimulate logical thinking on the part of students of college algebra. As further stimulation, I have compiled a set of home work problems for the students based on this treatment. Answers are given after the problems.

1. Prove that there is a set with a negative number of elements.
2. How would one prove the existence of a set with exactly one element?
3. Prove that if B precedes A, then A is the immediate predecessor of B.
4. Prove that $b \times 1$ is defined by the first sentence of Section 11 and does not need further definition.
5. What step was omitted in the proof of the Principle of Complete Induction?
6. State why the proof of the theorem at the top of p. 108 is incorrect, and give a correct proof.
7. Prove that Postulate 3 is not the Axiom of Zermelo.
8. Prove that the conditions of Postulate 4 do not characterize "linear order."

SOLUTIONS

1. Start with a set containing a single element. By Postulate 2 we can remove an element and get a set with no elements. By Postulate 2 again, we can remove another element and get a set with one less than no elements.

2. One wouldn't. All postulates concerning construction of sets start from given sets, but no postulate gives a set to start with. We cannot use Definition 1 since it gives us no authority to construct a set, but merely tells how to decide which sets are determined or defined. Indeed all postulates and definitions are consistent with the supposition that there are no sets at all, and so cannot possibly prove the existence of a set.

3. If B precedes A, then by Postulate 4 there does not exist an element C such that A precedes C and C precedes B. So by Definition 5, A is the immediate predecessor of B.

4. In the definition of sum at the bottom of p. 111, there is no stipulation as to the number of sets $A, B, C, \cdots, N$. By taking this number to be unity, we have a definition of sum for the case of a single summand. Hence $b \times 1$ is defined.

5. By (2) of Definition 8, either K is the first element or K has an immediate predecessor. Treatment of the case where K is the first element was omitted.

6. The Principle of Complete Induction provides for proofs of properties of elements of sets, and the theorem in question states a property of the entire set. Prove by induction on H the Lemma;

That portion of the set

$$X = A, B, C, \cdots, H, I, \cdots, L$$

consisting of H and all preceding elements can be made empty by removing its elements one at a time.

This Lemma states a property of the element H, and so can be proved by the Principle of Complete Induction, using essentially the reasoning given at the top of p. 108.

7. The Axiom of Zermelo (see *Untersuchungen über die Grundlagen der Mengenlehre* I, E. Zermelo, *Math. Annalen*, vol. 65 (1908), pp. 261–281) states that if one has given an infinite number of non-empty, non-overlapping sets, one can choose a unique element from each. The discrepancy with Postulate 3 is apparent.

8. The conditions of Postulate 4 characterize "partial order" (see p. 1 of *Lattice Theory* by G. Birkhoff, Revised Edition, Am. Math. Soc., 1948). For "linear order," an additional condition, "If A and B are distinct then either A precedes B or B precedes A," is needed (see p. 10 of *Lattice Theory*).

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 916. *Proposed by H. D. Larsen, Albion College*

As a variation of the popular game of *Twenty Questions*, suppose I think of a number which you are to determine by asking me not more than twenty questions, each of which can be answered by only "yes" or "no." What is the largest number that I should be permitted to choose that you may determine it in twenty questions?

E 917. *Proposed by Lawrence Ringenberg, Eastern Illinois State College*

Find all solutions of the equation $1/x^2+1/y^2=1/z^2$, where x, y, z are relatively prime positive integers.

E 918. *Proposed by N. A. Court, University of Oklahoma*

Two spheres are orthogonal if they cut a given line in pairs of harmonic points and if the two planes determined by that line and the centers of the two spheres are perpendicular.

E 919. *Proposed by Leo Moser, University of Manitoba*

Show that the greatest common divisor of a and b is given by

$$(a, b) = \sum_{m=0}^{a-1} \sum_{n=0}^{a-1} (1/a)e^{2\pi i b mn/a}.$$

E 920. *Proposed by T. G. Room, University of Sydney, Australia*

If S and T are any two square matrices of the same order, and the necessary matrices are non-singular, then

$$(I + S)^{-1}(S + T)(I + ST)^{-1}(I + S) = (I - S)(I + TS)^{-1}(S + T)(I - S)^{-1}.$$

SOLUTIONS

Minimal n Partitionable into Two Squares in Given Number of Ways

E 883 [1949, 552]. *Proposed by H. T. R. Aude, Colgate University.*

Find the smallest integer n which is the sum of the squares of two non-negative integers in exactly (a) three ways, (b) five ways, (c) six ways.

Solution by C. M. Sandwick, Easton High School, Easton, Pa. Let $P_2(n)$ denote the number of partitions of n into two non-negative integral squares. Then our problem is to find the minimum values of n that satisfy the equations $P_2(n) = 3$, $P_2(n) = 5$, $P_2(n) = 6$. If $P_2(n) > 1$, then $n = 2^r MQ^2$, where $r \geqq 0$, M is the product of two or more prime factors each of the form $4x+1$, and Q contains no prime factors of this form. The value of $P_2(n)$ is equal to the number of distinct ways in which M can be represented as the product of two positive integers.

If A, B, C are unequal primes of the form $4x+1$, it is easy to show that $P_2(n) = 3$ only when M has one of the forms A^2B, A^4, A^5; $P_2(n) = 5$ only when M has one of the forms A^2B^2, A^4B, A^8, A^9; and $P_2(n) = 6$ only when M has one of the forms A^2BC, A^3B^2, A^5B, A^{10}, A^{11}. If $C < AB$ and $B < A^2$, the first named form of M in each category yields the smallest value. These conditions are met when A, B, C take the values 5, 13, 17, respectively, the smallest primes of the form $4x+1$. Therefore the smallest values of n for which $P_2(n) = 3$, 5, 6, respectively, are (a) $(5^2)(13) = 325$, (b) $(5^2)(13^2) = 4225$, (c) $(5^2)(13)(17) = 5525$. In fact we have

$$325 = 1^2 + 18^2 = 6^2 + 17^2 = 10^2 + 15^2,$$

$$4225 = 0^2 + 65^2 = 16^2 + 63^2 = 25^2 + 60^2 = 33^2 + 56^2 = 39^2 + 52^2,$$

$$5525 = 7^2 + 74^2 = 14^2 + 73^2 = 22^2 + 71^2 = 25^2 + 70^2 = 41^2 + 62^2 = 50^2 + 55^2.$$

Also solved by B. H. Brown, W. E. Buker, Roger Lessard, C. S. Ogilvy, and the proposer.

Editorial Note. It may similarly be shown that $n=(5)(13)(16)=1105$ is the minimum n such that $P_2(n)=4$. If we decide to omit zero squares, then $n=(5^4)(13)=8125$ is the minimum n such that $P_2(n)=5$. The number of partitions of an integer into two squares is given in detail in Uspensky and Heaslet, *Elementary Number Theory*, pp. 336–42.

Concurrent Cevians of a Tetrahedron

E 885 [1949, 552]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In the tetrahedron $ABCD$ let A_1, B_1, C_1, D_1 divide a set of concurrent cevians AA', BB', CC', DD' in the same ratio, and let A_1B_1, A_1C_1, A_1D_1 pierce BCD in A_2, A_3, A_4. Show that triangles A_2CD, A_3DB, A_4BC have equal areas.

I. *Modification by the Proposer.* It has been found that the problem as originally stated is incorrect and we shall here prove the following modification:

In the tetrahedron $ABCD$ let A_1, B_1, C_1, D_1 be points dividing the cevians AA', BB', CC', DD', through a point Q in the same ratio $AA_1/A_1A'=\cdots=k$, and let the lines A_1B_1, A_1C_1, A_1D_1 pierce BCD in A_2, A_3, A_4. Then triangles A_2CD, A_3DB, A_4BC have equal areas when (1) *Q is arbitrary and $k=1$,* (2) *k is arbitrary and Q coincides with the centroid of $ABCD$.*

If we designate the barycentric coordinates of Q by (a, b, c, d), those of A_1, B_1, C_1 are of the form

$$A_1\colon (a', b, c, d), \qquad B_1\colon (a, b', c, d), \qquad C_1\colon (a, b, c', d),$$

where

$$a' = (b + c + d)/k, \qquad b' = (c + d + a)/k, \qquad c' = (d + a + b)/k.$$

On A_1B_1 one has the coordinates

$$a' + \lambda a, \qquad b + \lambda b', \qquad c(1 + \lambda), \qquad d(1 + \lambda),$$

whence those of A_2 are

$$0, \qquad ab - a'b', \qquad c(a - a'), \qquad d(a - a'),$$

and we have

$$A_2CD = BCD(ab - a'b')/[ab - a'b' + (a - a')(c + d)].$$

Similarly

$$A_3DB = BCD(ac - a'c')/[ac - a'c' + (a - a')(b + d)].$$

Consequently, the condition $A_2CD=A_3DB$ is equivalent to

$$(ab - a'b')(b + d) = (ac - a'c')(c + d),$$

or

$$a(b - c)(b + c + d) = a'[b'(b + d) - c'(c + d)],$$

or, substituting the values of a', b', c',

$$a(b - c)(b + c + d) = (b + c + d)(b - c)a/k^2,$$

or, finally,

$$a(b - c)(b + c + d)(1 - k^2) = 0.$$

This equality leads directly to the two announced cases, and the modified problem is established.

II. *Modification by Joseph Langr, Prague, Czechoslovakia.* Although the original problem is generally not true the following slight modification leads to a correct version.

In the tetrahedron $ABCD$ let A_1, B_1, C_1, D_1 be points on the cevians AA', BB', CC', DD' through a point Q such that

$$A'A/AA_1 = BB'/B'B_1 = CC'/C'C_1 = DD'/D'D_1 = u,$$

and let A_1B_1, A_1C_1, A_1D_1 pierce BCD in A_2, A_3, A_4. Then triangles A_2CD, A_3DB, A_4BC have equal areas.

Let the plane of the lines AA', BB' cut the edge CD in B''. Through A_1 draw the parallel to BB'' to cut AB in P and through B_1 the parallel to AB'' to cut BB'' in N and AB, of course, in P. Then

$$PA_1 = BA'/u, \qquad NB'' = BB''/u.$$

From the similar triangles A_1B_1P and A_2B_1N we have

$$A_2N = PA_1(B_1N/PB_1) = (BA'/u)(B'B''/AB'),$$

and consequently

$$A_2B'' = A_2N + NB'' = (BA'/u)(B'B''/AB') + BB''/u,$$

and

$$A_2B''/BB'' = (1/u)[1 + (B'B''/AB')(BA'/BB'')].$$

But, by the theorem of Menelaus applied to triangle $AA'B''$ and the transversal BB',

$$(B'B''/AB')(BA'/BB'') = AQ/QA',$$

whence

$$A_2B''/BB'' = (1/u)[1 + AQ/QA'] = v,$$

say. Therefore

$$A_2CD/BCD = A_2B''/BB'' = v.$$

Similarly

$$A_3DB/BCD = A_4BC/BCD = v,$$

and the modified problem is established.

Editorial Note. By considering the last equation in the solution to the proposer's modified problem, along with the two similar equations gotten by cyclically permuting b, c, d, we see that triangles A_2CD, A_3DB, A_4BC have equal finite areas in the following three cases: (1) Q arbitrary and $k=1$, (2) k arbitrary and Q on the line determined by A and the centroid of BCD, (3) k arbitrary and Q on the plane of BCD

If in Langr's modification, we take $u=2$, we obtain the first case of the proposer's modification, and this is the space analogue of the first part of E 879 [1949, 474]. Space analogues of the other two parts of E 879 may also be established.

A Circle and Variable Chord

E 886 [1949, 632]. *Proposed by P. L. Chessin and R. Gilman, New York City*

Let M be a fixed point on the radius OR of a circle of center O and set $OR/OM=k$. If AB is a variable chord such that $\sphericalangle AMR=\sphericalangle OMB$, then (1) AB passes through a fixed point on line OR, (2) $AB=k(MB-MA)$, (3) A, B, O, M are concyclic.

Solution by Roscoe Woods, State University of Iowa. Let MR and the perpendicular to OR at M cut AB in N and C respectively. Then MC bisects angle AMB, and $(NCAB)$ is harmonic. Thus N is the pole of line MC, and is therefore fixed. This establishes (1).

It is now obvious that the inverse of AB with respect to the given circle is a circle passing through A, B, M, and O. This demonstrates (3).

If the theorem of Ptolemy be employed with reference to the inscribed quadrilateral $ABOM$, we have $(OM)(AB)+r(MA)=r(MB)$, where r is the radius of the given circle. This relation reduces to (2) when use is made of the given condition $OR/OM=k$.

Also solved by Ragnar Dybvik, P. W. M. John, L. M. Kelly, Roger Lessard, C. S. Ogilvy, Nathan Schwid, A. Sisk, M. C. Stapp, and the proposers.

An Impossible Digit Relation

E 887 [1949, 632]. *Proposed by C. W. Trigg, Los Angeles City College*

Let N be an n-digit integer and M the kn-digit integer formed by writing N down k times. Show that there is no scale of notation in which $N^k=M$ if $k>1$.

Solution by N. G. Gunderson, University of Rochester. Let N be written to the base r, so that $0<N<r^n$, and suppose that $N^k=M=N\Sigma_{i=0}^{k-1} r^{ni}$ for some $k>1$. But this would imply $\Sigma_{i=0}^{k-1} r^{ni}=N^{k-1}<r^{n(k-1)}$, which is impossible.

Also solved by N. D. Lane, Roger Lessard, C. S. Ogilvy, C. M. Sandwick, and the proposer.

Passing a Cube through a Cube of Same Size

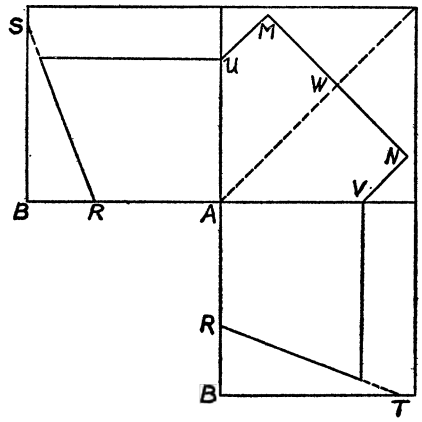
E 888 [1949, 632]. *Proposed by H. D. Grossman, New York City*

Show how to cut a hole in a cube through which another cube of equal size can pass.

I. *Solution by C. S. Ogilvy, Columbia University.* The projection of a unit cube on a plane perpendicular to a main diagonal is a regular hexagon whose inscribed circle has diameter $\sqrt{2}$. Hence any unit square hole (prism) whose axis is this main diagonal has its cross section circumscribed by the circle. Therefore such a hole is completely surrounded by material of the cube, provided only that it is not oriented with a diagonal of its cross section coinciding with a line joining the midpoints of two opposite edges of the cube.

II. *Solution by F. Bagemihl, University of Rochester.* Let the vertices of the given cube have rectangular coordinates (0, 0, 0), (1, 0, 0), (1, 1, 0), (0, 1, 0), (0, 0, 1), (1, 0, 1), (1, 1, 1), (0, 1, 1). The points A (0, .9, 1), B (.9, 0, 1), C (1, .1, 0), D (.1, 1, 0) are on the cube, and are the vertices of a rectangle with $AB=CD=\sqrt{1.62}>1$, $AD=BC=\sqrt{1.02}>1$, so that a square, s, can be drawn wholly interior to the rectangle $ABCD$, each side of the square having a length greater than unity. Through the given cube cut a hole with a square cross section, perpendicular to $ABCD$, and intersecting the latter in s. A unit cube can pass through this hole if four faces of the cube are parallel, respectively, to the four faces of the hole.

Also solved by R. V. Andree, Sholom Arzt, Sidney Kravitz, Mary Ann Lee, Roger Lessard, W. R. Ransom, L. A. Ringenberg, C. M. Sandwick, O. D. Smith, G. W. Walker, and D. G. Williams.



Williams furnished diagrams, based upon the symmetrical orientation in solution I, showing how to mark the intersections of an appropriate hole with the faces of the given cube. The essential information of his diagrams is contained in the accompanying figure, representing three faces of a trihedral angle of the

cube. By symmetry considerations the other three faces may be similarly marked. If e is the edge of the cube, then

$$AW = e\sqrt{3}/2, \qquad MW = WN = e/2,$$
$$AR = e\sqrt{6}/4, \qquad BS = BT = e(4 - \sqrt{6})/2.$$

Comparison of Some Solid Angles

E 889 [1949, 632]. *Proposed by Michael Golomb, Purdue University*

Let k be a circle with its interior, F_1F_2 one of its diameters, P a point on the axis of the circle, F_1PF_2 a circular arc, and A_1PA_2 a semiellipse with foci at F_1, F_2. Show that the solid angles subtended by k at the points of arc F_1PF_2 are no smaller, and at the points of arc A_1PA_2 are no greater, than the solid angle subtended by k at P.

Solution by the Proposer. If Q is any point in the plane F_1PF_2 and w is the bisector of $\sphericalangle F_1QF_2$, then a plane perpendicular to w cuts the cone $Q(k)$ in an ellipse whose center is on w, whose minor axis is in the plane F_1PF_2, and whose major axis is parallel to the plane of k.

Now let Q_1 be a point on arc F_1PF_2, w_1 the bisector of $\sphericalangle F_1Q_1F_2$, and O_1 the point on w_1 such that $Q_1O_1 = PO$. Then, since $\sphericalangle F_1PF_2 = \sphericalangle F_1Q_1F_2$, the elliptic cross section of the cone $Q_1(k)$ with center at O_1 and perpendicular to w_1 has a minor axis equal to the diameter of k. Hence the solid angle subtended by k at Q_1 is no smaller than that at P.

Next, let Q_2 be a point on arc A_1PA_2, w_2 the bisector of $\sphericalangle F_1Q_2F_2$, and O_2 the point of intersection of w_2 with F_1F_2. The major axis of the elliptic cross section of the cone $Q_2(k)$ with center O_2 and perpendicular to w_2 lies in the plane of k, and therefore is the chord $M_1O_2M_2$ of k which is perpendicular to F_1F_2. Since w_2 is the bisector of $\sphericalangle F_1Q_2F_2$, w_2 is normal to the semiellipse A_1PA_2. It is a general property of an ellipse that the ratio of the length of the normal Q_2O_2 to that of the chord M_1M_2 in the circle through the foci is independent of the position of Q_2 on the ellipse. Therefore $Q_2O_2/M_1M_2 = PO/F_1F_2$. Hence an elliptic cross section of the cone $Q_2(k)$ perpendicular to w_2 and at a distance from Q_2 equal to PO has a major axis equal to $(M_1M_2)(PO)/(Q_2O_2) = F_1F_2$, *i.e.*, equal to the diameter of the circle k. Consequently, the solid angle subtended by k at Q_2 is no greater than that at P.

Editorial Note. Let Σ be the locus, on the same side of k as P, of a point Q such that the solid angle subtended by k at Q is equal to that subtended by k at P. The above solution establishes, in an elementary fashion, that the surface of revolution Σ lies between the spherical cap obtained by rotating arc F_1PF_2 about PO, and the oblate hemispheroid obtained by rotating arc A_1PA_2 about PO.

A Composition of a Set of Integers

E 890 [1949, 632]. *Proposed by Leo Moser, University of Manitoba*

Let $a_1, \cdots, a_n$ be n integers with $a_1 = 1$ and $a_i \leqq a_{i+1} \leqq 2a_i$, $i = 1, \cdots, n-1$. Show that there exists a sequence $\{\epsilon_i\}$ of plus and minus ones such that $\Sigma_{i=1}^{n} \epsilon_i a_i = 0$ or 1.

Solution by S. H. Gould, Purdue University. If $f(k)$ denotes the smallest non-negative integer representable in the form $\Sigma_{i=n-k}^{n} \epsilon_i a_i$, then $f(0) = a_n$, and if we assume $f(k) \leqq a_{n-k}$, then

$$f(k+1) \leqq |f(k) - a_{n-k-1}| \leqq a_{n-k-1},$$

so that $f(n-1) \leqq a_1$

Also solved by A. L. Epstein, Paul Erdös, N. J. Fine, A. W. Fuller, N. G. Gunderson, Roger Lessard, D. J. Newman, A. Rosenthal, and the proposer.

Newman showed that, under the given hypothesis, one can find a sequence $\{\epsilon_i\}$ of plus and minus ones such that $\Sigma_{i=1}^{n} \epsilon_i a_i = k$ or $k+1$, where k is any integer such that $|k| < 2a_n$.

Erdös established the following more general theorem. "Let $0 < a_1 < \cdots < a_n$, be a sequence of real numbers such that $a_{i+1} \leqq 2a_i$, $i = 1, \cdots, n-1$. Then every closed interval (u, v) of length $2a_1$ satisfying

$$-2a_1 - \sum_{i=1}^{n} a_i \leqq u < v \leqq 2a_1 + \sum_{i=1}^{n} a_i$$

contains a number of the form $\Sigma_{k=1}^{n} \epsilon_k a_k$, $\epsilon_k = \pm 1$."

The proposer said that the problem was suggested by the following statement in Dickson, History of the Theory of Numbers, vol. 1, p. 436. "H. F. Scherk stated (1833) the empirical theorem: Every prime of odd rank can be composed by the addition and subtraction of all smaller primes, each taken once, thus $13 = 1+2-3-5+7+11$." This now follows easily from Bertrand's postulate (the existence of a prime between n and $2n$) and the result of the problem.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4395. *Proposed by David Gale, Princeton University*

Let S be a set of vectors (points) in n-space with the property that every vector in n-space can be expressed as a linear combination of vectors from S with non-negative coefficients. Show that one can find a subset R of S containing not more than $2n$ elements such that R has this same property.

4396. *Proposed by D. H. Browne, Buffalo, New York*

With the notation

$$P(s, t) = \prod_{m=0}^{s} (1 + mt),$$

let

$$S(n, t) = P(n, t) \sum_{k=0}^{n} k!t^k/P(k, t).$$

Show that for $n \geqq 1$

$$S(n, t) \equiv P(n - 1, t) \pmod{n + 1},$$

and that for $n > 2$, $t \leqq n$,

$$S(n, t) \equiv 0 \pmod{t + 1}.$$

4397. *Proposed by Albert Wilansky, Lehigh University*

Suppose $f''(0)$ exists, is finite, and is different from 0. Assume also $f(0) = 0$. For sufficiently small $h > 0$ we may write

$$\frac{f(-h)}{h} = f'(\xi), \qquad \frac{f(h)}{h} = f'(\zeta),$$

with $-h < \xi < 0 < \zeta < h$. Prove that

$$\lim_{h \to 0} \frac{\zeta - \xi}{h} = 1.$$

4398. *Proposed by Lucien Droussent, Clermont Ferrand, France*

The Lemoine point of a rectangular hyperbola for an inscribed triangle T lies on the orthic axis of T, and the polar of this point with respect to the hyperbola is tangent to the conic which touches the sides of T at the feet of its altitudes and has the symmedian point of T for center. (The tangents to a conic at the vertices of an inscribed triangle T form a triangle which is perspective to T. The center of perspectivity is called the *Lemoine point* of the conic for the triangle T.)

4399. *Proposed by Ky Fan, University of Notre Dame*

Let $\{f(n)\}$ and $\{g(n)\}$ be two sequences of natural numbers defined by the following three conditions:

(1) $f(1)=1$.

(2) $g(n)=na-1-f(n)$, a being an integer >4.

(3) $f(n+1)$ is the least natural number distinct from the $2n$ numbers $f(1)$, $f(2), \cdots, f(n); g(1), g(2), \cdots, g(n)$.

Prove that there exist constants α and β such that

$$f(n) = [\alpha n], \qquad g(n) = [\beta n].$$

The brackets denote, as usual, the greatest integer function.

SOLUTIONS

Morris' Lottery

4312 [1948, 505]. *Proposed by R. C. Lyness, Staffordshire, England*

In order to help our school savings campaign, Morris organized a lottery. Certain members of the savings group were each issued a book containing a gross of tickets. The tickets were sold for a penny each and the prize for the winning ticket was a number of six-penny stamps. When I asked Morris how many stamps the winner got, he said I could work it out for myself. All he need tell me was that before the winning ticket was drawn all the ticket sellers had assembled around a circular table and each had handed in his takings and his unsold tickets; that (a) no two had sold the same number: that (b) the number of unsold tickets returned by each seller was in every case equal to the product of the values in shillings of the tickets sold by his two neighbors at the table; and that (c) when the ticket sellers were rewarded with a penny each for their trouble the sum left over exactly bought the winner's stamps.

How many sellers were there and how many tickets did each sell?

Solution by N. J. Fine, University of Pennsylvania. Let $A_1, A_2, \cdots$, be the amounts, in pence, sold by the n sellers. Extend the sequence indefinitely, so that $A_{n+k}=A_k$. By condition (a), $A_j=A_k$ implies $j\equiv k \pmod{n}$. It is easily seen that $n\neq 1, 2$, and that

$$0 < A_k < 144. \tag{1}$$

Condition (b) may be written

$$144 - A_k = A_{k-1}A_{k+1}/144. \tag{2}$$

Given A_1, A_2, we compute A_3, A_4, A_5, A_6, from (2), and find that $A_6 = A_1$. Hence $n = 5$.

Now no A_k is prime to 144, for then A_{k+2} is divisible by 144, which contradicts (1). Furthermore, since $\sum_{k=1}^{5} A_k \equiv 5 \pmod 6$ by (c), at least one of the A_k, say A_1, is odd and therefore divisible by 3. Applying (2) for $k = 2, 5$, it is clear that both A_3 and A_4 are divisible by 16. Similarly, if A_2 were odd 16 would divide A_5, hence also

$$144 - A_5 = A_1A_4/144; \tag{3}$$

hence also $A_4/16$, since A_1 is odd, so that $A_4 \geqq 256$ contradicting (1). Therefore A_2 and A_5 (by similar argument) are both even. Using (3) again, $A_4/16$ is even, and so is $A_3/16$. Thus we have

$$A_1 = 3B_1, \quad A_2 = 2B_2, \quad A_3 = 32B_3, \quad A_4 = 32B_4, \quad A_5 = 2B_5,$$

with the additional conditions that B_1 is odd and that

$$1 \leqq B_3 \leqq 4, \quad 1 \leqq B_4 \leqq 4, \quad B_3 \neq B_4. \tag{4}$$

From (4), neither A_3 nor A_4 is divisible by 9; so from (2), with $k = 3, 4$, we find that both A_2 and A_5 are divisible by 3, hence by 6. Therefore $\sum_{k=1}^{5} A_k \equiv 3B_1 + 2B_3 + 2B_4 \equiv 5 \pmod 6$,

$$B_3 + B_4 \equiv 1 \pmod 3. \tag{5}$$

Also

$$\begin{aligned} 144 - A_3 &= A_4(A_2/144) < A_4, \\ 32(B_3 + B_4) &= A_3 + A_4 > 144, \\ B_3 + B_4 &> 9/2, \\ B_3 + B_4 &\geqq 5. \end{aligned} \tag{6}$$

The only combination of values satisfying (4), (5), (6) is 3 and 4. We may therefore take $A_3 = 96$, $A_4 = 128$, and from these by (2) obtain the complete solution

$$A_1 = 135, \quad A_2 = 54, \quad A_3 = 96, \quad A_4 = 128, \quad A_5 = 24.$$

(Interchanging the values of A_3 and A_4 gives the same set in reverse order.)

Also solved by the Proposer, whose major interest is in the illustration of the 5-cycle. He remarks that the recurrences, $u_{n+1}u_n = 1$, $u_{n+2}u_n = u_{n+1} + 1$, $u_{n+3}u_n = u_{n+2} + u_{n+1} + 1$ give respectively a 2-cycle, 5-cycle, and 8-cycle. The recurrence of the present problem is obtained by putting $u_k = -A_k/144$.

Subset of a Group

4317 [1948, 586]. *Proposed by Leo Moser, University of Manitoba*

Let G be an Abelian group and A a subset of order n, such that $a \in A$ implies $a^{-1} \in G - A$. Consider the n^2 (not necessarily distinct) elements of G of the form $a_i a_j$, a_i and a_j elements of A. Show that of these n^2 elements at most $\binom{n}{2}$ are elements of A.

Solution by Robert Steinberg, University of California, Los Angeles. For each $a_i a_j = a_k$ in A there is an $a_k a_j^{-1} = a_i$ in A, whence the number of elements $a_i a_j$ in A is the same as the number of elements $a_i a_j^{-1}$ in A. Consider the n^2 elements $a_i a_j^{-1}$. From the conditions on A, we see that $a_i a_i^{-1} = I$ is not in A and that $a_j a_i^{-1} = (a_i a_j^{-1})^{-1}$ is not in A if $a_i a_j^{-1}$ is in A. Thus at most $\frac{1}{2}(n^2 - n) = \binom{n}{2}$ elements $a_i a_j^{-1}$ are in A, and hence at most $\binom{n}{2}$ elements $a_i a_j$ are in A.

Also solved by Olga Taussky and John Todd, and by the Proposer.

Editorial Note. That $\binom{n}{2}$ is the best possible value is easily seen on consideration of a cyclic group G. The above proof shows also that the hypothesis that G be Abelian is not necessary.

Infinite Series

4318 [1948, 586]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Show that

$$\frac{1}{(\sinh \pi)^2} + \frac{1}{(2 \sinh 2\pi)^2} + \frac{1}{(3 \sinh 3\pi)^2} + \cdots = \frac{2}{3}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \cdots\right) - \frac{11\pi^2}{180}.$$

Solution by the Proposer. Take

$$\frac{\pi}{x} \coth \pi x = \frac{1}{x^2} + \frac{2}{x^2 + 1^2} + \frac{2}{x^2 + 2^2} + \cdots,$$

differentiate with respect to x and divide across by $-x$ to obtain

$$\frac{\pi}{x^3} \coth \pi x + \frac{\pi^2}{x^2 \sinh^2 \pi x} = \frac{2}{x^4} + \frac{4}{(x^2 + 1^2)^2} + \frac{4}{(x^2 + 2^2)^2} + \cdots,$$

and thence

$$\pi \sum_1^\infty \frac{\coth \pi n}{n^3} + \pi^2 \sum_1^\infty \frac{1}{(n \sinh \pi n)^2} = 2\zeta(4) + 4 \sum_1^\infty \sum_1^\infty \frac{1}{(m^2 + n^2)^2}.$$

Now

$$\sum_1^\infty \sum_1^\infty \frac{1}{(m^2 + n^2)^s} = \zeta(s)\left(\frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \cdots\right) - \zeta(2s),$$

and it remains to evaluate

$$\pi\sum_1^\infty \frac{\coth \pi n}{n^3} = \sum_1^\infty \frac{1}{n^4} + 2\sum_1^\infty \sum_1^\infty \frac{1}{n^2(n^2+m^2)}$$

$$= \sum \frac{1}{n^4} + 2\sum\sum \frac{1}{m^2}\left(\frac{1}{n^2} - \frac{1}{m^2+n^2}\right)$$

$$= \sum \frac{1}{n^4} + 2\sum \frac{1}{m^2}\sum \frac{1}{n^2} - 2\sum\sum \frac{1}{m^2(m^2+n^2)}$$

$$= \sum \frac{1}{n^4} + 2\sum \frac{1}{m^2}\sum \frac{1}{n^2} + \sum \frac{1}{m^4} - \pi\sum \frac{\coth \pi m}{m^3},$$

and hence

$$\pi\sum_1^\infty \frac{\coth \pi n}{n^3} = \frac{\pi^4}{90} + \frac{\pi^4}{36} = \frac{7\pi^4}{180}.$$

The desired result follows immediately upon substitution.

Exceptions to the Converse of Fermat's Theorem

4319 [1948, 586]. *Proposed by Paul Erdös, Syracuse University*

If p is a prime greater than 3 and if $n=(2^{2p}-1)/3$, show that 2^n-2 is divisible by n.

Solution by J. E. Wilkins, Jr., American Optical Company, Buffalo, New York. In

$$n - 1 = 4(2^{p-1}+1)(2^{p-1}-1)/3$$

the factor 4 is divisible by 2, the factor $2^{p-1}-1$ is divisible by p (Fermat's simple theorem, p being prime) and by 3 (since p is odd), hence by $3p$ when p is greater than 3. Therefore $n-1$ is divisible by $2p$.

By hypothesis $2^{2p}-1$ is divisible by n. Since $2p$ is a divisor of $n-1$ it follows that $2^{n-1}-1$ is divisible by n and we have the desired conclusion.

Also solved by Daniel Block, R. H. Breusch, A. W. Boldyreff, John Kelly, Leo Moser, Alex Rosenberg, N. C. Scholomiti, J. D. Swift, and the Proposer.

Editorial Note. For all $p>3$, the Proposer notes that n is composite and thus we have infinitely many exceptions to the converse of Fermat's theorem. Sierpinski proved (*Colloquium Math.* 1, 9) that if n is composite and divides 2^n-2, then 2^n-1 has the same property. Note also that the original proposal is true for $p=2$, although then n is not composite.

See also *On the Converse of Fermat's Theorem* by the Proposer, This MONTHLY, vol. 56 (1949) pp. 623–624.

Series with Irrational Sum

4321 [1948, 642]. *Proposed by Paul Erdös, Syracuse University*

Let $n_1<n_2< \cdots <n_k< \cdots$ be a sequence of integers such that

$$\lim_{k\to\infty} \frac{n_k}{n_1n_2 \cdots n_{k-1}} = \infty.$$

Show that $\sum_{i=1}^{\infty}1/n_i$ is irrational.

Solution by Robert Steinberg, University of California, Los Angeles. Suppose, if possible, that $\sum_{i=1}^{\infty}1/n_i=p/q$, in which p and q are integers. From the hypothesis we can choose k such that

$$\frac{n_k}{n_1n_2 \cdots n_{k-1}q} > 3$$

and $n_{i+1}>3n_i$ for $i\geqq k$. Then

$$pn_1n_2 \cdots n_{k-1} = \sum_{i=1}^{\infty} \frac{n_1n_2 \cdots n_{k-1}q}{n_i} = \sum_{i=1}^{k-1} \frac{n_1 \cdots n_{k-1}q}{n_i} + \sum_{i=k}^{\infty} \frac{n_1 \cdots n_{k-1}q}{n_i}.$$

Now, $pn_1 \cdots n_{k-1}$ and the finite sum on the right are integers, but

$$\sum_{i=k}^{\infty} \frac{n_1n_2 \cdots n_{k-1}q}{n_i}$$

is, because of our choice of k, term for term less than the series $1/3+1/9+ \cdots +1/3^i+ \cdots$ whose sum is $\frac{1}{2}$. By this contradiction the proposed result is established.

Also solved by Joshua Barlaz, N. J. Fine, W. Fulks, Stanley Katz and A. M. Peiser, J. B. Kelly, R. S. Lehman, Norman Miller, Leo Moser, Alex Rosenberg, and Ernest Trost.

Binomial Coefficients and Greatest Integer Function

4322 [1948, 642]. *Proposed by Pedro A. Piza, San Juan, Puerto Rico.*

If p is a prime, prove that

$$\binom{n}{p} \equiv [n/p] \pmod{p},$$

where $\binom{n}{p}$ and $[n/p]$ have their usual meanings.

Prove also that when $[n/p]$ contains p^s as a factor, then $\binom{n}{p}$ is divisible by p^s.

Solution by Joshua Barlaz, Rutgers University. Consider the n consecutive numbers, $n, n-1, \cdots, n-p+1$. These are congruent in some order to the numbers $1, 2, \cdots, p \pmod{p}$. Moreover, one of them, say N, is divisible by p, whence $[n/p]=N/p$. The remaining set, when we remove N, will be congruent to $1, 2, \cdots, p-1 \pmod{p}$ in some order.

Let Π equal the product of this set, that is

$$\Pi = \frac{n(n-1)\cdots(n-p+1)}{N}.$$

Then

$$\Pi \equiv (p-1)! \pmod{p}.$$

Multiplying by N/p, we have

$$\frac{N\Pi}{p} \equiv \frac{(p-1)!N}{p} \pmod{p}, \tag{1}$$

and dividing by $(p-1)!$ which is relatively prime to p, we have

$$\frac{N\Pi}{p!} \equiv \frac{N}{p} \pmod{p}, \tag{2}$$

where $N\Pi/p!$ is the integer $\binom{n}{p}$. This is the Proposer's first result.

If $N/p=[n/p]$ is divisible by p^s, then the congruences (1) and (2) become true modulo p^{s+1}. That $N\Pi/p=\binom{n}{p}$ is also divisible by p^s is a corollary.

Also solved by R. Bott, D. H. Browne, I. C. Diehm, N. J. Fine, Seymour Haber, O. H. Hoke, R. S. Lehman, L. C. Mathewson, Leo Moser, Alex Rosenberg, Azriel Rosenfeld, N. C. Scholomiti, Robert Steinberg, and Ernest Trost.

Editorial Note. Fine, in his solution, notes that the present results are corollaries of the following theorem of Lucas: *If p is a prime, and*

$$n = n_0 + n_1 p + \cdots + n_k p^k \qquad (0 \leqq n_r < p),$$

$$m = m_0 + m_1 p + \cdots + m_k p^k \qquad (0 \leqq m_r < p),$$

then

$$\binom{n}{m} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\cdots\binom{n_k}{m_k} \pmod{p}.$$

For proof see Fine, *Binomial Coefficients Modulo a Prime*, this MONTHLY, Dec. 1947.

A Double Limit

4324 [1948, 642]. *Proposed by W. O. Pennell, Exeter, New Hampshire.*

When $x \geqq 0$ what is the limit as $n \to \infty$ of

$$\phi(n, x) = e^{-x^n}\left[1 + \frac{x^n}{\frac{1}{n}+1} + \frac{x^{2n}}{\left(\frac{1}{n}+1\right)\left(\frac{1}{n}+2\right)} + \frac{x^{3n}}{\left(\frac{1}{n}+1\right)\left(\frac{1}{n}+2\right)\left(\frac{1}{n}+3\right)} + \cdots\right]?$$

Solution by Stanley Katz and A. M. Peiser, New York City. It is clear that we may take $x > 0$. Let

$$g_n(x) = xe^{x^n}\phi(n, x) = x + \sum_{k=1}^{\infty} \frac{n^k x^{kn+1}}{(n+1)\cdots(kn+1)}.$$

Then

$$\begin{aligned} g_n'(x) &= 1 + nx^n + \sum_{k=2}^{\infty} \frac{n^k x^{kn}}{(n+1)\cdots((k-1)n+1)} \\ &= 1 + nx^{n-1}g_n(x). \end{aligned}$$

Since $g_n(0) = 0$, the solution of this differential equation is

$$g_n(x)e^{-x^n} = \int_0^x e^{-u^n}du,$$

$$\phi(n, x) = \frac{1}{x}\int_0^x e^{-u^n}du.$$

For $0 < x \leqq 1$,

$$\lim_{n\to\infty} \phi(n, x) = \frac{1}{x}\int_0^x \lim_{n\to\infty} e^{-u^n}du = 1.$$

For $x \geqq 1$,

$$\lim_{n\to\infty} \phi(n, x) = \frac{1}{x}\int_0^1 \lim_{n\to\infty} e^{-u^n}du + \frac{1}{x}\int_1^x \lim_{n\to\infty} e^{-u^n}du = \frac{1}{x}.$$

Also solved by Robert Steinberg, N. J. Fine, and the Proposer.

Powers of 2

4327 [1949, 39]. *Proposed by R. J. Walker, Cornell University*

In attempting to solve the preceding problem* we tried to show that for some positive integer s no power of 2 had its last s digits all powers of 2. Show that this attempt had to fail; in particular, show that for each s there exist powers of 2 each of whose last s digits is either 1 or 2.

Solution by E. P. Starke, Rutgers University. The proof proceeds by induction. Noting that $2^5=32$ and $2^9=512$, let us suppose that 2^n is a number whose last s digits are exclusively 1's and 2's, so that

$$2^n \equiv z \pmod{10^s},$$

where the s-digit number z is composed entirely of 1's and 2's. From the properties of the Euler ϕ-function we have

$$2^{\alpha(s)} \equiv 1 \pmod{5^s},$$

where $\alpha(s)=\phi(5^s)$, and hence

$$2^{\alpha(s)}A \equiv A \pmod{2\cdot 10^s} \tag{1}$$

provided A is divisible by 2^{s+1}. Thus A and $2^{\alpha(s)}A$ have the same s final digits, and their critical digits are of like parity. (We shall refer to the digit preceding the s final digits of any number as its critical digit.)

Now consider the numbers

$$2^{n+k\alpha(s)}, \qquad k = 0, 1, 2, 3, 4. \tag{2}$$

These five numbers have distinct critical digits; for, otherwise we would have

$$2^{n+k\alpha(s)} \equiv 2^{n+k'\alpha(s)} \pmod{10^{s+1}}, \qquad 5 > k > k' \geqq 0,$$

whence

$$2^{(k-k')\alpha(s)} \equiv 1 \pmod{5^{s+1}};$$

but this is impossible since $(k-k')\alpha(s)<5\phi(5^s)=\phi(5^{s+1})$, whereas the fact that 2 is a primitive root, mod 5, implies

$$2^r \not\equiv 1 \pmod{5^{s+1}} \qquad \text{for } r < \phi(5^{s+1}).$$

This means that if the critical digit of 2^n is odd (or even) then the critical digits of the numbers (2) are, in some order, 1, 3, 5, 7, 9, (or 0, 2, 4, 6, 8). Since by (1) the five numbers (2) have the same s final digits as 2^n, one of them has its $s+1$ final digits exclusively 1's and 2's. This completes the induction.

* No. 4326 by the same proposer: Prove or disprove: 128 is the only power of 2 of two or more digits each of which is a power of 2. (No solution or discussion has been submitted.)

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

The Mathematical Analysis of Logic. By George Boole, Macmillan, Barclay, and Macmillan, 1847. Reprinted. Philosophical Library, New York, 1948. 6+82 pp. $3.75.

A hundred years ago logic was still a "literary" science, about which people could talk at length, and oftentimes foolishly, without its being possible for anyone to prove them wrong. Like ethics and politics, logic rested on no solid foundation, and almost everything was subject for controversy.

But just when political thought was taking a turn for the worse (with the publication of the *Communist Manifesto* in 1848), the prospects for logic were beginning to brighten. Boole and De Morgan were loosening the bonds of scholastic philosophy—and showing that logic could be made into an exact mathematical science.

The historically minded reader can get a good notion of the difficulties confronting these early logicians, from Boole's *Mathematical Analysis of Logic*, now happily reprinted by the Philosophical Library. How difficult it must have been to devise an appropriate symbolism for logic when one had first to present arguments in favor of using any symbolism at all, even in mathematics—arguments against the objections which "have lately been very strongly urged against this practice, on the ground, that by obviating the necessity of thought, and substituting a reference to general formulae in the room of personal effort, it tends to weaken the reasoning faculties."!

The less serious reader will find a source of amusement in some of Boole's queer proofs, and a ground of self-satisfaction in living in so superior an age. For such a purpose, I cannot recommend anything better than the weird proof (page 60) of the law of expansion for Boolean functions: a proof which proceeds by expanding an arbitrary Boolean function in a Maclaurin's series and this, when the author has not defined derivatives of Boolean functions nor division of them by integers.

J. C. C. MCKINSEY

Principles of Mechanics. By J. L. Synge and B. A. Griffith. Second Edition. New York, McGraw-Hill, 1949. 16+530 pages. $5.00.

That a second edition of this intermediate text in theoretical mechanics should be required after a relatively short interval is evidence of its deserved popularity. The authors state in the preface that a textbook on mechanics must attempt to reconcile "two opposing goals—the reduction of the subject to a compact and classified form and its exposition with sufficient fullness to make the arguments easy to follow." Indeed, any textbook upon this most elegant

and complete branch of mathematical physics must also embody a compromise among three points of view: that of the mathematician, who cultivates the general theory of mechanics as a deductive system, and directs his efforts mainly toward transformation theory and questions of existence and stability; that of the physicist, who, seeking to describe the behavior of actual matter, desires on the one hand a thorough familiarity with the simplest situations in mass-point mechanics to serve as a starting point for physical models of crystals, atoms, *etc.*, and on the other, a general formulation of classical mechanics which is suggestive for the newer theories, so as to leave it behind at the earliest opportunity in his pursuit of high velocities and small dimensions; and that of the engineer, who desires to turn his equations to the design of practical devices. The resulting compromise leads to the science of *rational mechanics*, which many European universities deservedly recognize by separate chairs—a science unfortunately neglected on this continent. Every good book on mechanics represents this compromise and the present one is no exception. The following discussion is to be seen in the light of these remarks, not being meant as criticism of the general plan, but rather as presentation of an alternative point of view with regard to a few details.

In reviewing the first edition, J. W. Campbell (*Bulletin of The Am. Math. Soc.*, 49, 208–211, (1943)) offered five principal criticisms: (1) the basic concepts are insufficiently discussed for a first course; (2) vectors are defined incorrectly (or at best inconsistently) (pp. 18, 20); (3) the term "impulsive force" is used instead of "impulse," although this quantity is not a kind of force at all and is not even of the dimensions of force; (4) it is incorrectly stated (p. 88) that the coefficient of static friction $\mu \leqq 1$ experimentally and (5) while the theory of dimensions is treated in an appendix, there is little use of dimensional principles in the text. The authors have chosen to accept this criticism only in the fifth instance, where it was least well taken, and thus have inserted a number of remarks and easy exercises concerning dimensions at various points. In fact, in the theoretical mechanics of mass-points and rigid bodies the theory of dimensions is less important than in any other branch of mathematical physics, and may be altogether omitted without any loss. Once having chosen to mention it however, it is unfortunate that the authors merely reproduce the usual vague remarks to be found in physics and engineering texts, not going even so far as the pi-theorem, instead of showing that indeed dimensional theory is a simple clearly defined algebra like the algebra of real numbers, which we all use in a more or less unconscious fashion—an algebra underlying all mathematical physics very much as the algebra of logic underlies mathematics. As for the first criticism, perhaps the authors have noticed that it is only quite advanced students who begin to understand the fundamentals of a subject, whatever may or may not be said in an elementary course, whose main function is to give the student facility and structural orderliness in handling the facts with which one builds a science "comme un maison avec des pierres," without which he cannot appreciate the strength or weakness of those facts. It is unfortunate, however, that the authors did not see fit to be more careful regarding the purely mathe-

matical definitions, as in the case of a vector. It would seem also that here the student might first learn that every theory is an abstract mathematical model, rather than being allowed to fancy that he is somehow dealing directly with the phenomena the theory is designed to describe.

In his review of the first edition W. Prager [*Math. Rev.* 3, 213 (1942)] remarked upon the sparing use of vectorial methods, and in particular upon the fact that the authors' eschewance of tensorial or dyadic notations renders their treatment of the motion of rigid bodies quite lengthy. In the first edition the authors were content to state that since the equation of any ellipsoid could be reduced to standard form by a proper choice of axes, the existence of principal axes of inertia followed. In the second edition they have given a proof of this statement, but it is unfortunate that in what may be the student's first encounter with the characteristic value problem he will have to read a long and special treatment of a special case, instead of learning once and for all a general theorem that would be of frequent usefulness. In any case, the moments and products of inertia are brought in by the heels rather than shown to arise most naturally in the problem of reducing the equations of rigid dynamics to a simple and perspicuous form.

The second edition differs from the first mainly in the addition of a few words of explanation (as in the case of d'Alembert's principle), a few figures, and a few problems, amounting to fifteen pages in all. In addition to the one major change already mentioned, the only other is the insertion of a careful and very interesting treatment (pp. 388–403) of the motion of a charged particle near the axis of symmetry of an electromagnetic field, including applications to the electromagnetic lens.

To return again to the attitude expressed in the opening sentences of this review, there is one respect in which the book leans too far toward the engineering side. Lagrange's equations are introduced only after 450 pages, the applications made of them are not important, and their extension to rigid dynamics is not mentioned. The authors say "However, it seems probable that as time passes the method of Lagrange will work its way from the end to the beginning of textbooks on mechanics. The easier, but more cumbrous, methods are becoming a luxury for which we cannot afford the time." It is unfortunate that the authors did not actualize their own position. Since many a physics curriculum nowadays affords only one course in mechanics, it would seem that Hamilton's canonical equations and the Hamilton-Jacobi equation should be included, although no mention of them occurs in this book.

An elementary textbook cannot be loaded with references and attributions, but it would seem a pleasant duty of the writers of such a book to connect a number of the methods and theorems with the great names responsible for them. Although there is hardly a great mathematician or physicist from 1650 to 1850 who did not make some important contribution to their subject, the authors mention only about twenty-five classical names, and those very sparingly.

Any textbook serves principally as a guide for the instructor and students,

who may make a reasonable number of additions or omissions. In this respect the present book is admirable, for the additions one is likely to suggest are few, and the omissions only such as time enforces.

C. TRUESDELL

Introduction to Analytic Geometry and The Calculus. By H. M. Dadourian. New York, The Ronald Press, 1949. 10+246 pages. $3.25.

This is a textbook designed "for use in combined courses of Freshman mathematics, such as are offered by an increasing number of colleges and universities." For a terminal course, the author points out that trigonometry is not presupposed except in a single section (which can be omitted), thus making possible an exclusive emphasis on analytic geometry and calculus. Otherwise, for the student who will study mathematics further, a brief course in trigonometry would seem to be desirable before undertaking the subject matter of this text.

An instructor in choosing a textbook will, naturally, take into account not only the subject matter he plans to teach but also the preparation and background of his students, that is, the general intellectual level of his audience. It should be noted, therefore, that the volume under review "is only an introduction to the simplest elements of analytic geometry and the calculus." And that the author has attempted to present these elements with a continuing and conscientious regard for the difficulties that students, beginning in these fields, always meet.

Following an introductory chapter on how to study mathematics and on solving problems, the section on analytic geometry includes chapters on functions and graphs, the linear function and the straight line, and the quadratic function and conics. The author's restriction to the "simplest elements" has resulted apparently in the omission of any study of the general second degree equation, and other omissions of material usually considered in a first course will be noted. Nevertheless, there is considerable information for the student here. This reviewer, however, would have preferred that some of this information might have been the result of "derivation" rather than mere "statement of fact," and, as a consequence, the author's admonition for the student "to think" the better motivated.

Five chapters on the differential calculus cover fairly well the usual theory and applications of a first course, at least as well as one can expect when limited to the derivative of powers of algebraic functions. Under an analogous limitation, the integral calculus is developed in theory and applications in the four final chapters.

Keeping in mind the author's aim to present this introductory material to college freshmen, which is of course an entirely reasonable and justifiable aim, it would nonetheless seem pertinent here to criticize the execution on several counts: The lack of *explicit* definitions in many cases where the consequences are based entirely on the definition (cf., pp. 165, 182); loose or incomplete state-

ments and definitions (cf., pp. 77, 81); and a tendency (mentioned above) to state facts rather than to deduce results. In particular, the last chapter would seem to suffer especially in these respects (cf., "negative mass," p. 199, and Sections 142–143).

At the same time one can have only praise for the author's choice of illustrative material, his clear exposition of this material, and his problem lists. In general, the text reads well and possesses an admirable conciseness of statement. Only one misprint was observed (p. 110), but one which will probably not be condoned by the devotees of Lewis Carroll!

B. C. PATTERSON

NEW BOOKS RECEIVED

College Algebra. By. E. B. Miller and R. M. Thrall. New York, The Ronald Press. 493 pp. $3.75.

Plane and Spherical Trigonometry. By M. Richardson. New York, Macmillan, 1950. 15+138 pp. $3.75 ($3.40, without tables).

Modern College Geometry. By D. R. Davis. Cambridge, Addison-Wesley, 1949. 10+238 pp. $4.50.

Elements of Mathematical Analysis. By S. E. Urner and W. B. Orange. Boston, Ginn and Company, 1950. xi+562 pp. $4.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS 1948–1949

Pi Mu Epsilon, University of Georgia

The following papers were presented before the *Pi Mu Epsilon* fraternity at the University of Georgia during 1948–49:

Honor courses at the University of Georgia, by Prof. Tomlinson Fort

Sturms' theorem, by Prof. Tomlinson Fort

Unit fractions of early mathematics, by Howard Burnette

The geometry of the triangle and some of its applications, by Prof. Hugh Henry of the Department of Physics

The history of Pi Mu Epsilon, by Prof. Tomlinson Fort

Certain theorems on matrices, by Prof. J. A. Ward

The symbiosis of knowledge, by Prof. E. H. Dixon of the Department of Physics

G. H. Hardy's solution of a problem in genetics, by Prof. R. J. Levit

Geometry, a series of colloquium lectures, by Prof. C. C. MacDuffee of the University of Wisconsin.

A colloquium lecture was also given each month by one of the professors.

The officers for the coming year are: Director, George Peters; Vice-Director, William Shy; Secretary-Treasurer, Ann Taylor; Faculty Advisor, Prof. Alonzo Cohen; Corresponding Secretary, Prof. W. S. Beckwith.

Mathematics Club, University of Buffalo

The topics discussed by the University of Buffalo *Mathematics Club* were:

Construction by compass alone, by C. Collins and M. Kallett

Geometric approach to some algebra expressions, by C. Christenson

Mathematical curiosities, by George Rudroff

A collinear set in a lattice of points, by George Rudroff

Diophantus solves an algebra problem, by Thomas Huestis

Construction of a differential analyzer, by Brent Haslam

Determination of elementary symmetrical polynomials, by Brent Haslam

Arithmetic short cuts, by Richard Smurthwaite and Harold Cutcher

Derivation of formulas for obtaining the sums of powers of the roots, and the elementary symmetrical polynomials in terms of each other, by Albert Fadell.

The officers for the year 1949–50 are: President, Roger Hill; Vice-President, Richard Smurthwaite; Treasurer, Fred Gregory; Secretary, Wilbur Hastings; Sponsor, Miss June McArtney.

Kappa Mu Epsilon, Drake University

The activities of *Iowa Beta* chapter of *Kappa Mu Epsilon* for the past year were devoted to reorganization meetings, initiations, business meetings and a banquet. A modification of the chapter's by-laws took several meetings.

Three delegates attended the national convention of Kappa Mu Epsilon at Topeka, Kansas.

The officers elected are: President, William Lafferty; Secretary, June Kagawa; Treasurer, Marvin Van Cleave; Corresponding Secretary, Prof. Earle Canfield.

Pi Mu Epsilon, University of Wisconsin

The *Wisconsin Beta* chapter of *Pi Mu Epsilon*, inactive during the war period, was reorganized during the academic year 1947–48 as a graduate student club to which a number of graduating seniors are elected each year in honor of their high scholarship.

The program of the chapter during 1948–49 included the following talks:

Computation by punched cards, by Fred Gruenberger

Atomic elements in Boolean algebras, by Joseph Engel

Duality in differential calculus and projective geometry, by Rodney Hood

Loops and their isotopes, by Donald Norton

The P-adic and G-adic number systems, by Richard Andree.

A get-acquainted open house was held for the mathematics faculty, graduate students and guests. Other social events included an initiation banquet and another party.

Officers for 1948–49 were: President, James Renno; Vice-President, Herman Cohen; Secretary-Treasurer, Violet Hachmeister; Faculty Adviser, Prof. R. E. Fullerton.

Mathematics Society, University of Miami

The *Mathematics Society* of the University of Miami was organized in July 1948. During the year 1948–49 fifteen regular meetings and a banquet were held. Among the papers presented were:

Rhind papyrus, by W. G. Franzen

Probability curve, by L. G. Barbee

Boolean algebra, by Prof. Ira Rosenbaum

Natural law of growth and the number e, by D. J. R. Foulis

Magic squares, by Miss Ethelmae Miller

Notation in differential and integral calculus, by J. H. Maecher.

The officers for 1948–49 were: President, William Franzen; Vice-President, Eugene P. Goldberg; Recording Secretary, Marjorie Stern; Corresponding Secretary, Gloria Lichtenstein; Treasurers, Arlo Shaver and John Maecher; Faculty Advisor, Prof. Mary Dean Clement.

Kappa Mu Epsilon, Nebraska State Teachers College

The *Nebraska Alpha* chapter of *Kappa Mu Epsilon* held two initiation meetings during 1948–49. Other meetings consisted of business meetings and the following programs:

Salesmanship, by Donald Simpson

Certain aspects of mathematics in high school, by President V. P. Morey.

Seven members of *Kappa Mu Epsilon* attended the State meeting of the National Council of Teachers of Mathematics at which Dr. Morey was the luncheon speaker.

The only new officer elected for the coming year is James Callaway, who is the Organizing Chairman.

Mathematics Club, Berea College

The *Mathematics Club* of *Berea College* had regular monthly meetings. Some of the topics discussed were:

Possibilities for assistantships in graduate school, by Donald W. Pugsley

The abacus, by George S. Noss

Oddities in mathematics, by Jack Hale

Relationship between philosophy and mathematics, by W. G. Ross

Short-cuts in mathematics, by V. D. Roberts.

The elected officers were: President, Howard Stevens; Vice-President, Robert Manning; Secretary-Treasurer, Jacky Gene Hopper.

Pi Mu Epsilon, University of Nebraska

The *Nebraska Alpha* chapter of *Pi Mu Epsilon* reports that, for the year 1948–49, eight meetings were held including business meetings and the two initiation banquets. The following talks were given:

Special theory of relativity, by Prof. Theodore Jorgensen
Boolean algebra, by Eugene Luchei
Quantum mechanics, by Dr. J. H. Cohn
Calculating machines, by Fred Pelton
Television, by Mr. D. D. Holmes
"*Murder incorporated*," by Dr. Frederick Ludwig.

The chapter sponsored classes to extend aid in mathematics to the slower students. Members of the chapter volunteered their services as instructors. The plan proved highly satisfactory as it benefited both the students and the tutors.

A pamphlet was prepared which introduced *Pi Mu Epsilon* to students not familiar with the organization.

The annual freshman and sophomore competitive mathematics examinations were held; the sophomore award was won by Robert Kleppinger, who also won the freshman award last year, and Chris Kuyatt was the winner of the freshman award.

The newly elected officers to serve for the coming year are: President, Cecil Doubt; Vice-President, Robert Ceasari; Secretary, Donald Cochran; Treasurer, David Olive; Faculty Advisor, Dr. W. G. Leavitt.

Kappa Mu Epsilon, Baldwin-Wallace College

Papers presented to the *Ohio Gamma* chapter of *Kappa Mu Epsilon* during 1948–49 were:

Boolean algebra, by Donald Parrish
The mathematics of audio amplification, by Ralph Paul
The duodecimal system, by Robert Jungles
Report of the national convention of Kappa Mu Epsilon, by Donald Parrish and Howard Mielke
Probability—the law of averages, by Dr. R. W. Wagner of Oberlin College
Mathematics and aviation, by Cook Cleland.

The annual banquet of *Kappa Mu Epsilon* was held in May at which Dr. R. W. Wagner spoke.

The society visited the National Advisory Committee on Aeronautics Laboratory at the Cleveland Airport. Various applications for mathematics were shown. One of the many interesting things seen was the world's largest supersonic wind tunnel.

Officers elected for 1949–50 are: President, Fred Rakowsky; Vice-President, Dwain Diller; Secretary, Anita Constantine; Treasurer, Curt Liebert; Corresponding Secretary and Sponsor, Prof. Paul Annear.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

CONFERENCES AND SYMPOSIA

A Symposium on the Theory of Electromagnetic Wave Propagation will be held at New York University on June 6–8, 1950.

The Second Berkeley Symposium on Statistics and Probability will be held at the University of California, Berkeley, from July 31 to August 12. This meeting will be given to theory and to several fields of application. Up to the present time the following persons have accepted invitations to participate in the Symposium: Professors W. G. Cochran, Harald Cramér, J. L. Dobb, G. C. Evans, William Feller, Harold Hotelling, Mark Kac, Paul Lévy and Abraham Wald.

The University of Chicago announces that a Conference on Topology will be held on May 24–27, 1950.

SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1950:

Boston University. May 31 to July 8: Dr. Giever, topics in the theory of numbers; Dr. Scheid, calculus of variations. July 10 to August 19: Professor Johanson, introduction to higher algebra; Professor Sobczyk, introduction to modern mathematics, seminar in analysis; Professor Chand, the design of experiments, introduction to mathematical statistics.

Catholic University of America. June 26 to August 5: Professor Rice, advanced calculus and statistics; Professor Finan, higher algebra; Mr. Clark, partial differential equations of mathematical physics; Dr. Markowitz, mathematical astronomy I and II; Dr. Moller, theory of equations; Professor Ramler, college geometry, differential equations and synthetic projective geometry.

Columbia University. July 10 to August 18: Professor Ritt, differential equations; Professor Levi, theory of functions of a complex variable; Professor Koopman, Fourier series and integrals; Professor Eilenberg, introduction to modern topology; Professor Dieudonné, introduction to classical groups; Professor Lorch, introduction to mathematical logic.

Northwestern University. The following courses will be offered in both the six-week session, June 23 to August 5, and the nine-week session, June 23 to August 26: differential equations, fundamental concepts of analysis; definite integrals; non-euclidean geometry; theory of statistics; introduction to the theory of numbers; algebra for teachers; the history and teaching of mathematics; functions of a complex variable; the continuum; introduction to modern algebra; seminar; independent study; thesis. A course, the teaching of mathematics in the secondary school, will be offered during the six-week session only.

University of Arkansas. June 5 to July 14: Professor Richardson, college geometry, theory of equations; Professor Perry, advanced calculus, Laplace transform; Mr. Hull, differential equations. July 17 to August 26: Professor Gundlach, history of mathematics, vector analysis.

University of California, Berkeley. Department of Mathematics. June 19 to July 29: Professor Besicovitch, extremal problems in geometry; Professor McDonald, coordination in mathematics for secondary schools.

University of California, Berkeley. Department of Mathematical Statistics. June 19 to July 29: Professor Lehmann, recent developments in the theory of estimation; Professor Cochran, design of experiments, seminar on current problems in analysis of variance; Professor Lévy, random variables and random functions, seminar in probability theory.

University of Chicago. June 26 to September 2: Professor Barnard, analysis E, vector analysis; Professor Kaplansky, fundamental concepts of geometry, algebra II; Professor Schilling, algebra V, theory of rings; Professor Halmos, theory of sets, seminar in probability; Professor Zygmund, Lebesque and Stieltjes integrals, regular functions in the unit circle; Professor Segal, partial differential equations; Professor Stone, approximation in the complex domain; Professor Weil, functions of several complex variables, historical seminar.

University of Oklahoma. June 8 to August 7: Professor Court, college geometry, history of mathematics; Dr. Huff, ordinary differential equations; Dr. Andree, solid analytical geometry; Professor Bernhart, ordinary and partial differential equations; Mr. LaFon, advanced calculus; Professor Grau, theory of groups; Professor Springer, partial differential equations; Professor Goffman, infinite series; Professor Whitney, celestial mechanics.

University of Southern California. June 26 to August 4: Professor Albert, modern higher algebra, seminar in algebra; Professor Dugundji, advanced calculus; Professor Henkin, introduction to mathematical logic; Professor Phillips, integral equations; Dr. Shniad, vector analysis; Professor White, advanced calculus; Professor Whiteman, theory of numbers, matrix theory.

University of Virginia. June 26 to August 19: Professor Klee, advanced calculus and applied mathematics; Professor Botts, differential equations and applied mathematics; Professor Whyburn, analytic topology; Professor Floyd, advanced differential equations.

PERSONAL ITEMS

Professor C. O. Oakley of Haverford College and Professor G. C. Webber of the University of Delaware were appointed to serve as delegates of the Association at the Annual Meeting of the American Academy of Political and Social Science which was held in Philadelphia on April 14–15, 1950.

Professor W. G. Scobert of the University of Idaho represented the Association at the Centennial Celebration of the University of Utah, February 27–28, 1950.

Professor Einar Hille of Yale University has been elected Vice-President of the American Association for the Advancement of Science and Chairman of the Mathematics Section; Professor R. W. Brink, University of Minnesota, has been re-elected Secretary of the Mathematics Section.

Southern Illinois University announces the following appointments for the 1950 summer session: Professor Emeritus W. B. Carver of Cornell University as Professor of Mathematics; Associate Professor T. H. Southard of Wayne University as Associate Professor of Mathematics.

Stanford University reports that the following persons will be associated with the Mathematics Department during the Summer Quarter, 1950: Harold Davenport, University College, London; Werner Fenchel, Polytechnical School, Copenhagen; Paul Garabedian, University of California, Berkeley; W. K. Hayman, Exeter College, England; Charles Loewner, Syracuse University; W. W. Rogosinski, King's College, Newcastle, England; H. L. Royden, Harvard University.

University of California, Berkeley, announces the following promotions, appointments and leaves of absence for the year 1949–50: Professor C. B. Morrey has been appointed Chairman of the Department of Mathematics; Associate Professors Sophia L. McDonald, A. P. Morse, R. M. Robinson have been promoted to professorships; Dr. S. P. Diliberto, Dr. J. L. Hodges, Jr., and Dr. R. C. James have been promoted to assistant professorships; Dr. H. B. Ribeiro, Marianne F. Smith and Virginia W. Wakerling have been promoted to instructorships; Dr. P. R. Garabedian has been appointed to an assistant professorship; R. M. Lakness, Ting-Kwan Pan and Esther Seiden have been appointed Lecturers; Dr. H. B. Mann has been appointed Visiting Professor; Professor D. H. Lehmer and Professor R. M. Robinson are on sabbatical leave for the year; Professor Jerzy Neyman is on leave for the spring term 1950 and will deliver lectures at University College, London, Institut Poincaré and University of Paris, Paris; Assistant Professor Charles Stein holds a National Research Council Fellowship and is spending his leave in Paris; Professor Alfred Tarski was on leave in March 1950 to lecture at University College, London, and the University of Amsterdam.

University of Florida reports: Assistant Professors B. F. Dostal and G. B. Lang have been promoted to associate professorships; Instructor R. G. Blake has been promoted to an assistant professorship.

University of Georgia announces the following: Dr. A. S. Householder of the Oak Ridge National Laboratory gave a series of talks on numerical methods to the mathematics colloquium on February 15–17, 1950; during March 27 to April 4 Professor J. L. Walsh of Harvard University gave a series of colloquium lectures on approximation by polynomials in the complex domain, a Sigma Xi lecture on surface navigation, and a Pi Mu Epsilon lecture on maximum and minimum problems.

Assistant Professor L. A. Aroian of Hunter College is on leave the spring semester. He is serving as Research Physicist in the Department of Electronics

and Guided Missiles of the Hughes Aircraft Company, Culver City, California, in charge of computations.

Professor R. V. Blair of Vanderbilt University was reported erroneously in this department, February, 1950, to have the rank of associate professor. He was promoted to a professorship in 1948.

Associate Professor N. R. Bryan of the University of Georgia has been appointed to an associate professorship at Clemson College.

Mr. Roland Deaux has assumed the editorship of *Mathesis*.

Instructor Watson Fulks of the University of Minnesota has been promoted to an assistant professorship.

Mr. A. G. Hansen has been appointed to an instructorship at the University of Maryland.

Professor Eberhard Hopf, formerly of New York University, has been appointed to a professorship at Indiana University.

Mr. D. W. Hullinghorst has accepted a position as director of the Civil Works Investigation Project, South Pacific Division, Engineering Corps, Oakland Army Base, Oakland, California.

Mr. Sydney Kaplan, formerly a research mathematician in the Applied Mathematics Division, Naval Ordnance Laboratory, has accepted the position of Statistician, Bureau of Employment Security, United States Department of Labor.

Dr. Knox Millsaps of the Air Materiel Command, Wright Field, has been appointed to a professorship of theoretical physics at the Alabama Polytechnic Institute.

Mr. S. F. Neustadter has been appointed to an instructorship at Harvard University.

Professor Rufus Oldenburger of DePaul University has resigned to devote more time to the Woodward Governor Company, Rockford, Illinois, as mathematician-engineer.

Dr. Irene Price, formerly statistician for the United States Air Force, has been transferred to Holloman Air Force Base, New Mexico, to direct mathematical research on guided missiles.

Dr. L. T. Ratner has been appointed to an assistant professorship at Vanderbilt University.

Professor C. J. Rees, director of graduate study, University of Delaware, has been appointed Director of Summer Session.

Dr. H. M. Schwartz of Brookhaven National Laboratory has accepted an associate professorship at the University of Arkansas.

Dr. Edward Silverman of the Institute for Advanced Study has been appointed to a visiting assistant professorship at Kenyon College.

Assistant Professor F. E. Smith of Brooklyn College has been promoted to an associate professorship.

Professor A. W. Tucker of Princeton University has been spending the

academic year 1949–50 at Stanford University as Visiting Professor of Mathematics.

Professor W. D. A. Westfall of the University of Missouri has retired.

Miss Bertha E. Frankenbush, emeritus high school teacher of New Orleans, Louisiana, died February 7, 1950.

Professor Emeritus W. J. Humphreys of George Washington University died November 10, 1949.

Professor Emeritus F. M. McGaw of Cornell College died on February 6, 1950. He was a charter member of the Association.

Mr. A. Mineur, editor of *Mathesis*, died on February 4, 1950.

Professor Emeritus C. E. Wilder, Dartmouth College, who was a charter member of the Association, died on February 9, 1950.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

OCTOBER MEETING OF THE OKLAHOMA SECTION

The October meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the convention of the Oklahoma Education Association in Oklahoma City on Friday, October 14, 1949. Professor R. W. Veatch, Chairman of the Section, presided.

One hundred twenty persons attended the meeting, including the following forty-three members of the Association: E. F. Allen, R. V. Andree, Arthur Bernhart, J. C. Brixey, W. C. Brown, H. N. Carter, R. L. Caskey, N. A. Court, R. B. Deal, A. H. Diamond, R. C. Dragoo, J. R. Ellis, D. A. Gorsline, L. D. Gregory, O. H. Hamilton, Claire A. Harrison, J. O. Hassler, E. E. Heimann, L. R. Holland, S. L. Hull, P. W. M. John, W. G. Johnson, J. T. Krattiger, W. D. Krentel, J. E. LaFon, M. L. Lawson, Eunice Lewis, H. W. Linscheid, R. D. McDole, Dora McFarland, J. C. C. McKinsey, R. D. McKnelly, G. E. Meador, R. R. Murphy, C. L. Perry, C. J. Pipes, C. M. Pirrong, Herbert Scholz, H. W. Smith, T. C. Smith, C. E. Springer, R. W. Veatch, J. H. Zant.

At the business session the following officers were elected: Chairman, J. T. Krattiger, Southeastern State College; Vice-Chairman, S. L. Hull, University of Arkansas; Secretary, J. C. Brixey, University of Oklahoma.

The program consisted of the following seven papers:

1. *Orientation of the new teacher*, by L. W. Lavengood, Director of Mathematics, Tulsa Public Schools, introduced by Professor Veatch.

Mr. Lavengood discussed methods used to help orient new teachers in the Tulsa Public Schools

and pointed out that new teachers should be made to feel welcome in the community, and have a responsible part in the school organization.

2. *Essential and inessential transformations of point sets into n-spheres*, by Professor O. H. Hamilton, Oklahoma A. and M. College.

Professor Hamilton gave a new proof of the theorem that every continuous transformation of an n-cell into an n-sphere is inessential, and discussed the relation between the concept of an inessential transformation and fixed point theorems.

3. *Old and new on the centroid*, by Professor N. A. Court, University of Oklahoma.

A historical discussion of the notion of centroid, and a new treatment of centroid, were given in this paper.

4. *Axiomatic foundations of the theory of games*, by Professor J. C. C. McKinsey, Oklahoma A. and M. College.

5. *L'Hospital's rule*, by Professor Arthur Bernhart, University of Oklahoma.

Criticisms were given of the treatments of L'Hospital's rule appearing in several of the popular elementary calculus texts.

6. *The summation of a Dirichlet series*, by Professor H. N. Carter, University of Tulsa.

The ordinary and general Dirichlet series were defined, and a statement was made regarding their introduction into analysis and application in analytic number theory.

7. *Contributions of Hilbert space theory to applied mathematics*, by Professor A. H. Diamond, Oklahoma A. and M. College.

Hilbert space theory was applied to the problem of determining the frequencies of vibration of a square clamped plate. The procedure consisted of a generalization by Aronszajn of a method devised by A. Weinstein for finding lower bounds for the eigenvalues.

J. C. BRIXEY, *Secretary*

OCTOBER MEETING OF THE MINNESOTA SECTION

The fall meeting of the Minnesota Section of the Mathematical Association of America was held at the University of North Dakota in Grand Forks, North Dakota, on Saturday, October 15, 1949. Sessions were held in the forenoon, at luncheon, and in the afternoon. Professors R. C. Staley, A. G. Hill and Kenneth May (Chairman of the Section) presided at the respective sessions.

Sixty-one persons attended the meeting, including the following twenty-five members of the Association: H. M. Anderson, F. J. Arena, L. E. Bush, E. J. Camp, Sister Seraphim Gibbons, Ruby M. Grimes, K. L. Hankerson, A. G. Hill, Karlis Kaufmanis, S. L. Mason, Kenneth May, W. H. McBride, W. R. McEwen, A. G. Montgomery, S. Mundhjeld, J. C. Peterson, P. A. Rognlie, L. W. Sheridan, F. C. Smith, I. W. Smith, R. C. Staley, Florence S. Stennes, A. G. Swanson, E. L. Swanson, Christine Westgate.

Professor E. J. Camp made a progress report for the Committee investigating

the advisability of establishing a high school mathematics contest. At the request of this Committee, a resolution was passed committing the Section to the sponsorship of such a high school contest as soon as arrangements can be made for it. Another resolution was passed authorizing the Executive Committee of the Section to act for the Section on all questions arising between meetings of the Section in connection with the arrangement of such a contest.

By invitation of the Executive Committee, Professor Peter Scherk of the University of Saskatchewan delivered an address at the morning session. The title of his address was *What are Tensors?* An abstract of this address follows:

A complete vector space is a vector space over the real field and with a scalar product such that the null-vector is the only vector perpendicular to the whole space. In the first part of his talk, Professor Scherk showed that the vectors in euclidean 3-space form such a space. Making use of ideas of Bourbaki's "algébre multilinéaire," he then developed the tensor algebra in an arbitrary complete vector space. The tensors of order p, for example, may be introduced as finite sums of ordered p-tuples of vectors, provided the equality of two such sums is defined in a suitable fashion.

The following eight short papers were presented:

1. *The star of Bethlehem from the astronomical point of view*, by Professor Karlis Kaufmanis, Gustavus Adolphus College.

In this paper, the author discussed several attempts to explain the star mentioned in the second chapter of St. Matthew as an astronomical phenomenon. He emphasized particularly the amazing agreement between the message of the Gospel and the conjunction of the planets Jupiter and Saturn in the sign of the Fish, which took place in the year 7 B.C.

2. *Substandard endowments with graded amounts of insurance*, by Professor F. C. Smith, College of St. Thomas.

If n-year endowment contracts with standard premiums $P_{x\,\overline{n}|}$ and standard reserves ${}_tV_{x\,\overline{n}|}$ are issued to a group of applicants whose mortality will follow a $100k$ per cent mortality table, then it is well-known that the amount of the insurance benefit for the t-th year is

$$\frac{1 + (k-1)\,{}_tV_{x\,\overline{n}|}}{k}.$$

In this paper, the author presents the usual derivation of this result by use of Fackler's accumulation formula, and then gives a proof of the fact that the present value of the insurance benefits is equal to the present value of the premiums. The proof, which is believed to be new, depends upon showing that two rather complicated polynomials are identical.

3. *Further remarks on the estimation of a function of several variables by graphical techniques*, by Professor L. W. Sheridan, The College of St. Thomas.

If X is a function of several variables, its value can be estimated by combining the several variables into a single variable, Z, by graphic techniques. Next Z and X are grouped into classes, and a cumulative table, with X expressed in per cent, is formed. The set of curves obtained from the cumulative table gives an estimate of the most probable class in which X may occur, and additional information as to the probability of its occurrence in adjacent classes.

4. *Problem on crystal vibrations*, Professor R. H. Cook, South Dakota School of Mines (introduced by the Secretary).

The theoretical determination of frequency for normal modes of vibration of crystal plates with two parallel lateral boundaries is presented. The method is one in which the differential equations are exactly satisfied, and the boundary conditions are approximated. It is shown that the method gives results practically identical to those of infinite plate theory for very high harmonics.

5. *The adjusted premium method of determining surrender values*, by Mr. I. W. Smith, Actuary, Pioneer Mutual Life Insurance Company, Fargo, North Dakota.

Prior to January 1, 1948, the cost of surrender of a life insurance policy, regardless of type, was fixed at a maximum of twenty-five dollars per one thousand dollars of insurance, or two and one-half per cent of the policy. This restriction applied only to the first twenty years duration of the policy. The cost was a graduated scale decreasing from approximately twenty dollars the first policy year to zero the twentieth policy year. The surrender value was the full terminal reserve the twentieth policy year.

Competition was the basis for some life insurance companies to return the full terminal reserve the tenth policy year. Surrender practice became irregular, the only restriction being the maximum cost of two and one-half per cent of the amount of the policy.

With the adoption of the Commissioners Standard Ordinary Mortality Table, effective January 1, 1948, actuaries have derived a method of determining the surrender cost by the adjusted premium. The adjusted premium is a new term in actuarial science. It is a function of mortality, duration of policy, and a maximum cost. It is expressed algebraically for an ordinary life policy as,

$$A_{P_x} = \frac{A_x + 20 + .4A_{P_x} + .25A_{P_x}}{a_x}$$

with the limiting value of 40 per cent of A_{P_x} not greater than sixteen dollars, and 25 per cent of A_{P_x} for any type of policy not more than twenty-five per cent of the adjusted premium for an ordinary life policy nor greater than ten dollars.

When the adjusted premium has been determined, the cash value is calculated by the prospective method of valuation. There will be a surrender charge for the entire premium paying period for any policy.

6. *Some elementary determinations of Pythagorean numbers*, by Professor Ruby M. Grimes, North Dakota Agricultural College.

This speaker proved by means of trigonometric relations and also by algebra that any three integers in the ratios of $2ab$, a^2-b^2, a^2+b^2, $(a>b)$ can be the three sides of a right triangle. A demonstration of the same property was accomplished by use of analytic geometry, using the unit circle, and this generalizes the other two proofs.

7. *Complex planes and Pauli matrices*, by Professor W. L. Woodley, North Dakota Agricultural College (introduced by the Secretary).

This speaker began with the fact that the circle $x^2+y^2=a^2$ can be represented by the equation $u\bar{u}=a^2$, where $u=x+iy$, and $\bar{u}$ is the complex conjugate of u. This fact was made the basis of certain generalizations of which Pauli matrices are a special case. Pauli used these matrices in studying the spin of an election.

8. *On some properties of the trinomial coefficients*, by Professor F. J. Arena, North Dakota State College.

If n is a positive integer, and if

$$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n},$$

the following relation connects the trinomial coefficients a_Ω with the binomial coefficients ${}_nC_\Omega$:

$$a_\Omega = \sum_{i=b}^{\Omega} {}_iC_{2b-i} \cdot {}_nC_i, \qquad \text{if } \Omega = 2S$$

$$a_\Omega = \sum_{i=S+1}^{\Omega} {}_iC_{2S-1} \cdot {}_nC_i, \qquad \text{if } \Omega = 2S + 1.$$

Other properties are stated and proved.

L. E. BUSH, *Secretary*

NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at Haverford College, Haverford, Pennsylvania, on Saturday, November 26, 1949. Professor G. E. Raynor, Chairman of the Section, presided.

There were sixty-eight present, including the following forty-eight members of the Association, C. B. Allendoerfer, L. J. Burton, P. A. Caris, John Cawley, J. O. Chellevold, E. J. Cogan, Mary Louise Constable, E. H. Cutler, James Elmer Davis, F. L. Dennis, Arnold Dresden, H. H. Goldstine, Theodore Hailperin, R. C. Haseltine, C. E. Heilmen, Hyman Kamel, J. C. Lanz, V. V. Latshaw, Joseph Lehner, Marguerite Lehr, F. L. Manning, Edith A. McDougle, S. S. McNeary, A. E. Meder, Jr., W. R. Murray, A. B. Neale, C. A. Nelson, C. O. Oakley, E. R. Ott, J. C. Oxtoby, G. E. Raynor, R. D. Schafer, I. J. Schoenberg, Francis Aloysius C. Sevier, Benjamin Slepin, J. C. Smith, W. M. Smith, R. R. Stoll, D. L. Thomsen, Jr., Michael Tikson, R. R. Townsend, R. M. Walter, G. C. Webber, Anna Pell Wheeler, Albert Wilansky, A. H. Wilson, R. H. Wilson, Jr., H. M. Zerbe.

At the business meeting the following officers were elected for the coming year: Chairman, G. C. Webber, University of Delaware; Secretary, C. O. Oakley, Haverford College. The Program Committee for the next meeting will be: Theodore Hailperin (Chairman), Lehigh University, Joseph Lehner, University of Pennsylvania, and C. B. Allendoerfer, Haverford College. The 1950 meeting of the Section will be held at Lehigh University on November 25.

The program consisted of the following papers:

1. *Some observations relative to mathematics in research and development organizations*, by Dr. A. D. Hestenes, Franklin Institute, introduced by the Secretary.

This paper dealt with the process by which equipment is developed and studies are made in typical industrial research and development organizations, and stressed the role a mathematician may perform in such processes. The speaker described various types of mathematics encountered in such organizations. An interesting locus problem arose in the course of the author's industrial research activities. This locus, whose equation is

$$r_1 - a = c\,(r_2 - b)$$

where a, b and c are constants, and r_1 and r_2 are distances from fixed points p_1 and p_2 respectively, is a generalization of the well known expressions for the circle, ellipse and hyperbola.

2. *Some problems in numerical analysis*, by Dr. H. H. Goldstine, Institute for Advanced Study.

There are four major sources of error in a numerical computation: those arising from the idealizations and simplifications of a given physical problem; those arising from the fact that certain parameters may not be known precisely; those arising from the approximate character of the numerical form given the mathematical problem, *i.e.*, truncation errors; and finally, those arising from the errors of the computing machine itself, *i.e.*, the "noise" or rounding errors. The main purpose of the present exposition is to treat this last category, and to discuss its great importance when large scale calculations are undertaken. Several examples are given to indicate generally how analyses of rounding errors can be performed.

3. *Minimal sets*, by Professor J. C. Oxtoby, Bryn Mawr College.

The theory of the structure of minimal sets due to Mariano Garcia, Jr., and G. A. Hedlund (*Bull. Amer. Math. Soc.*, vol. 54, (1948), pp. 954–964) was discussed, and it was shown that in the example there constructed of a 0-reducible minimal set which is not completely reducible the set of non-regularly almost periodic points actually consists of just two orbits, namely the orbit of the point β used in defining the set, and that of the point β' which is equal to β except for $n=0$.

4. *On smoothing operations*, by Professor I. J. Schoenberg, University of Pennsylvania.

C. O. OAKLEY, *Secretary*

CALENDAR OF FUTURE MEETINGS

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December 30, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Allegheny College, Meadville, May 6, 1950.
ILLINOIS, Southern Illinois University, Carbondale, May 12–13, 1950.
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Richmond, May 6, 1950.
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, Macalester College, St. Paul, May 6, 1950.
MISSOURI
NEBRASKA, Nebraska Wesleyan University, Lincoln, May 6, 1950.
NORTHERN CALIFORNIA, University of San Francisco, January 27, 1951.
OHIO
OKLAHOMA
PACIFIC NORTHWEST, University of Washington, Seattle, June 16, 1950.
PHILADELPHIA, Lehigh University, Bethlehem, Pennsylvania, November 25, 1950.
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WISCONSIN, Marquette University, Milwaukee, May 13, 1950.

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THE PUPIL'S ADVOCATE*

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Once upon a time this Association appointed a committee to consider the rival claims of courses in mathematics and of professional courses in education in a fifth year program for prospective teachers of secondary mathematics. This committee reported that it naturally favored mastery of subject matter, but that it was bound to admit that the continued existence and growth of instruction in educational psychology, in the principles of teaching, and in allied subjects indicated some merit in those areas also. It recognized that instruction in "how to teach" was often feeble, and that therefore it was easy to see why such instruction was generally scorned by the academic mind. Nevertheless this committee took the position that where there was smoke there must also be fire, and decided to recommend for the fifth year an almost equal division between the claims of mathematics and professional training in education. They presumed, I suppose, that a prospective teacher whose training included both mathematics and some knowledge of the learning process would blend both ingredients in an effective teaching procedure. You can read the report of this committee in this MONTHLY, volume 42, May, 1935, page 263.

If this committee of your own Association could take such a position concerning secondary mathematics in 1935, it seems to me reasonable to expect that many members of this Association would be sympathetic to the idea that prospective teachers of freshman and sophomore mathematics in our colleges today should blend with their mastery of subject matter some knowledge of how the human mind acquires new ideas, how it schools itself in new procedures, and how the acquisition of these new ideas and procedures is helped or hurt by the attitudes and emotions that accompany them. I do not propose to tell you how you might alter the training of prospective college teachers from its present pattern so as to include some of the ideas on "how to teach" that I believe would be helpful. Three things, however, I will say: first, that the important ideas and attitudes that I have in mind can be put into such compact form that they can be picked up in relatively short time; second, that even a small amount of properly concentrated instruction in how to teach could result in great improvement in teaching in freshman and sophomore classes; and third, that it is futile to advocate the inclusion of such instruction in the training of prospective college teachers until college departments of mathematics see the worth of it and show that they want it. Once they decide that they want it, it will not be difficult to provide it. Until they decide that they want it, the obstacles to providing it will be numerous.

Instead, therefore, of wasting my time on proposals for modifying the present training program, I will endeavor to show the sorts of changes in instruction you might expect if prospective college teachers were brought to consider the

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human mind and how it learns. If you don't like the changes that I predict, you need only sit tight. If, in time, you come to the point where you can see advantage to the cause of college mathematics in making certain changes, you will find a way to do so.

The changes that I believe would result from giving more thought to the pupil and to the manner in which he learns would necessarily involve some changes in courses of study and in text books. These changes, as I envisage them, would be small, but their effect on the pupil would be great. The changes that interest me most all stem from one idea, namely, proper motivation of the subject-matter. This, broadly conceived, includes as a corollary the idea of withholding certain details of rigorous exposition until the pupil appreciates their import. When I say "withhold," I do not mean "suppress." However, I do mean something that you will probably not enjoy, even though you come in time to accept the idea. For it implies some disruption and some dislocation of the purely logical presentation of a subject. It implies beginning some subjects in the middle and working both ways from that first plunge. After all, has not much of mathematics been developed in this same way, beginning on the ground floor and then advancing to higher levels while new and better foundations were tucked in underneath?

The first attack on a subject, instead of being straightforward and logically ordered, would in some instances be fragmentary and messy. It would require more artful exposition than merely proceeding in logical order; for, in the interest of truth, we should wish to indicate where some of the truth is withheld, and to indicate at times also the general nature of the truth that we withhold. It implies some repetition also, in the form of orderly recapitulation; because, once the pupil appreciates the inadequacies of the initial attack, which I have described as fragmentary and messy, we do indeed want him to see the argument set down in order, with as much attention to rigor as he can accept. This explains what I had in mind by disruption and dislocation of the usual logical presentation.

The reason that I suggest a dual procedure of this sort with respect to certain topics is that the psychological approach to a subject is often quite different from the logical. I advocate the psychological approach in order to win and to hold the pupil's attention; I advocate a subsequent logical treatment because that is the goal of all our effort. I believe that if we proceed in this way the pupil will be more likely to accept and to heed the nice distinctions that interest us, and that he too will become interested in these fine points when he appreciates their significance. I think that our present approach causes us to lose many pupils, some by physical withdrawal, and others by mental and spiritual withdrawal. In one or two instances (overplaying limit of function and continuity at the start of the calculus, and overprotection against certain familiar pathological functions later on) the severely logical approach of some teachers, who seem to the pupil to personify mathematics itself, and of some authors seems to me to savor overmuch of the "Damn you, love me" technique that brings the young

lover to the front pages of our newspapers, but indicates to most of us that that is no way to woo.

The reason for my prediction above, that you will not enjoy the prospect of a dual treatment of certain topics, is that what I have called "merely proceeding in logical order" already demands exposition of a high order, as you well know; the even greater demands on the teacher's powers of exposition to provide an extra higglety-pigglety treatment out of deference to the pupil's psyche will really test the depth of your devotion to the pupil and your interest in connecting him in some way with mathematics. In case these demands seem to you to be just too much, please recall that my suggestions are aimed at only a few crucial topics in mathematics and that I am counting on great changes on the part of the pupil in return for only a few changes on our part.

Please understand that I think I appreciate the conflict in interest and attitude that are inherent in the teacher's task of bringing together the severely logical subject of mathematics, that requires thinking, just thinking all unadorned, and the mind of the human animal, that finds pure thinking difficult and is easily diverted to other methods of solving problems that to greater or less degree involve the emotions. We meet this same conflict in acute form when we attempt to use congenial exploratory and inductive methods to lure secondary pupils to contemplate the less congenial deductive aspects of demonstrative geometry. The problem there is how to teach deduction inductively.

I am sure that many college teachers regard it as altogether obvious that in beginning a new subject, say the infinitesimal calculus, they should build a good foundation and that the pupil for his part should review any bits of earlier instruction that will be needed in the new. The introductory chapter of text book after text book is built on that obvious pattern. It defines important ideas such as sequence, limit of a sequence, variable, function, limit of a function, continuous function, and announces that, if two variables have limits, the limit of the sum of the two variables is the sum of their two limits. Often the author of the text book shows that he recognizes that the pupil will consider such an approach altogether dreary. He suggests that the pupil skip these introductory ideas but remember where he can find them if ever he needs them. So the pupil is led without introduction, or by a highly repellent introduction that is worse than no introduction, to contemplate the limit of the ratio of two variables each of which is tending toward zero. And now observe the trouble we are in. The pupil who has been conducted through the introductory chapter has met the necessary ideas, but has acquired a distaste for them. The pupil who has been protected against an unfortunate emotional set toward the subject by being allowed to skip the introductory chapter finds now that he must grub around in it after all. It is quite possible that a sympathetic teacher, and there are many such, will not probe too embarrassingly at the start into the pupil's ideas concerning the limit of delta-x and the limit of delta-y. The teacher may deem it better not to molest a pupil who happens to think of delta-x as assuming only rational values as it tends toward zero. But however sympathetic the teacher,

he must face this fundamental conflict: that you can't begin the calculus without emphasizing the idea of limit, and that, whether or not the pupil has studied the introductory chapter, you can't count on his having an accurate notion of limit, ready for immediate use. To my mind, one of the most important outcomes of instruction in the infinitesimal calculus is the pupil's ultimate acquisition of sound ideas concerning variable and limit, and his appreciation of the vital role that these ideas play in establishing methods of calculating those quantities from geometry and physics that are the chief concern of the calculus and that the pupil recognizes as important and interesting. But the pupil will require time and patient handling if he is ultimately to acquire these ideas in the form that we desire. My plea is not for a cheapening of the ultimate goal of our instruction. I urge merely a compassionate approach that would save as many souls as possible at the beginning and along the way. At the beginning, therefore, I favor a method that relies heavily on the pupil's intuition and only gradually reveals to him that his intuitive ideas will bear examination and revision. I believe that I am correct in asserting that the usual formal introductory approach to the calculus is a direct affront to the pupil's intuition and for that reason is psychologically bad. I repeat, I think that such an approach antagonizes many pupils unnecessarily; that it loses us many pupils who could be saved to understand and enjoy mathematics; and that by saving them we could render a distinct service to the cause of mathematics.

I believe that the way to save them is easy; that it requires very little time and effort; and would necessitate very little change in courses of study or in the printed page. It may indeed require a considerable change in attitude at certain points. And this brings me back to my basic idea, the proper motivation of the subject-matter. The word motivation implies respect for the pupil's interests and attitudes; and it is good strategy for us to respect them, for interests and attitudes provide the drive to action. If you will trust my expressed intention that all the treasured ideas of limit, continuous function, and so on are eventually to be saved; and if my hope is realized that more pupils than formerly will be saved to appreciate these ideas; then why not introduce the pupil to the calculus by first showing him one or two examples of what the calculus can do and how it does it? Give him some reason for studying the calculus; appeal to his interests; stimulate in him a spirit of inquiry. For example, we can invent some plausible reason for wishing to know the maximum value of $5+6x-x^2$ as x varies—say, the number of inches $f(x)$ of growth of pole beans in terms of the number of inches x of rain in June—and show the bare technique by which this maximum value is computed. No mention of variable secant, no deltas, no limit; nothing but the rigmarole of passing from $5+6x-x^2$ to $6-2x$, of attaching to this latter expression the idea of slope of tangent, and then eliciting from the pupil the connection between relative maximum and zero slope, and allowing him to finish the computation himself then and there. The pupil's earlier instruction in algebra has probably already given him an idea or two concerning the critical point of a quadratic function against which he can check this new

technique of the calculus. And then, in another few minutes, another example involving a cubic function set equal to y, where our new technique, called differentiation, produces y', the slope of the tangent. And then, with equal abandon, a third example in which the pupil employs this new technique to pass from $s=16t^2$ to $s'=32t$, and is given immediately a brief picture of the connection between slope of tangent and rate of change of a function in general. In this last example it is conceivable that the pupil can suggest that rate in this case can be interpreted as velocity; at least the teacher will invite him to try to make this association. And then the teacher can suggest the possibility of reversing this operation so that, given the relation between velocity and time, one can infer a relation between distance and time. There is no attempt here at explanation; the sole intent is to give a quick preview of some of the ideas that will occupy the pupil in his study of the calculus; to show him quickly and with very simple examples how the calculus is linked to geometry and physics; to contrast the two main operations of the calculus; and, above all else, to make the subject appear in prospect to be easy. The pupil is not expected to understand much of what he has been shown. The object is rather to show him that it involves concepts already familiar to him and is the sort of thing that he can understand. A further object is to establish a feeling of confidence between pupil and teacher; for at this same time the teacher must say that certain steps in reasoning that support the simple techniques just exhibited will seem to the pupil to involve nice distinctions, the need for which is not immediately apparent, but the teacher understands this and does not expect immediate and complete acceptance of every new idea on the part of the pupil; that allowance will be made for a period of incubation. I am aware that this sounds a bit childish. But if so, is it not because we commonly forget how close the adult is to the child in matters of this sort? In brief, I am suggesting that if we defer somewhat to the psychological child in the pupil before us, we increase the likelihood that he will accept the adult mathematics that we set before him. Now, just before you start to tear this suggestion to pieces, allow me to insist that I am not discussing at this point the teaching of the calculus, and that I do not want my remarks with respect to $5+6x-x^2$ and thereafter to be taken as a sample of how I would teach those topics in the calculus. We are not in the calculus yet; only on the doorstep. I am discussing the introduction to the calculus; and not so much how to introduce the idea of limit as how to introduce the pupil. The word calculus still means to me all that it has ever implied at the beginning level with respect to the idea of limit.

I sympathize with the desire to parade at the beginning of a book on the calculus the materials of the present introductory chapter. From the point of view of the learner, however, I incline to favor the dissemination of this material throughout the book as need for it arises. That is one of the disruptions that I mentioned earlier. Then, considering the learner and also the teacher, I am inclined to favor a collection of all this material for ready reference in the appendix. And that is one of the dislocations that I mentioned.

If only the introduction could show the pupil why he is studying the calculus, and lead him at once to the derivative as usually explained, and to exercises and problems involving the derivative, I think the pupil would be better served and better pleased. The exposition of the derivative would suffer no change. The pupil would continue to be as naïve as formerly about variable and limit, but his morale would be better. Instead of being given the idea that he was doing less than expected and was getting off to a bad start because he had not entirely understood the necessary preliminaries about variable and limit, he would be given the idea that he was doing all that was expected at this time, that it was adequate for the present, but that he must expect very soon to acquire greater sensitivity to the techniques involved and to the ideas on which they rest. Most books require the pupil to apply the fundamental definition of the derivative to a series of ten or more expressions like $4x^2+3$ and $3x^2-5x+4$. I would allow the pupil to consider the earlier ones in this list as aimed chiefly at giving him facility in the technique of translating $f(x+\Delta x)-f(x)$ into terms appropriate to a given exercise. And then in subsequent exercises I would expect the teacher to be more demanding of the pupil with respect to the limit idea. Just how demanding depends at this stage on how much of this the pupil can accept.

It is possible, of course, to take the point of view that if we defer always to the pupil, he will stage a sit-down strike and baulk us in our attempt to reach our goal. Actually, I think the human animal reacts quite differently: that when pushed and crowded he does indeed sit-down and baulk; but when held responsible for displaying his own powers under conditions that give him a fair chance to do so, he surprises us by doing more than we expect. I believe it is all too easy to find examples of pupils who sit passively in our class-rooms and respond only briefly to our best efforts. I have an idea that if we should pay more attention to how pupils learn, we could discover ways of converting our best efforts into even better and more successful efforts.

In similar vein, would it not be worth while to devote a few minutes at the beginning of infinite series to explain why we interrupt the steady course of the calculus to include this topic? And why we make so much fuss over the matter of convergence? It is a rare book that lets the pupil in on the secret that $x \tan x$ cannot be integrated in terms of elementary functions; that admits that the strange and brief assortment of curves presented as exercises under length of arc are the best that can be assembled, again because of difficulties with integration; or that discloses why certain differential equations are offered as exercises and why others, much like them, are not offered. I believe that it would cheer the pupil immeasurably to learn that author and teacher are less than omnipotent, that there are things they cannot do. Why not tell him? In this connection I ask why we list in a table of integrals only those expressions that we can integrate in terms of elementary functions; why not include some of the closely related functions that cannot be handled so readily, with a simple notation to

that effect? I believe that the pupil would take hold of infinite series much better if we told him at the outset that one reason for introducing the study of infinite series into the calculus is to circumvent this barrier of non-integrability. And then, instead of proceeding to a formal exposition of the first ideas on infinite series—handing it out cold, as the pupil would say—my own inclination would be to lead the pupil to do a bit of experimenting at this point. I suggest that he try to employ the first few terms of the binomial expansion in an effort to evaluate $(2+1)^{-2}$; $(1+2)^{-2}$; $(4+1)^{1/2}$; $(1+4)^{1/2}$; $1/(1+x)$ for $x=\frac{1}{2}$; and $1/(1+x)$ for $x=2$. Nothing would be said at this time about the validity of employing the binomial expansion in this way. Further, I would suggest that the pupil integrate from $x=0$ to $x=h$ the first six terms of the expansion of $1/(1+x)$ in order to approximate the integral $\int_0^h dx/(1+x)$ and try in this way to find $\log(1+\frac{1}{2})$ and $\log(1+2)$. And I wouldn't be above asking him to contrast the expansion of $1/(1+x)$ when x equals 1 with the value of his series for $\log(1+h)$ when h equals 1. I should expect that the pupil would see some point now in considering whether a given series converges or not.

With respect to convergence, there is some danger of piling up ingenious examples that run far beyond any use that the pupil will make of this important idea. In this respect this topic is reminiscent of factoring in algebra. With respect to both topics it can be said, and truly, that the fussy cases are introduced to test the pupil's grasp of basic principles. If we recall why we introduced series and why we decided to inquire into the matter of convergence, it will be easy to determine when the pupil has had enough. Certain series serve admirably to test the pupil's grasp of the idea of limit of a sequence and deserve consideration for that reason if for no other. By now the pupil has progressed far beyond the point where we need be solicitous about his attitude toward the idea of limit; our concern at this stage is to seek occasions that will cause him to revert to the idea and test his mastery of it.

Another opportunity to appeal to the pupil's interest is afforded by the hyperbolic functions, but I do not recall any text that seizes this opportunity. To me it seems almost unavoidable that the pupil, comparing the integrals

$$\int_0^x \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

and

$$\int_0^x \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log_e\left(\frac{x}{a} + \frac{\sqrt{a^2 + x^2}}{a}\right)$$

and their geometric counterparts right-triangle-plus-circular-sector and right-triangle-plus-hyperbolic-sector, as in Figure 1, would be compelled by the striking analogy to ruminate on the result of relabeling $\log_e(x/a+\sqrt{a^2+x^2}/a)$, which

we shall call u for short, as $\sinh^{-1} x/a$. He would then be curious as to what $\sinh u$,

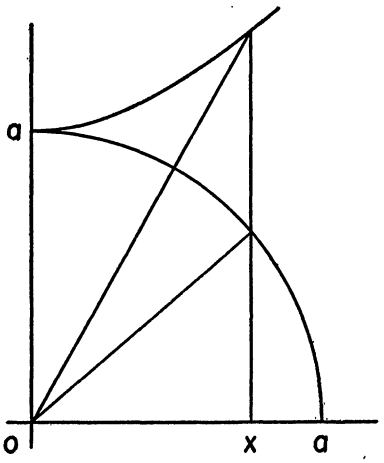


FIGURE 1

equal to x/a, might look like. He can easily satisfy his curiosity about x/a by converting

$$\log_e\left(\frac{x}{a}+\frac{\sqrt{a^2+x^2}}{a}\right)=u$$

into

$$\frac{x}{a}+\frac{\sqrt{a^2+x^2}}{a}=e^u,$$

which yields

$$\frac{x}{a}=\frac{e^u-e^{-u}}{2}=\sinh u.$$

Similarly, by comparing the negative integral

$$\int_a^x \sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{a^2}{2}\cdot\frac{\pi}{2}$$

$$=\frac{x}{2}\sqrt{a^2-x^2}-\frac{a^2}{2}\cos^{-1}\frac{x}{a}$$

and the positive integral

$$\int_a^x \sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log_e\left(\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right)$$

and their geometric counterparts right-triangle-minus-(larger)circular-sector and right-triangle-minus-(smaller)hyperbolic-sector, as shown in Figure 2, the

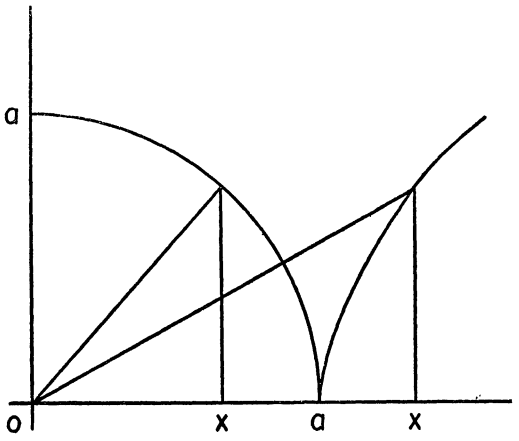


FIGURE 2

pupil would be tempted by the analogy to try the result of writing

$$\log_e\left(\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right)=v=\cosh^{-1}\frac{x}{a}\,.$$

From

$$\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}=e^v$$

he gets

$$\frac{x}{a}=\frac{e^v+e^{-v}}{2}=\cosh v.$$

Giving further rein to his fancy he invents the hyperbolic tangent, the derivatives of the hyperbolic sine, cosine, and tangent, and pokes around for the anti-hyperbolic tangent. He gets this by twisting

$$\tanh v=\frac{e^v-e^{-v}}{e^v+e^{-v}}=w$$

into

$$e^{2v}=\frac{1+w}{1-w},$$

from which he derives

$$v = \tanh^{-1} w = \frac{1}{2} \log \frac{1+w}{1-w}.$$

This brings the integrals

$$\int \frac{dx}{1+x^2} = \tan^{-1} x \quad \text{and} \quad \int \frac{dx}{1-x^2} = \frac{1}{2} \log \frac{1+x}{1-x}$$

much closer together; and the pupil will find further support for the results of this fanciful inquiry when he works out the series developments of these functions.

This takes a bit more time than just "handing it out cold"; but I believe that it pays dividends in the long run. My object is to arouse the pupil's interest, to insist that he sees the meaning of what he does, and to lead him to go on his own whenever possible. Neither you nor I want him to accept procedures blindly, trusting the word of teacher or of the book. But sometimes we get behind in our program, possibly just because we have listened long and patiently to pupils' difficulties, objections, doubts, and inquiries. Remembering the demands of the syllabus of the course we are teaching, we are tempted then to stifle questions and proceed with the program. Sometimes, of course, we must do this. But the pupil knows whether our attitude in general is to prefer his interest to the demands of the syllabus, or whether we tend to give the syllabus the right of way. What good does it do if the teacher completes the course on time, but loses the pupil on the way?

I never cease to wonder how some instructors can proceed blandly on with the discussion of topics B and C when the pupils in front of them are not yet satisfied with important details of topic A. Cannot such an instructor imagine the discontent such treatment breeds in the pupil? Does he not see that the pupil who is still dissatisfied with respect to topic A is ripe to be taught; that he is emotionally set to inquire, to test one alternative after another; in short, is emotionally primed really to study? Here is an opportunity not to be missed. It does not mean necessarily more debate in class at this time; it may be better to suggest that the pupil—it is almost time to call him the "student"—follow this or that line of inquiry and report on it at the next meeting. Almost anything will serve that pays attention to the pupil at such a time. It is important that the instructor remember on his part to follow up the pupil's inquiry at their next meeting.

Ignoring a pupil who is emotionally set to learn tends to turn him against the instructor and against the course. What is the syllabus for, anyhow, that it is more important than the pupil?

What does the pupil bring to your class? Some proficiency; some misconceptions; some interest; some fear. It is the teacher's task to increase this proficiency; to replace misconception with proper understanding; to hold whatever interest the pupil may have had and to add to this interest. The teacher can do

little of any of these, however, unless he dispels the pupil's fear. Well, what is the pupil afraid of? He is afraid that he will not succeed with worded problems; that he will not really understand the "why" and "what for" of the various mathematical maneuvers he is expected to perform. He inclines to seek the security of a sure rule of thumb. You know where that will lead him! Consequently the teacher's chief task is to dispel these doubts; to create an atmosphere of confidence, in which the pupil will expose his ignorance and misconceptions freely, and will even welcome the teacher's probing to uncover other weaknesses that the pupil has not confessed. In such an atmosphere doubtful points in the teacher's exposition and in the book will be cleared up. Exercises that encourage the pupil to discover mathematics himself in his own imperfect way are an aid to this end; and so are exercises that allow for plenty of learning by actual doing. Admittedly these things all take time. But it is amazing how rapidly a pupil can acquire the information we want him to acquire if only we take pains to give him the proper emotional set first of all. Teachers in general know that they ought to provide an effective motive for learning; that they ought to allow some time for the pupil to discover some of mathematics for himself; and that there is an efficacy in the pupil's own doing that outweighs that which is done for him. But there are schedules to be maintained; the class must cover the ground. In many cases, if only the teacher would begin by concentrating on the pupil and ignoring the syllabus, he would discover that the pupil finally will rescue the syllabus for himself. There is much to be said for a slow beginning, to save as many souls as possible; those who survive can tolerate cheerfully a more rapid pace at the end.

It is my belief that many teachers and authors of textbooks recognize the merit of these ideas. Why, then, are they not more generally adopted? I will hazard the guess that it is partly because of ignorance and partly because of fear. Our knowledge of mathematics is relatively sure; our knowledge of human beings is uncertain. Then, too, it is more respectable to be concerned with the subject than with the pupil. If the teacher teaches, or the author writes, with his attention chiefly on the pupil, he risks suspicion as one who does not "know his stuff." I think that there are ways by which one who is really master of his subject can protect his scholarly reputation by adroit qualifications and occasional foot-notes. Some authors have shown that they can do this. I wish that more would follow suit.

The environment in which the college teacher works is concerned chiefly with his subject. What forces act on him in behalf of the pupil? If only it could be generally understood that every author has heard of the so-called "pathological functions" and was guaranteed free of attack because he seemed to ignore them, I think the pupil would be pleased and would progress more rapidly to the point where he too could join safely in the sacred rites surrounding them. I recognize the important part that exceptional cases and famous puzzles have played in the development of mathematics. But we can never attain a completely rigorous presentation on the adult level even; we must draw the line

somewhere. You can see that I think that we commonly draw it with too little regard for the pupil.

In this plea for greater consideration of the pupil's point of view I have drawn my illustrations from the calculus and have said nothing with respect to the subjects that normally precede it. You will not be surprised, however, at my opinion that, from the point of view of the subject, the best way to begin analytic geometry is in terms of directed line segments and their projections on the axes; but that, from the point of view of the pupil, such a beginning is unfortunate. My own inclination would be to rely at first on the pupil's intuition concerning the general validity of x_2-x_1 as the projection on the x-axis of the directed line segment P_1P_2. I am fully aware that the generality of the algebraic processes that we employ in analytic geometry forms the very heart of the subject; but I prefer to insinuate this idea gradually as the course develops. I believe that direct frontal assault causes the loss of too many of the participants; that repeated small forays at different times and places will bring more participants to the desired goal. Furthermore, directed line segments do not quite live up to the high hopes we have of them. It is a bit of an art, indeed one of the trickiest aspects of analytic geometry for the beginnner, to know when directed line segments are to be respected and when they can be safely ignored. I might even say "profitably ignored," having in mind the possible loss of the inner loop of the limaçon $r=3-4\cos\theta$ by too stuffy insistence on positive r's only. You know when you think it well to consider length, area, and volume as unsigned quantities and when you would be relieved of this restriction in order to win some important generalization. To the pupil the rules of the game are not so clear.

Again, in trigonometry, the majority of college teachers who write texts in this subject seem to think that if they begin with general angle they can clean up on the elementary functions once and for all. In my opinion the pupil would prefer to be introduced to the elementary functions as they appear in right triangles, with some subsequent extension short of the general angle to accommodate obtuse-angled triangles. I am aware that the style has turned a bit from the solution of triangles and that analytic trigonometry is more in favor. I think I know why, and I agree. I mention, merely, that I think that the approach through triangles means more to the pupil; and I would not ignore the pupil's point of view.

In summary, I advocate giving the game back to the pupil. Let him find out a few things for himself in the trial-and-error method by which we all learn. I would not be so solicitous for his safety in the mathematical world. A good soaking may teach him more than he would ever learn if he started out with full equipment of rubbers, raincoat, and umbrella. Actually this metaphor does not express my whole meaning, because I want him to return home after the first shower and get more equipment, once he sees what he is in for. Of course, it would make things easier at home if he would start out in the first place with all the equipment he might possibly require, even including crampons, and perhaps

water and lunch, in case he is marooned on a glazed highway and cannot move. That is the way some recent textbooks impress me. They care for every emergency in advance. That way everything is neat and orderly. But most of it is done by the guide; little initiative is left to the pupil.

Changing the metaphor, I would say that on the whole our mathematical house is in good order and that we take pride in keeping it so. I am not asking for a thorough house-cleaning. Quite the contrary. The house is spick and span now and needs no house-cleaning. I think, however, that it does need a little room that the children can call all their own, where a certain amount of disorder is tolerated, provided that from time to time the children are expected to tidy up the room according to their own standards of what is neat and orderly. They will not lack examples of adult standards of neatness and order in all other parts of the house.

A CLASS OF NON-DESARGUESIAN PLANE GEOMETRIES*

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1. Introduction. F. R. Moulton has shown† that Desargues' theorem is not a consequence of certain of Hilbert's axioms‡ by exhibiting a simple non-desarguesian plane geometry fulfilling these axioms. The points of this geometry are points of the euclidean plane, the lines of non-positive slope are euclidean lines, and the lines of positive slope are euclidean broken lines, broken at the x-axis so that the slope in the upper half-plane is a positive constant (not unity) times the slope in the lower half-plane.§

The present paper generalizes Moulton's paper in two ways. First, his geometry is generalized by consideration of the broken-line plane geometries defined by an arbitrary functional relationship between the upper and lower half-plane angles. The class of functions which define projective plane geometries is determined, and the non-desarguesian projective planes are found. Second, the non-desarguesian property of Moulton's geometry is generalized to show that Desargues' theorem is one of a large class of theorems which are not valid in each of the non-desarguesian projective planes.

2. The broken-line projective geometries. The points of our geometries consist of points of a real euclidean plane, and the lines include a euclidean axis creating an upper and lower plane and the euclidean parallels to this axis. The remaining lines of our geometries are the broken lines meeting the axis at an an-

* The author is indebted to Professor R. H. Bruck for his encouragement and advice in the preparation of this paper.

† A simple non-desarguesian plane geometry, Trans. Amer. Math. Soc., 3, 1902, pp. 192–195.

‡ Axioms I 1–2, II, III, IV 1–5, V. See Hilbert, Foundations of Geometry, 1902, chap. I.

§ The slope ratio is usually taken as two to one. See M. Pasch and M. Dehn, Vorlesungen über Neuere Geometrie, 1926, p. 256.

gle θ in the upper plane and an angle ϕ in the lower plane, as shown in Figure 1. Evidently, it is only necessary to consider a relationship between the angles θ and ϕ of these lines to be given by a single-valued function $\phi=f(\theta)$, with a single-valued inverse, such that $0\leqq\theta\leqq\pi$, $0\leqq\phi\leqq\pi$, $f(0)=0$, and $f(\pi)=\pi$. Ordinary euclidean geometry is thus defined by $f(\theta)\equiv\theta$, $0\leqq\theta\leqq\pi$.

As plane axioms of projective geometry we take Veblen and Young's assumptions of alignment,* which are as follows:

A1. If A and B are distinct points, there is at least one line on both A and B.

A2. If A and B are distinct points, there is not more than one line on both A and B.

A3. If A, B, C are points not all on the same line, and D and E $(D\neq E)$ are points such that B, C, D are on a line and C, A, E are on a line, there is a point F such that A, B, F are on a line and also D, E, F are on a line.

Since all the geometries satisfying these assumptions are desired, we seek a necessary and sufficient condition on the function $\phi=f(\theta)$ so that the geometries defined by the functions satisfy these assumptions. For this purpose, define S as the class of single-valued strictly monotonic increasing continuous functions in

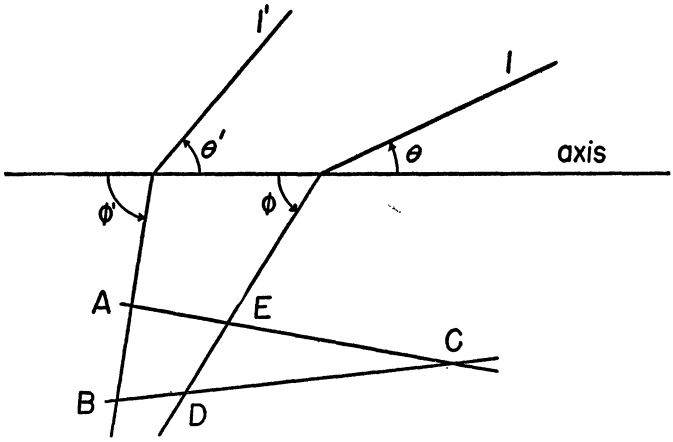


FIGURE 1

the closed interval $(0, \pi)$, with endpoints $(0, 0)$ and (π, π). We will prove

THEOREM 1. *A necessary and sufficient condition that a function $\phi=f(\theta)$ define a geometry satisfying the assumptions of alignment is that $\phi=f(\theta)$ be a member of class S.*

First, assume that a function $\phi=f(\theta)$ defines a geometry satisfying the assumptions of alignment. By definition, the function has the endpoints $(0, 0)$ and (π, π), and the function, and its inverse, are single-valued in the closed interval $(0, \pi)$. The strictly increasing monotonicity can be shown by considering lines l and l' (Fig. 1) for two values θ and θ', with $\theta'>\theta$. Lines CEA and CDB can be drawn as indicated, so lines l and l' must intersect by **A3.** If $\varphi'<\phi$ lines l and l'

* Projective Geometry, Veblen and Young, vol. I, 1910, p. 16.

would not interesct, and if $\phi' = \phi$ the inverse function would not be single-valued. Thus $\phi' > \phi$, as was to be shown. Finally, **A1** requires that the function take on every value in the open interval $(0, \pi)$ for θ in that interval.

We have, then, that $f(\theta)$, for the closed interval $(0, \pi)$, is a single-valued strictly monotonic increasing function, with endpoints $(0, 0)$ and (π, π), taking on every value in the interval. Such a function is continuous in the closed interval, since the monotonicity requires that it can have simple discontinuities at most, in which event it could not take on all values in the interval.* The function $\phi = f(\theta)$ is thus of class S and the necessity of the condition is established.

We now assume that a geometry is defined by a function $\phi = f(\theta)$ of class S. Assumptions **A1** and **A2** will be satisfied if there is one and only one line through any two distinct points A and B. This is obvious when both A and B are in the same half-plane and when one or both points are on the axis. The remaining case is when A and B are in different half-planes, say A in the upper plane and B in the lower plane. The problem is to show that there is a unique breaking point along the axis such that the angles (θ, ϕ) formed by connecting the breaking point to A and B, respectively, satisfy the given functional relationship $\phi = f(\theta)$. It is evident geometrically that the function $\phi = g(\theta)$, obtained by varying the breaking point along the axis, is a single-valued strictly monotonic decreasing continuous function in the open interval $(0, \pi)$. This can be verified by dropping perpendiculars from A and B on the axis and noting that the function has the form $\cot \phi = a + b \cot \theta$, $b < 0$.† Defining $g(0) = \pi$ and $g(\pi) = 0$, we can describe $g(\theta)$ in the same terms used in defining $f(\theta)$, if we substitute "decreasing" for "increasing" and reverse the endpoint values. If the graphs of two functions so related were drawn on the same coordinate axes, the existence of the desired unique simultaneous solution would be geometrically evident. An analytic proof is obtained by consideration of the difference function $h(\theta) = g(\theta) - f(\theta)$. Since $h(\theta)$ is a single-valued strictly monotonic decreasing continuous function in the closed interval $(0, \pi)$ with endpoints $(0, \pi)$ and $(\pi, -\pi)$, it takes on the value zero once and only once, in the open interval $(0, \pi)$. Thus **A1** and **A2** are satisfied.

Assumption **A3** is satisfied if any two lines l and l', intersect. This is evident when at least one of the lines is the axis or a line parallel to the axis (ideal points exist as in euclidean geometry). When neither l nor l' is the axis or parallel to it, they both intersect the axis in ordinary points. If these two intersection points are not distinct this common point is the required intersection. For distinct intersections with the axis, refer to Figure 1, showing l and l'. If $\theta' < \theta$, there is an ordinary intersection in the upper plane, and if $\theta' = \theta$, there is an ideal intersection. If $\theta' > \theta$, we have $\phi' > \phi$ by the strict monotonicity condition on our function, and there is an ordinary intersection in the lower plane. In any event, an intersection exists and **A3** is satisfied. Thus the sufficiency of the condition is established, and we conclude that class S is the class of functions which define

* See Hardy, Pure Mathematics, 1925, p. 181.

† In this paper, the values of the inverse cotangent are not necessarily principal values, but are taken in the first and second quadrants.

the geometries satisfying the assumptions of alignment.

Note that the geometries defined by the functions of class S are actually plane geometries, since all points are on lines joining points of the axis to a particular point not on the axis.*

3. The non-desarguesian geometries. In what follows, we will refer interchangeably to a geometry and the function defining the geometry. All geometries considered will be understood to be of class S. Terminology involving an angle couple (θ, ϕ) of a geometry will refer to a pair of corresponding values of these angles in the open interval $(0, \pi)$ of the function defining the geometry. Note that two couples (θ, ϕ) and (θ', ϕ') of a geometry will necessarily have a strictly monotonic increasing relation to each other, *i.e.* $\phi' < \phi$ according as $\theta' \geqq \theta$.

Consider two distinct couples of a geometry, (θ_1, ϕ_1) and (θ_2, ϕ_2), and draw the corresponding lines through a point P in the upper plane as shown in Figure 2. For any other couple (θ, ϕ) of the geometry, draw the line through P and construct triangles $A_1B_1C_1$ and $A_2B_2C_2$ as shown, with $A_1B_1 \| A_2B_2$ and $A_1C_1 \| A_2C_2$. If the geometry is assumed to be desarguesian, the two triangles, being perspective from the point P, would be perspective from the line at infinity. Then clearly, a second desarguesian geometry having the couples (θ_1, ϕ_1) and (θ_2, ϕ_2)

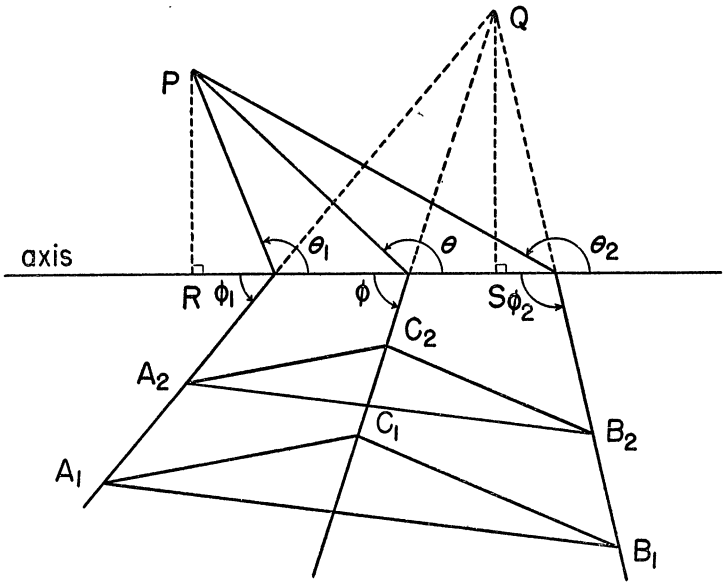


FIGURE 2

would have the couple (θ, ϕ) also, because C_1C_2P would be a line in the second geometry. Thus the two desarguesian geometries would be identical since (θ, ϕ)

* Reference is to Veblen and Young's conception of a plane and a plane geometry (*l.c.*, chap. I).

was any couple of the original desarguesian geometry other than the two common couples (θ_1, ϕ_1) and (θ_2, ϕ_2). We therefore arrive at

THEOREM 2. *With the possible exception of but one geometry, all the geometries of class S having two given distinct couples are non-desarguesian.*

COROLLARY. *All the geometries of class S having at least two couples in common with euclidean geometry (except euclidean geometry itself) are non-desarguesian.*

The corollary follows from the fact that euclidean geometry is desarguesian. As an application of the corollary, we note that Moulton's geometry is a non-desarguesian geometry of class S, since it is defined by

$$\begin{cases} \tan\phi = c\tan\theta, & \text{for} \quad 0 < \theta < \dfrac{\pi}{2}, \ c > 0, \ c \neq 1 \\ \phi = \theta, & \text{for} \quad \theta = 0 \text{ and } \dfrac{\pi}{2} \leqq \theta \leqq \pi, \end{cases}$$

and has at least two couples in common with euclidean geometry (defined previously by $\phi = \theta$, $0 \leqq \theta \leqq \pi$).

From the foregoing discussion of non-desarguesian geometries, it may be thought likely that all geometries of class S might be non-desarguesian (with the exception of euclidean geometry). This is not true, and the non-desarguesian proper subclass of S will now be determined.

Define T as the class of functions given by the two parameter family, $\cot\phi = a + b\cot\theta$, $b > 0$, $0 < \theta < \pi$, with unique endpoints $(0, 0)$ and (π, π). We will prove

THEOREM 3. *The functions of class T define desarguesian geometries of class S.*

It is evident that a function of class T defines a geometry of class S. In order to show that Desargues' theorem holds for the geometries of class T, a general principle for these geometries will be established.* Consider a transformation taking any point P, at a height $p = PR$ above the axis, to a point Q, at a height $q = QS$ above the axis and at a distance $d = RS$ from P measured parallel to the axis with positive direction to the right (refer to Figure 2). Let the transformation equations be $q = (1/b)p$, and $d = (a/b)p$, where a and b are the parameters of a function of class T. If we take any couples of the geometry of this function of class T, such as (θ_1, ϕ_1), (θ_2, ϕ_2), and (θ, ϕ), it is seen that the euclidean prolongations of the lower plane segments of the corresponding lines through P will go through Q. We thus see that any configuration of points and lines in a geometry of class T can be transformed, by euclidean prolongations of the lower plane line segments, into a corresponding euclidean configuration such that concurrent lines go into concurrent lines and collinear points go into collinear points (the case of an ideal intersection is obvious, and the case of a line parallel to the axis is

* This principle was suggested by Professor S. B. Jackson.

handled by the first transformation equation). The converse situation is evidently true also. That is, upper plane lines and points of a euclidean configuration can be transformed back in the same way to yield a corresponding configuration in a geometry of class T. This establishes the general principle that a theorem relating to points of intersection is true in the geometries of class T if and only if it is true in euclidean geometry. In particular, then, we have Desargues' theorem true in the geometries of class T.

Noting that there is a geometry of class T having any two given distinct couples of a geometry of class S, we can remove the uncertainty in Theorem 2, and get

THEOREM 4. *With the exception of one desarguesian geometry (of class T), all the geometries of class S having two given distinct couples are non-desarguesian.*

Defining class $U \equiv S - T$, we have the theorem which may be regarded as the objective of this section.

THEOREM 5. *Class U is the class of non-desarguesian geometries of class S.*

4. A property of the non-desarguesian geometries. Pappus' theorem does not hold in our non-desarguesian projective planes by virtue of Hessenberg's proof that Desargues' theorem is a consequence of Pappus' theorem.* Hilbert and others have shown that the validity of Desargues' theorem in a projective plane is equivalent to the possibility of coordinatizing the plane by the von Staudt construction with elements from a skew-field, and that the skew-field will be a commutative field if and only if Pappus' theorem holds.†

The question arises as to what theorems can hold in our geometries of class U. We note that the method of proof used in the preceding section to show that these geometries are non-desarguesian can also be used to show that Pappus' theorem does not hold. The method can be used in general to show that all non-trivial constructible intersection point theorems of euclidean geometry are not true in our non-desarguesian planes. A definition of non-trivial constructible intersection point theorems is given by Moufang.‡ (The theorems of Desargues and Pappus and the theorem of the complete quadrilateral are examples of theorems of this type.)

To apply the method in general, consider a configuration for such a theorem of euclidean geometry constructed in a euclidean plane, with an axis drawn such that one of the intersection points of the configuration is isolated in the upper half-plane while the remaining intersection points are in the lower plane. An intersection point can be isolated by an axis in this manner since the definition

* G. Hessenberg, Beweis des Desarguesschen Satzes aus dem Pascalschen, Math. Ann., 61, 1905, pp. 161–172.

† See M. Pasch and M. Dehn (*l.c.*), and Marshall Hall, Projective planes, Trans. Amer. Math. Soc., 54, 1943, pp. 229–277.

‡ Ruth Moufang, Zur Struktur der projektiven Geometrie der Ebene, Math. Ann., 105, 1931, pp. 536–601.

of an intersection point theorem requires that there be only a finite number of intersection points in the configuration. Thus it is seen that an intersection point at the maximum distance from any particular point of the plane can be so isolated. Note that the definition of a non-trivial intersection point theorem also requires at least three lines on a point and three points on a line so that we have at least three lines of the configuration concurrent at the isolated point and cutting the axis in ordinary points. With the configuration of the theorem thus established in a non-desarguesian plane, it is seen that the arguments of the preceding section apply as before to show that the theorem is false in every one of the geometries of class U.

Thus far we have only considered non-trivial constructible intersection point theorems of euclidean geometry. However, we can show that possible theorems which do not hold in euclidean geometry cannot hold in our non-desarguesian planes either. For, consider the configuration of such a theorem constructed in a euclidean plane. Establish the configuration in a non-desarguesian plane by drawing an axis isolating it completely in the lower half-plane. Then, if the theorem were valid in the non-desarguesian geometry, it would hold in euclidean geometry also. We may now state

THEOREM 6. *All non-trivial constructible intersection point theorems are false in each of the geometries of class U.*

This yields, by elementary means, the result of Moufang (*l.c.*) to the effect that all non-trivial constructible intersection point theorems are independent of the plane axioms of projective geometry. It is interesting to note that a common geometric model (Moulton's geometry, for example) can be exhibited to show this independence for all non-trivial constructible intersection point theorems, and that each geometry of class U will serve as such a model.

CARDAN AND THE PASCAL TRIANGLE

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It is well known that the arithmetic triangle, in one or another variant of the form

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

appeared long before Pascal composed his famous *Traité du triangle arithmétique*

(published posthumously in 1665). The device may have been known to Omar Khayyam (c. 1100), it appeared in China in 1303, and it was published in Europe by Apian in 1529. [1] Similar forms of the triangle were given before Pascal by numerous men, including Stifel (1544), Scheubel (1545), Peletier (1549), Rudolph (1553), Tartaglia (1556), Stevin (1585), Girard (1629), Oughtred (1631), and Briggs (1633). [2] However, it seems still to be widely believed [3] that "the triangular array was investigated by Pascal (1654) under a new form, substantially as follows":

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots & & \\ 1 & 3 & 6 & \cdots & & & \\ 1 & 4 & \cdots & & & & \\ 1 & \cdots & & & & & \end{array}$$

So significant did Cantor, Wieleitner, and Tropfke regard the difference in form that they declined to recognize any dependence of Pascal upon the earlier instances. [4] There also is a general impression that to Pascal is due the first study of the relationships exhibited by the triangle and their application to questions in the theory of probability. [5] It is the purpose of this note to call attention to the work of one whose name has not been associated with the arithmetic triangle but who anticipated Pascal with respect both to the form and the study of the triangle. This man, perhaps the greatest mathematician of his day, was Jerome Cardan (1501–1576).

The *Ars magna*, which appeared in 1545, contains no reference to the arithmetic triangle; but in 1570 Cardan published his *Opus novum de proportionibus*, and in this work the Pascal triangle appears in both forms and with varying applications. In connection with the problem of the determination of roots of numbers, Cardan used the familiar earlier form, citing Stifel as the putative discoverer. Here he gave the numbers in the triangle through $n=17$, and he pointed out the relationship, known to Stifel, equivalent to

$$\binom{m}{n}+\binom{m}{n+1}=\binom{m+1}{n+1}$$

[6]. Later on in the *Opus novum* the question of combinations and probabilities is taken up, and then Cardan gave the arithmetic triangle, through $n=11$, in virtually the form later made famous by Pascal. [7] Even in the case of Cardan, however, this form was not entirely original, for in 1556 Tartaglia had published it in a somewhat similar square array, [8] and had used it in determining the coefficients in the expansion of the twelfth power (*cubo-censo-censo*) of a binomial (*una quantita divisia in due parti*). In connection with the newer arrangement of the triangle Cardan reiterated the familiar formula known by his name and enunciated by him many years before—the total number of different combina-

tions which can be formed from n different objects, taking as many at a time as one pleases, is given by 2^n-1. As an illustration of the usefulness of his result, he determined the number of possible combinations of 25 different objects, the result being $2^{25}-1=16777215$. Following this, Cardan gave something more important which seems to have been overlooked by historians and which incorrectly has been ascribed to Pascal. [9] This is the rule of succession, equivalent to

$$\binom{n}{r}=\frac{n-r+1}{r}\binom{n}{r-1}.$$

The rule is not given symbolically but is expressed rather awkwardly as follows: To obtain the third element in the row corresponding to $n=11$, for example, subtract one from eleven, divide by two, and then multiply by eleven. The result, 55, is the number desired. To obtain the next number, subtract two from eleven, divide by three, and multiply by 55. Continuing in this manner, Cardan obtained all of the elements in the sequence of combinations of eleven objects. [10] Had Cardan applied his rule to the expansion of binomials, he would have anticipated the binomial theorem for positive integral powers. Instead he emphasized the connection between the numbers of the arithmetic triangle and "mixed proportions," *i.e.*, progressions of higher order, and the applicability of these to musical theory. There can be little doubt but that Cardan, like Tartaglia, was aware that the elements in the triangle are coefficients in the expansions of binomials. That a clear-cut statement of Cardan's rule of succession as applied to the binomial theorem should have been delayed for about another century is one of the anomalies in the development of mathematics, and that the arithmetic triangle should be named for Pascal rather than for one of his many anticipators is largely an accident of history.

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6. Jerome Cardan, Opera omnia (10 vols., Lugduni, 1663), IV, 529.

7. *Ibid.*, IV, 557.

8. Niccolo Tartaglia, General trattato di numeri et misure (part II, Vinegia, 1556), book I, fol. 17; book II, fols. 69–73. H. G. Zeuthen, Geschichte der Mathematik im XVI. und XVII. Jahrhundert (German ed., Leipzig, 1903), pp. 102, 169, and Guillaume Libri, Histoire des sciences

mathématiques en Italie (4 vols., Paris, 1838–1841), III, 158, mention the work of Tartaglia but not that of Cardan.

9. Encyclopédie des sciences mathématiques, I (1), 1 (1904), pp. 83–84. Tropfke, *loc. cit.*, ascribes Cardan's rule of succession to Briggs in 1633.

10. Opera omnia, IV, 558.

THE FINITE FOURIER SERIES AND ELEMENTARY GEOMETRY

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Every freshman knows, or soon learns, that the periodic sequence $x_0=1$ $x_1=-1$, $x_2=1$, $x_3=-1$, . . . has a general term x_n which may be written as $x_n=(-1)^n$. This is a very special case of representation of a periodic sequence by the so-called finite Fourier Series. The finite Fourier series is the analogue for periodic sequences of the ordinary Fourier series expansion of periodic functions. Its only mention in our textbook literature is under the heading of practical harmonic analysis, or trigonometric interpolation, as found in books on applied mathematics or numerical methods; and yet, the range of applications of the finite Fourier series extends beyond this important practical problem. It is intimately related to the Gaussian sums and has been used for number theoretic purposes by Eisenstein and more recently by H. A. Rademacher. The following lines present the basic properties of the (complex) finite Fourier series stressing its geometric significance and followed by a few applications to extremal problems of elementary geometry.

§1. The Finite Fourier Series

1. The finite Fourier series. We recall the general problem of polynomial interpolation: If we are given k distinct complex numbers $\omega_0, \omega_1, \cdots, \omega_{k-1}$, and a second set of k arbitrary complex numbers $z_0, z_1, \cdots, z_{k-1}$, then there is one and only one polynomial $P(x)=\zeta_0+\zeta_1 x+\cdots+\zeta_{k-1}x^{k-1}$ satisfying the equations $P(\omega_\nu)=z_\nu$ $(\nu=0, 1, \cdots, k-1)$ or

$$(1) \qquad z_\nu = \zeta_0 + \zeta_1\omega_\nu + \zeta_2\omega_\nu^2 + \cdots + \zeta_{k-1}\omega_\nu^{k-1} \qquad (\nu = 0, 1, \cdots, k-1). \quad [1]$$

Indeed, the system of linear equations (1) in the unknowns ζ_ν has a non-vanishing (Vandermonde) determinant and has therefore a unique solution. We obtain the finite Fourier series (abbreviated in the sequel to f.F.S.) if we choose the numbers ω_ν to be the kth roots of unity; so for the remainder of this paper we shall assume that

$$(2) \qquad \omega_\nu = e^{2\pi i\nu/k} \qquad (\nu = 0, 1, \cdots, k-1).$$

This equation defines ω_ν for all integral values of ν, a fact occasionally used later.

The equations (1) now represent, by definition, the f.F.S. expansion of the given sequence $z_0, \cdots, z_{k-1}$, while the coefficients $\zeta_0, \cdots, \zeta_{k-1}$, are called the f.F. coefficients of the given sequence (z_ν).

In order to solve the system (1), *i.e.* find the f.F. coefficients ζ_ν, a few properties of the roots of unity (2) are necessary. Thus (2) readily implies such relations as

$$\omega_\nu^\alpha = \omega_\alpha^\nu, \qquad \bar{\omega}_\nu^\alpha = \omega_\nu^{-\alpha}, \qquad \omega_1^\alpha = \omega_\alpha, \qquad \omega_{\alpha+1} = \omega_\alpha \omega_1, \tag{3}$$

which will be used freely. Especially important for us are the so-called orthogonality relations

$$\sum_{\nu=0}^{k-1} \omega_\nu^\alpha \bar{\omega}_\nu^\beta = \begin{cases} k & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta, \end{cases} \qquad (\alpha, \beta = 0, 1, \cdots, k-1). \tag{4}$$

They allow us to determine the f.F. coefficients as follows: If we multiply (1) or

$$z_\nu = \sum_{\mu=0}^{k-1} \zeta_\mu \omega_\nu^\mu$$

by $\bar{\omega}_\alpha^\nu = \bar{\omega}_\nu^\alpha$ and sum over all ν, we find

$$\sum_{\nu=0}^{k-1} z_\nu \bar{\omega}_\alpha^\nu = \sum_{\mu,\nu} \zeta_\mu \omega_\nu^\mu \bar{\omega}_\nu^\alpha = \sum_{\mu=0}^{k-1} \zeta_\mu \sum_{\nu=0}^{k-1} \omega_\nu^\mu \bar{\omega}_\nu^\alpha = \zeta_\alpha \cdot k$$

by (4). The f.F. coefficients are therefore given by the relations

$$\zeta_\alpha = \frac{1}{k}(z_0 + z_1\bar{\omega}_\alpha + z_2\bar{\omega}_\alpha^2 + \cdots + z_{k-1}\bar{\omega}_\alpha^{k-1}), \quad (\alpha = 0, 1, \cdots, k-1). \tag{5}$$

2. Geometric interpretation. It is both suggestive and desirable for future purposes to plot the numbers z_ν in the complex plane where they may be thought of as the successive vertices of a closed polygon $\Pi = (z_\nu)$. In particular, the sequence (ω_ν) represents there a regular k-gon $\Pi_1 = (\omega_\nu)$, while the sequence (ω_ν^α) represents a k-gon $\Pi_\alpha = (\omega_\nu^\alpha)$ obtained from Π_1 by starting from $\omega_0 = 1$ and taking every αth vertex. Π_α is a starred regular k-gon if and only if α is relatively prime to k; otherwise it is a regular polygon of lower order described several times so as to result in a total of k vertices. We also include the polygon $\Pi_0 = (\omega_\nu^0)$ all of whose k vertices coincide with $\omega_0 = 1$. A glance at the f.F.S. (1) now shows that our polygon $\Pi = (z_\nu)$ has been analyzed harmonically, *i.e.* has been represented as a linear combination of the fundamental k-gons Π_α in a manner suggested by the symbolic relation $\Pi = \zeta_0\Pi_0 + \zeta_1\Pi_1 + \cdots + \zeta_{k-1}\Pi_{k-1}$.

It is sometimes desirable to re-write the expansion (1) by grouping its terms as follows:

$$z_\nu = \zeta_0 + (\zeta_1\omega_\nu + \zeta_{k-1}\omega_\nu^{k-1}) + (\zeta_2\omega_\nu^2 + \zeta_{k-2}\omega_\nu^{k-2}) + \cdots,$$

or

$$z_\nu = \zeta_0 + (\zeta_1\omega_\nu + \zeta_{k-1}\bar{\omega}_\nu) + (\zeta_2\omega_\nu^2 + \zeta_{k-2}\bar{\omega}_\nu^2) + \cdots, \tag{6}$$

where the last term is the central term of (1), or the sum of the two central terms of (1), depending on whether k is even or odd. Also the individual terms of (6) have a simple geometric interpretation which I learned orally from Fritz John. Let us look at the first term

$$z_\nu' = \zeta_1\omega_\nu + \zeta_{k-1}\bar{\omega}_\nu, \qquad (\nu = 0, \cdots, k-1). \tag{7}$$

Separating reals and imaginaries by setting

$$z_\nu' = x_\nu' + iy_\nu', \quad \zeta_1 = a_1 + ib_1, \quad \zeta_{k-1} = c_1 + id_1, \quad \omega_\nu = \cos\frac{2\pi\nu}{k} + i\sin\frac{2\pi\nu}{k},$$

we find the coordinates of the point (7) to be

$$x_\nu' = (a_1 + c_1)\cos\frac{2\pi\nu}{k} + (-b_1 + d_1)\sin\frac{2\pi\nu}{k},$$

$$y_\nu' = (b_1 + d_1)\cos\frac{2\pi\nu}{k} + (a_1 - c_1)\sin\frac{2\pi\nu}{k}.$$

If we write

$$a' = a_1 + c_1, \qquad b' = -b_1 + d_1,$$
$$c' = b_1 + d_1, \qquad d' = a_1 - c_1,$$

it is clear that the polygon Π_1', of vertices (7), is the image of the regular k-gon $\Pi_1 = (\omega_\nu)$ by the affine transformation

$$\begin{aligned} x' &= a'x + b'y, \\ y' &= c'x + d'y. \end{aligned} \tag{8}$$

Conversely, by following our relations backwards we see that we may start with an arbitrary affine transformation (8) and find that the image of Π_1, by (8), may be written in the form (7). Generally, if α is any integer in the range $1 \leqq \alpha \leqq k/2$, we find the αth term of (6), or

$$z_\nu^{(\alpha)} = \zeta_\alpha\omega_\nu^\alpha + \zeta_{k-\alpha}\bar{\omega}_\nu^\alpha \qquad (\nu = 0, \cdots, k-1), \tag{9}$$

to represent the image $\Pi_\alpha^{(\alpha)}$, of $\Pi_\alpha = (\omega_\nu^\alpha)$, by a uniquely defined affine transformation

$$\begin{aligned} x^{(\alpha)} &= a^{(\alpha)}x + b^{(\alpha)}y, \\ y^{(\alpha)} &= c^{(\alpha)}x + d^{(\alpha)}y, \end{aligned} \tag{10}$$

whose coefficients are related to those of (9) by precisely the same relations which connected (8) to (7).

We may state our result as follows: *The expansion* (6) *represents our given polygon* Π *as a uniquely defined sum of a constant* ζ_0 (*the centroid of* Π) *plus* $[k/2]$ *k-gons* $\Pi_\alpha^{(\alpha)}$ $(\alpha=1, \cdots, [k/2])$, *where* $\Pi_\alpha^{(\alpha)}$ *is an affine image of the fundamental k-gon* $\Pi_\alpha=(\omega_\nu^\alpha)$. [2]

The expansion (6) should be called the real f.F.S. of (z_ν), in contradistinction to (1) which is the complex f.F.S. These terms are justified by the following remarks:

1. If all z_ν are real then ζ_0 is real and $\zeta_{k-\alpha}=\bar{\zeta}_\alpha$ $(\alpha=1, 2, \cdots, k-1)$, and conversely. Indeed, the direct statement follows from (5), the converse one from (6).

2. If the z_ν are real then all terms of the real f.F.S. (6) are real. Evidently so, because $\zeta_\alpha\omega_\nu^\alpha+\zeta_{k-\alpha}\bar{\omega}_\nu^\alpha$ is the sum of two complex conjugate terms. The same result can be seen from (9) which turns out to be a degenerate affine transformation with $y^{(\alpha)}=0$, *i.e.* mapping the whole plane onto the real axis.

3. The analogue of the Parseval relation. [3] Besides the f.F.S.

$$z_\nu = \sum_{\alpha=0}^{k-1} \zeta_\alpha \omega_\nu^\alpha, \qquad (\nu = 0, \cdots, k-1), \tag{11}$$

let

$$x_\nu = \sum_{\beta=0}^{k-1} \xi_\beta \omega_\nu^\beta, \qquad (\nu = 0, \cdots, k-1), \tag{12}$$

be the f.F.S. of a second sequence (x_ν). We now find

$$\sum_{\nu=0}^{k-1} z_\nu \bar{x}_\nu = \sum_{\alpha,\beta,\nu} \zeta_\alpha \omega_\nu^\alpha \bar{\xi}_\beta \bar{\omega}_\nu^\beta = \sum_{\alpha,\beta} \zeta_\alpha \bar{\xi}_\beta \sum_\nu \omega_\nu^{\alpha-\beta}$$

and therefore, by the orthogonality relations (4),

$$\sum_{\nu=0}^{k-1} z_\nu \bar{x}_\nu = k \cdot \sum_{\alpha=0}^{k-1} \zeta_\alpha \bar{\xi}_\alpha.$$

In particular, if $x_\nu=z_\nu$ we find that

$$\sum_{\nu=0}^{k-1} |z_\nu|^2 = k \cdot \sum_{\alpha=0}^{k-1} |\zeta_\alpha|^2. \tag{13}$$

This is the finite Parseval relation which we shall use repeatedly.

4. The convolution of sequences. Let again (z_ν) and (x_ν) be the two sequences (11) and (12). Out of them we form a new sequence (z_ν') defined by the equations

$$z_\nu' = \sum_{\alpha=0}^{k-1} z_\alpha x_{\nu-\alpha} \qquad (\nu = 0, \cdots, k-1), \tag{14}$$

where we observe the convention of treating (x_ν) as an infinite sequence of period k, *i.e.* $x_{-1}=x_{k-1}$, $x_{-2}=x_{k-2}$, and so forth. In explicit form, the relations (14) are

$$(15)\qquad \begin{aligned} z_0' &= z_0x_0 + z_1x_{k-1} + z_2x_{k-2} + \cdots + z_{k-1}x_1 \\ z_1' &= z_0x_1 + z_1x_0 + z_2x_{k-1} + \cdots + z_{k-1}x_2 \\ &\vdots \\ z_{k-1}' &= z_0x_{k-1} + z_1x_{k-2} + z_2x_{k-3} + \cdots + z_{k-1}x_0. \end{aligned}$$

We say that (z_ν') is obtained from the sequences (z_ν) and (x_ν) by convolution, an operation which we indicate by the symbolic relation $(z_\nu')=(z)_\nu*(x_\nu)$. The f.F.S. of (z_ν') is readily found as follows: From (14), (11) and (12) we have

$$z_\nu' = \sum_\alpha z_\alpha x_{\nu-\alpha} = \sum_{\lambda,\mu,\alpha} \zeta_\lambda \omega_\alpha^\lambda \xi_\mu \omega_{\nu-\alpha}^\mu = \sum_{\lambda,\mu,\alpha} \zeta_\lambda \xi_\mu \omega_\nu^\mu \omega_\alpha^\lambda \bar{\omega}_\alpha^\mu$$

$$= \sum_{\lambda,\mu} \zeta_\lambda \xi_\mu \omega_\nu^\mu \sum_{\alpha=0}^{k-1} \omega_\alpha^\lambda \bar{\omega}_\alpha^\mu,$$

and finally, by (4),

$$z_\nu' = \sum_{\lambda=0}^{k-1} (\zeta_\lambda \xi_\lambda k)\omega_\nu^\lambda, \qquad (\nu = 0, \cdots, k-1).$$

This being the f.F.S. of (z_ν'), we see that the f.F. coefficients (ζ_ν') of the convolution $(z_\nu')=(z_\nu)*(x_\nu)$ are given by

$$(16)\qquad \zeta_\nu' = \zeta_\nu \xi_\nu k \qquad (\nu = 0, \cdots, k-1).$$

We wish to state this result in a somewhat more convenient form. We may look on the convolution operation $(z_\nu')=(z_\nu)*(x_\nu)$ as the linear transformation (15) from the variables (z_ν) to the new variables (z_ν'). (15) may be described as a cyclic transformation because the successive rows of its matrix are obtained from its first row by successive cyclic permutations. Now (16) shows that the f.F. coefficients of (z_ν') are obtained by multiplying those of (z_ν), *i.e.* the (ζ_ν), by the factors

$$\xi_\nu k = x_0 + x_1\bar{\omega}_\nu + x_2\bar{\omega}_\nu^2 + \cdots + x_{k-1}\bar{\omega}_\nu^{k-1}$$

which we prefer to write in the equivalent form

$$\xi_\nu k = x_0 + x_{k-1}\omega_\nu + x_{k-2}\omega_\nu^2 + \cdots + x_1\omega_\nu^{k-1}.$$

If we change our notation by defining $a_0=x_0$, $a_1=x_{k-1}$, $a_2=x_{k-2}$, $\cdots$, $a_{k-1}=x_1$, we may state our result as

THEOREM 1. *If* (ζ_ν) *are the f.F. coefficients of a sequence* (z_ν), *and if we subject the sequence* (z_ν) *to the cyclic transformation*

$$(17)\qquad \begin{aligned} z_0' &= a_0z_0 + a_1z_1 + a_2z_2 + \cdots + a_{k-1}z_{k-1} \\ z_1' &= a_{k-1}z_0 + a_0z_1 + a_1z_2 + \cdots + a_{k-2}z_{k-1} \\ &\vdots \\ z_{k-1}' &= a_1z_0 + a_2z_1 + a_3z_2 + \cdots + a_0z_{k-1} \end{aligned}$$

then the f.F. coefficients of the new sequence (z_ν') *are*

$$(18)\qquad \zeta_\nu' = \zeta_\nu f(\omega_\nu) \qquad (\nu = 0, \cdots, k-1),$$

where

$$(19)\qquad f(z) = a_0 + a_1z + \cdots + a_{k-1}z^{k-1},$$

and will be referred to as the representative polynomial of the cyclic transformation (17). [4]

By the finite Parseval relation (13) we have

$$\sum |z_\nu'|^2 = k\sum_\nu |\zeta_\nu'|^2.$$

If we substitute on the right-hand side the values (18) we obtain the following

COROLLARY. *The relations* (17) *imply the identity*

$$(20)\qquad \sum_{\nu=0}^{k-1} |z_\nu'|^2 = k\cdot\sum_{\nu=0}^{k-1} |f(\omega_\nu)|^2 |\zeta_\nu|^2,$$

where (ζ_ν) *are the f.F. coefficients of the sequence* (z_ν), *while* $f(z)$ *is defined by* (19).

The relation (20) should be regarded as an identity in the k variables (z_ν), the (ζ_ν) being given in terms of the (z_ν) by (5). It has a number of applications to elementary geometry to which we now turn.

§2. GEOMETRIC APPLICATIONS

5. The MONTHLY Problem no. 3547. The problem referred to was proposed by Martin Rosenman in 1932 and may be stated as follows:

Let $\Pi = P_0P_1 \cdots P_{k-1}$ *be a closed polygon in the plane* $(k \geqq 2)$. *Denote by* $P_0', P_1', \cdots, P_{k-1}$, *the midpoints of the sides* $P_0P_1, P_1P_2, \cdots, P_{k-1}P_0$, *respectively, obtaining the first derived polygon* $\Pi' = P_0'P_1' \cdots P_{k-1}'$. *Repeat the same construction on* Π^1 *obtaining the second derived polygon* Π'', *and finally, after n constructions, obtain the nth derived polygon* $\Pi^{(n)} = P_0^{(n)}P_1^{(n)} \cdots P_{k-1}^{(n)}$. *Show that the vertices of* $\Pi^{(n)}$ *converge, as* $n \to \infty$, *to the centroid of the original points* $P_0, P_1, \cdots, P_{k-1}$. [5].

Let $z_0, \cdots, z_{k-1}$ denote the complex coordinates of the vertices of Π. It is then clear that the coordinates of the vertices of the polygon Π' are obtained from (z_ν) by means of a linear cyclic transformation (17) of coefficients $a_0 = a_1 = 1/2$, $a_2 = \cdots = a_{k-1} = 0$. Its representative polynomial (19) now becomes

$$f(z) = (1 + z)/2,$$

and by (20) we have the identity

$$\sum_{\nu=0}^{k-1} |z'_\nu|^2 = k \cdot \sum_{\nu=0}^{k-1} \left|\frac{1+\omega_\nu}{2}\right|^2 \cdot |\zeta_\nu|^2, \tag{21}$$

where (ζ_ν) are the f.F. coefficients of (z_ν). We now make the further assumption

$$\sum_{0}^{k-1} z_\nu = 0, \tag{22}$$

which means that the centroid of the vertices of Π is at the origin. By (5), our assumption (22) is equivalent to $\zeta_0 = 0$, so that (21) now implies

$$\begin{aligned} \sum_{0}^{k-1} |z'_\nu|^2 &= k \cdot \sum_{\nu=1}^{k-1} \left|\frac{1+\omega_\nu}{2}\right|^2 \cdot |\zeta_\nu|^2 \leqq \max_{\nu=1,\cdots,k-1} \left|\frac{1+\omega_\nu}{2}\right|^2 \cdot k \sum_{\nu=1}^{k-1} |\zeta_\nu|^2 \\ &= \cos^2 \frac{\pi}{k} \cdot \sum_{0}^{k-1} |z_\nu|^2. \end{aligned} \tag{23}$$

In justifying the last equality sign of (23) we have used firstly, the finite Parseval relation (13) with $\zeta_0 = 0$, secondly, that the maximum occurring in (23) is $\cos^2(\pi/k)$. This last fact is immediately seen geometrically, because the points $(1+\omega_\nu)/2$, which are the midpoints of the segments joining $z=1$ to $z=\omega_\nu$, are the vertices of a regular polygon inscribed in the circle having the segment joining $z=0$ to $z=1$ as diameter. This proves the following

THEOREM 2. *Let $z_0, z_1, \cdots, z_{k-1}$ be the vertices of a k-gon such that $\sum z_\nu = 0$, and let $z'_0, z'_1, \cdots, z'_{k-1}$ be the midpoints of its sides, then*

$$\sum_{\nu=0}^{k-1} |z'_\nu|^2 \leqq \cos^2 \frac{\pi}{k} \cdot \sum_{\nu=0}^{k-1} |z_\nu|^2, \tag{24}$$

with equality if and only if

$$z_\nu = \zeta_1 \omega_\nu + \zeta_{k-1} \bar{\omega}_\nu, \qquad (\nu = 0, \cdots, k-1), \tag{25}$$

i.e. when our polygon Π is an affine transform of an ordinary regular k-gon with its center at the origin.

The inequality (24) follows from (23). To find the cases when we have the equality sign in (24) we need the further geometrically obvious fact that the equality

$$\left|\frac{1+\omega_\nu}{2}\right|^2 = \max_{\nu=1,\cdots,k-1}\left|\frac{1+\omega_\nu}{2}\right|^2 = \cos^2\frac{\pi}{k}$$

is true only if $\nu=1$, or $\nu=k-1$. Now we have equality in (24), *i.e.* in (23), if and only if $\zeta_\nu=0$ for $\nu=2, 3, \cdots, k-2$, which means that the f.F.S. of (z_ν) indeed reduces to (25).

As special cases we mention the following: 1. If $k=3$ we always have equality in (24). 2. If $k=4$ we have equality in (24) if and only if the quadrilateral Π is a parallelogram.

A solution of Problem no. 3547 is now immediate; indeed, it is clear from (18) for $\nu=0$ (or also directly) that we have $\zeta_0'=0$, hence $\sum z_\nu'=0$. We may therefore apply the inequality (24) to any pair of consecutive derived polygons $\Pi^{(s-1)}$, $\Pi^{(s)}$, obtaining

$$\sum_{\nu=0}^{k-1}\left|z_\nu^{(s)}\right|^2 = \cos^2\frac{\pi}{k}\cdot\sum_{\nu=0}^{k-1}\left|z_\nu^{(s-1)}\right|^2.$$

From these equations for $s=1, 2, \cdots, n$ we obtain

$$\sum_{\nu=0}^{k-1}\left|z_\nu^{(n)}\right|^2 \leqq \cos^{2n}\frac{\pi}{k}\cdot\sum_{\nu=0}^{k-1}\left|z_\nu\right|^2. \tag{26}$$

As $n\to\infty$, the right-hand side tends to zero, because $\cos(\pi/k)<1$, showing that

$$\lim_{n\to\infty} z_\nu^{(n)} = 0, \qquad (\nu = 0, \cdots, k-1),$$

as required by the problem.

Our last inequality (26) gives some idea of the rapidity of convergence and of its deterioration as k increases. We actually have equality in (26) for all n if and only if Π is an affine image of a regular polygon. More precise information concerning the sequence of polygons $\Pi^{(n)}$ is furnished by (18). For instance if $k=4$, then $f(\omega_2)=(1+\omega_2)/2=0$, therefore Π' and all succeeding $\Pi^{(n)}$ are parallelograms, a fact which is evident geometrically.

6. A generalization of Problem no. 3547. Our generalization is as follows. We start as before from a polygon $\Pi=(z_\nu)$ and derive from it the new polygon $\Pi'=(z_\nu')$, not by the special midpoint construction as heretofore, but by means of a fixed cyclic transformation (17). Let again $\Pi^{(n)}$ be the nth polygon derived by repeating the construction (17) n times. Under what circumstances will the vertices $P_0^{(n)}, \cdots, P_{k-1}^{(n)}$ of the polygon $\Pi^{(n)}$ converge to limit positions for an arbitrary choice of the original polygon Π? [6]

This question is readily settled if we employ the finite Fourier series of our successive polygons and Theorem 1. Indeed, if $(\zeta_\nu^{(n)})$ are the f.F. coefficients of $\Pi^{(n)}$ $(n=0, 1, \cdots)$, we find by a repeated application of (18) that

$$\zeta_\nu^{(n)} = (f(\omega_\nu))^n\zeta_\nu, \qquad (\nu = 0, \cdots, k-1). \tag{27}$$

Therefore the $\lim \zeta_\nu^{(n)}$, as $n \to \infty$, will exist for all ν and arbitrary values of (ζ_ν) if and only if, for each ν, we have either $f(\omega_\nu) = 1$, or else $|f(\omega_\nu)| < 1$. We may state our result as

THEOREM 3. *Let* $\Pi = (z_\nu)$ *be a plane k-gon and let* $\Pi' = (z_\nu')$ *be the polygon derived from* Π *by the construction described by the cyclic linear transformation* (17). *Let* $\Pi'' = (z'')$ *be the polygon obtained from* Π' *by the same construction* (17) *and finally, let* $\Pi^{(n)} = (z_\nu^{(n)})$ *be the polygon obtained after n iterations of the construction* (17). *The k limits*

$$\lim_{n \to \infty} z_\nu^{(n)} = z_\nu^{(\infty)}, \qquad (\nu = 0, \cdots, k-1), \tag{28}$$

will exist for every polygon Π *if and only if all of the k roots of unity* (ω_ν) *fall into two classes* (ω_ρ) *and* (ω_σ) *such that*

$$\begin{aligned} f(\omega_\rho) &= 1 \quad \textit{for the roots of the first class } (\omega_\rho), \\ |f(\omega_\sigma)| &< 1 \quad \textit{for the roots of the second class } (\omega_\sigma). \end{aligned} \tag{29}$$

If these conditions are satisfied, then the vertices of the limit polygon are given by

$$z_\nu^{(\infty)} = \sum_{(\rho)} \zeta_\rho \omega_\nu^\rho \qquad (\nu = 0, \cdots, k-1), \tag{30}$$

which is the sum of precisely those terms of the f.F.S. (1) *of the original polygon* $\Pi = (z_\nu)$ *which correspond to the first class of subscripts* (ρ) *according to* (29). *In particular, the limit of* $\Pi^{(n)}$ *is the centroid of the vertices of* Π *if and only if the f.F.S.* (30) *reduces to the constant term* ζ_0 *of* (1), *which is the case if and only if*

$$f(1) = 1, \qquad |f(\omega_\nu)| < 1 \quad \textit{if} \quad \nu = 1, 2, \cdots, k-1. \tag{31}$$

This theorem has already been established up to the conditions (29) inclusive. If they are verified we get, letting $n \to \infty$ in (27), the values

$$\zeta_\rho^{(\infty)} = \zeta_\rho, \qquad \zeta_\sigma^{(\infty)} = 0,$$

for the f.F. coefficients of the limit polygon $\Pi^{(\infty)}$. But then the f.F.S. of $\Pi^{(\infty)}$ is indeed (30).

7. An extremal problem. The argument which led from the identity (20) to Theorem 2 may be readily extended so as to furnish a general inequality, of the same type as (24), in terms of a general cyclic transformation (17). We prefer, however, to discuss only one further special case. Starting as before from the polygon $\Pi = (z_\nu)$, we consider the special cyclic transformation

$$z_\nu' = z_{\nu+1} - z_\nu \qquad (\nu = 0, \cdots, k-1; z_k = z_0). \tag{32}$$

This is indeed identical with (17) if $a_0 = -1$, $a_1 = 1$, $a_2 = \cdots = a_{k-1} = 0$, hence

$$f(z) = -1 + z.$$

But then (20) reduces to the identity

$$\sum_{\nu} |z_{\nu+1} - z_{\nu}|^2 = k \cdot \sum_{\nu} |\omega_{\nu} - 1|^2 \cdot |\zeta_{\nu}|^2. \tag{33}$$

From this we draw two conclusions: Firstly

$$\begin{aligned} \sum_{\nu} |z_{\nu+1} - z_{\nu}|^2 &= k \sum_{\nu} |\omega_{\nu} - 1|^2 \cdot |\zeta_{\nu}|^2 \\ &\leqq \left(\max_{\nu} |\omega_{\nu} - 1|^2\right) \cdot k \sum_{\nu} |\zeta_{\nu}|^2 \\ &= \left(\max_{\nu} |\omega_{\nu} - 1|^2\right) \cdot \sum_{\nu} |z_{\nu}|^2, \end{aligned} \tag{34}$$

by the finite Parseval relation. Secondly, assuming that $\sum z_{\nu} = 0$, or $\zeta_0 = 0$, we have

$$\begin{aligned} \sum_{\nu} |z_{\nu+1} - z_{\nu}|^2 &= k \cdot \sum_{\nu=1}^{k-1} |\omega_{\nu} - 1|^2 \cdot |\zeta_{\nu}|^2 \\ &\geqq \left(\min_{\nu=1,\cdots,k-1} |\omega_{\nu} - 1|^2\right) \cdot k \sum_{\nu=1}^{k-1} |\zeta_{\nu}|^2 \\ &= \left(\min_{\nu=1,\cdots,k-1} |\omega_{\nu} - 1|^2\right) \cdot \sum_{0}^{k-1} |z_{\nu}|^2. \end{aligned} \tag{35}$$

This proves

THEOREM 4. *Let* $\Pi = P_0P_1 \cdots P_{k-1}$ *be a plane closed* k*-gon having the point* 0 *as centroid of its vertices. The following inequalities then hold*

$$4 \sin^2 \frac{\pi}{k} \leqq \frac{\sum_{\nu} (P_{\nu}P_{\nu+1})^2}{\sum_{\nu} (OP_{\nu})^2} \leqq 4 \sin^2 \left(\frac{\pi}{k} \cdot \left[\frac{k}{2}\right]\right), \tag{36}$$

where we have equality on the left side if and only if Π *is an affine image of an ordinary regular* k*-gon; the equality sign holds on the right side under circumstances which depend on the parity of* k *as follows:*

1. *If* $k = 2p$, *we have equality on the right if and only if*

$$P_0 = P_2 = \cdots = P_{2p-2} \quad \text{and} \quad P_1 = P_3 = \cdots = P_{2p-1},$$

i.e. our polygon is a segment described to and fro p *times. The right-hand side of* (36) *reduces to the value* 4.

2. *If* $k = 2p+1$, *we have equality on the right if and only if* Π *is an affine image of that starred regular* k*-gon, inscribed in the unit circle, which has the largest side among all such starred regular* k*-gons. The right-hand side of* (36) *reduces to the value* $4 \cos^2 (\pi/2k)$.

The conditions of equality in (36) follow by inspection of the multiple relations (34) and (35). Thus the extreme right member of (36) is the value of the maximum occurring in (34). Equality occurs on the right in (36) if and only if we have equality in (34). If $k=2p$, this is the case if and only if $\zeta_0=\cdots=\zeta_{p-1}=\zeta_{p+1}=\cdots=\zeta_{k-1}=0$, *i.e.* the f.F.S. of Π reduces to

$$z_\nu = \zeta_p\omega_\nu^p = \zeta_p(-1)^\nu.$$

If $k=2p+1$ we have equality in (34) if and only if the f.F.S. of Π reduces to

$$z_\nu = \zeta_p\omega_\nu^p + \zeta_{p+1}\omega_\nu^{p+1} = \zeta_p\omega_\nu^p + \zeta_{p+1}\bar{\omega}_\nu^p,$$

which is an affine image of the starred polygon $\Pi_p=(\omega_\nu^p)$ described in the theorem (compare with (9)). The conditions of equality on the left side of (36) are obtained by a similar argument involving (35).

A few special cases are worth mentioning separately:

If $k=3$, the extreme members of (36) are equal and are 3, whence the identity

$$\sum_{\nu=0}^{2}(P_\nu P_{\nu+1})^2 = 3\cdot\sum_{\nu=0}^{2}(OP_\nu)^2.$$

If $k=4$ we have

$$2 \leqq \frac{\sum_0^3 (P_\nu P_{\nu+1})^2}{\sum_0^3 (OP_\nu)^2} \leqq 4,$$

with equality on the left if and only if $P_0P_1P_2P_3$ is a parallelogram, and equality on the right if and only if $P_0=P_2$, $P_1=P_3$.

8. The area of polygons. Let A denote the oriented area of the polygon $\Pi=(z_0, z_1, \cdots, z_{k-1})$. Let $z_\nu=x_\nu+iy_\nu$; then

$$\text{Area of triangle } Oz_\nu z_{\nu+1} = \tfrac{1}{2}(x_\nu y_{\nu+1} - x_{\nu+1}y_\nu) = \tfrac{1}{2}\Im(\bar{z}_\nu z_{\nu+1}).$$

The sum of the areas of these k triangles being the area of the k-gon, we have

$$A = \frac{1}{2}\sum_{\nu=0}^{k-1}\Im(\bar{z}_\nu z_{\nu+1}) = \frac{1}{4i}\sum_{\nu=0}^{k-1}(\bar{z}_\nu z_{\nu+1} - z_\nu\bar{z}_{\nu+1}). \tag{37}$$

The area A may now readily be expressed in terms of the f.F. coefficients of Π. Indeed, by (1) we have

$$\sum_\nu \bar{z}_\nu z_{\nu+1} = \sum_{\alpha,\beta,\nu}\bar{\zeta}_\alpha\bar{\omega}_\nu^\alpha\zeta_\beta\omega_{\nu+1}^\beta = \sum_{\alpha,\beta}\bar{\zeta}_\alpha\zeta_\beta\omega_1^\beta\sum_\nu\omega_\nu^\beta\bar{\omega}_\nu^\alpha$$

and by the orthogonality relations (4) we get

$$\sum_{\nu} \bar{z}_\nu z_{\nu+1} = k \cdot \sum_{\alpha} |\zeta_\alpha|^2 \omega_\alpha.$$

Taking the imaginary parts we obtain, in view of (37), the final expression

$$A = \frac{k}{2} \sum_{\alpha=0}^{k-1} |\zeta_\alpha|^2 \sin \frac{2\pi\alpha}{k}. \tag{38}$$

This remarkable formula reveals a few facts at a glance:

1. The area A is equal to the sum of the oriented areas of the $k-1$ *regular* k-gons $\zeta_1\Pi_1, \zeta_2\Pi_2, \cdots, \zeta_{k-1}\Pi_{k-1}$ into which Π is analyzed by its finite Fourier series (1). For indeed, the oriented area of the polygon $\zeta_\alpha\Pi_\alpha = (\zeta_\alpha\omega_\nu^\alpha)$ is visibly equal to

$$k \cdot \frac{1}{2} |\zeta_\alpha| \cdot |\zeta_\alpha| \sin \frac{2\pi\alpha}{k} = \frac{k}{2} |\zeta_\alpha|^2 \sin \frac{2\pi\alpha}{k},$$

which is the general term of (38). A second proof is obtained if we observe that A must reduce to the area of $\zeta_\nu\Pi_\nu$ if we assume in (38) that all $\zeta_\alpha = 0$ except ζ_ν.

2. The area A depends only on the absolute values $|\zeta_\alpha|$ of the f.F. coefficients of Π and not on their arguments.

We may finally derive an extremal property of the area. Indeed, from (38) we obtain by a now familiar argument

$$\begin{aligned} A = \frac{k}{2} \sum_{\alpha} |\zeta_\alpha|^2 \sin \frac{2\pi\alpha}{k} &\leqq \left(\max_\alpha \frac{1}{2} \sin \frac{2\pi\alpha}{k}\right) \cdot k \sum_{\alpha} |\zeta_\alpha|^2 \\ &= \left(\max_\alpha \frac{1}{2} \sin \frac{2\pi\alpha}{k}\right) \cdot \sum_{0}^{k-1} |z_\nu|^2. \end{aligned} \tag{39}$$

If we set

$$\gamma = \frac{1}{2} \max_\alpha \sin \frac{2\pi\alpha}{k},$$

we may conclude that

$$A \leqq \gamma \sum_{0}^{k-1} |z_\nu|^2,$$

where we have equality if and only if we have equality in (39), which is the case if $\zeta_\alpha = 0$, except for those values of α for which $\frac{1}{2}\Im\omega_\alpha = \gamma$. This proves

THEOREM 5. *Let $\Pi = P_0P_1 \cdots P_{k-1}$ be a closed k-gon in the oriented plane with not all vertices coalescent, O a further point in the plane. A being the oriented area of* Π, *consider the ratio*

$$R = \frac{A}{\sum (OP_\nu)^2}.$$

There are three different cases

1. *If* $k=4p$ *then*

$$R \leqq \frac{1}{2},$$

with equality if and only if our polygon is $z_\nu=\zeta_p\omega_\nu^p=\zeta_p\omega_p^\nu=\zeta_p i^\nu$, *with* O *at the origin, i.e.* Π *is a square of center* O *described* p *times.*

2. If $k=4p\pm1$, we have

$$R \leqq \frac{1}{2}\sin\frac{2\pi p}{k}=\frac{1}{2}\cos\frac{\pi}{2k},$$

with equality if and only if Π *is the starred* k*-gon* $z_\nu=\zeta_p\omega_p^\nu$, *and* O *is its center.*

3. *If* $k=4p+2$, *we have*

$$R \leqq \frac{1}{2}\sin\frac{2\pi p}{k}=\frac{1}{2}\cos\frac{\pi}{k},$$

with equality if and only if Π *is of the form* $z_\nu=\zeta_p\omega_\nu^p+\zeta_{p+1}\omega_\nu^{p+1}$, *with* O *at its center.*

In the particular cases when $k=3$ and $k=4$, we have the following statements:

Let $P_0P_1P_2$ *be a triangle,* O *a point in its plane, then*

$$\text{Area of } P_0P_1P_2 \leqq \frac{\sqrt{3}}{4}((OP_0)^2+(OP_1)^2+(OP_2)^2),$$

with equality if and only if the triangle is equilateral with O *at its center.*

Let $P_0P_1P_2P_3$ *be a quadrilateral,* O *a point in its plane, then*

$$\text{Area of } P_0P_1P_2P_3 \leqq \frac{1}{2}((OP_0)^2+(OP_1)^2+(OP_2)^2+(OP_3)^2),$$

with equality if and only if the quadrilateral is a square with O *at its center.*

9. The isoperimetric inequality for equilateral polygons. We shall now assume our polygon $\Pi=(z_0, z_1, \cdots, z_{k-1})$ to be equilateral, which means that

$$|z_{\nu+1}-z_\nu| = a, \qquad (\nu = 0, \cdots, k-1).$$

If we denote by $L=ka$ its perimeter, then

$$\sum_\nu |z_{\nu+1}-z_\nu|^2 = ka^2 = L^2/k,$$

so that by (33) we obtain

$$L^2 = \sum_{\nu=1}^{k-1} 4k^2\sin^2\frac{\pi\nu}{k}\cdot|\zeta_\nu|^2.$$

On the other hand, by multiplying (38) by $4k\tan(\pi/k)$ we have

$$\left(4k \tan\frac{\pi}{k}\right)A = \sum_{\nu=1}^{k-1} 4k^2 \tan\frac{\pi}{k} \sin\frac{\pi\nu}{k} \cos\frac{\pi\nu}{k} \cdot |\zeta_\nu|^2.$$

Notice that the coefficients of $|\zeta_1|^2$ in these two expansions agree; hence if we subtract these two equations, those terms cancel and we obtain

$$L^2 - \left(4k \tan\frac{\pi}{k}\right)A = \sum_{\nu=2}^{k-1} 4k^2 \sin\frac{\pi\nu}{k}\left(\sin\frac{\pi\nu}{k} - \tan\frac{\pi}{k}\cos\frac{\pi\nu}{k}\right) \cdot |\zeta_\nu|^2.$$

Since all coefficients on the right-hand side are positive, this relation proves the following last

THEOREM 6 (Due to J. Steiner). *If* $\Pi = P_0P_1 \cdots P_{k-1}$ *is a plane equilateral closed k-gon of area A and perimeter L then*

$$L^2 - \left(4k \tan\frac{\pi}{k}\right)A \geqq 0, \tag{40}$$

with equality if and only if Π *is an ordinary regular k-gon.*

The inequality (40) is called isoperimetric because it implies that among all equilateral k-gons of given perimeter L (and therefore having all sides equal to L/k), the regular k-gon has the largest area. [7]

Notes and References

1. The significance of the finite Fourier series (1) for the geometry of the triangle ($k=3$) is briefly discussed in the instructive book of Ernesto Cesàro, Elementares Lehrbuch der algebraischen Analysis und der Infinitesimalrechnung, Leipzig, 1904.

2. We denote, as usual, by $[x]$ the greatest integer not exceeding x.

3. For purposes of motivation we use here and in the next paragraph terms which are familiar from the theory of the Fourier series, a fact which of course in no way presupposes a knowledge of Fourier series theory.

4. This result may also be phrased in terms of matrices and linear transformations as follows: If we denote the linear transformations (1) and (17) by $(z) = \Omega(\zeta)$ and $(z') = A(z)$ respectively, then the matrix $\Omega^{-1}A\Omega$ assumes the diagonal form

$$\Omega^{-1}A\Omega = \begin{Vmatrix} f(\omega_0) & 0 & \cdots & 0 \\ 0 & f(\omega_1) & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & f(\omega_{k-1}) \end{Vmatrix},$$

a fact which is readily verified directly. In this form our result is substantially equivalent to a theorem of Spottiswoode on cyclic determinants; see Th. Muir, Theory of Determinants, vol. 2, p. 405. I owe this last reference to Alfred Brauer.

5. Martin Rosenman, Problem no. 3547, this MONTHLY, vol. 39, 1932, p. 239. An elegant solution was given by R. E. Huston in this MONTHLY, vol. 40, 1933, pp. 184–185.

6. Stated in terms of matrices, our question is to decide if the nth power A^n, of a given cyclic matrix A, tends to a limit as $n \to \infty$. This problem was recently solved for an arbitrary square matrix A of complex elements by R. Oldenburger, Infinite powers of matrices and characteristic roots, Duke Mathematical Journal, vol. 6, 1940, pp. 357–361, and A. Dresden, On the iteration of linear homogeneous transformations, Bulletin of the American Math. Society, vol. 48, 1942, pp. 577–579.

7. A wealth of geometric applications of the Fourier series was given by Adolf Hurwitz in his famous paper: Sur quelques applications géométriques des séries de Fourier, Mathematische Werke, Basel, vol. 1, 1932, pp. 509–554. The above given derivation of the inequality (40) by means of the finite Fourier series is due to W. Blaschke, Kreis und Kugel, Leipzig, 1916, pp. 13–20. Blaschke avoids the use of complex numbers, a fact which robs the method of much of its elegance and requires a distinction of two cases according to the parity of k.

MATHEMATICAL NOTES

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ON ALMOST PRIMES*

P. ERDÖS

1. Introduction. D. H. Lehmer [1] and others have studied odd composite numbers n which behave like primes in that they satisfy the congruence

$$2^n \equiv 2 \pmod{n}.$$

For brevity, we call such numbers almost primes. In a previous note [2] we proved that for every k there exist infinitely many square free almost primes having k distinct prime factors; this generalizes a result of Lehmer for $k \leqq 3$. In the present note we estimate from above the number of almost primes less than a given limit.

2. Theorem. Our result is the following.

THEOREM. *Let $f(x)$ denote the number of almost primes $\leqq x$. Then, for x sufficiently large, we have*

$$f(x) < x \exp\left\{-\tfrac{1}{3}(\log x)^{1/4}\right\}.$$

Remark. Since the number of primes $\leqq x$ is asymptotic to $x/\log x$, our theorem implies that the number of almost primes $\leqq x$ is very much less than the number of actual primes.

3. Proof. Let $g(n)$ be the least positive exponent e such that

$$2^e \equiv 1 \pmod{n}.$$

We separate the almost primes $n \leqq x$ into two classes C_1 and C_2. The class C_1 consists of those n's for which

$$g(n) \leqq [\exp((\log x)^{1/2})] = H,$$

while C_2 consists of all the other almost primes $\leqq x$.

The members of C_1 are divisors of

* Revised by D. H. Lehmer.

$$P = \prod_{r=1}^{H} (2^r - 1).$$

Let $q_1, q_2, \cdots, q_k$ be all the prime factors of P. Then the members of C_1 are included in the class Γ_1 of integers $\leqq x$ having prime factors taken from the set $q_1, \cdots, q_k$ only. Since $2^r - 1$ has less than r prime factors, we have

$$k < \sum_{r=1}^{H} r \leqq H^2. \tag{1}$$

We now separate the members of Γ_1 into two subclasses $\Gamma_{1,1}$ and $\Gamma_{1,2}$, where $\Gamma_{1,1}$ consists of those members of Γ_1 which have less than

$$W = \frac{1}{10} (\log x)^{1/2}$$

distinct prime factors. From the fact that if $m \leqq x$ and if g^α divides m, then $\alpha \leqq (\log x)/\log 2$, it follows from (1) that the number of members of $\Gamma_{1,1}$ is less than

$$(\log x/\log 2)^W \sum_{t=1}^{W} \binom{k}{t} < W k^W (\log x/\log 2)^W,$$

a quantity less than $x^{1/4}$ for all sufficiently large x.

We consider next the class $\Gamma_{1,2}$. Let $d(m)$ denote the number of divisors of m, and let $v(m)$ be the number of distinct prime factors of m. If m belongs to $\Gamma_{1,2}$, then

$$d(m) \geqq 2^{v(m)} = \exp \{v(m) \log 2\} > \exp (W/2).$$

Hence, if N is the number of members of $\Gamma_{1,2}$, we have

$$2x \log x > x \sum_{m \leqq x} m^{-1} \geqq \sum_{m \leqq x} \left[\frac{x}{m}\right] = \sum_{m \leqq x} d(m) \geqq \sum_{m \subset \Gamma_{1,2}} d(m) \geqq N \exp (W/2).$$

That is, we have

$$N \leqq 2x(\log x) \exp (-W/2).$$

Therefore, if x is sufficiently large, the total number of members of C_1 is less than

$$x^{1/4} + N \leqq x^{1/4} + 2x(\log x) \exp (-W/2) < x \exp \left\{-\frac{1}{30} (\log x)^{1/2}\right\}. \tag{2}$$

We take up now the class C_2 which we separate into two classes $C_{2,1}$ and $C_{2,2}$. The class $C_{2,1}$ consists of those members n of C_2 which have a prime factor p such that the greatest common divisor δ of $n-1$ and $p-1$ satisfies

$$\delta = (n - 1, p - 1) \geqq \exp ((\log x)^{1/4}) = T.$$

In other words, for each member n of $C_{2,1}$ there is a prime p and an integer m such that

$$n = pm, \qquad p = \delta t + 1, \qquad m = \delta u + 1, \qquad x/p \geqq m > 1.$$

The last inequality follows from the fact that n is composite. If p and δ are fixed, the number of choices for m is at most $x/(\delta p)$. Hence the number of members of $C_{2,1}$ does not exceed

$$\begin{aligned}(3)\qquad \sum_{\delta>T} \sum_{\substack{p\leqq x \\ p=\delta t+1}} x/(\delta p) &< x \sum_{\delta>T} \sum_{t<x/\delta} (t\delta^2)^{-1} < x \sum_{\delta>T} 2(\log x/\delta)\delta^{-2} \\ &< 2x(\log x)T^{-1} < x \exp\left\{-\tfrac{1}{2}(\log x)^{1/4}\right\},\end{aligned}$$

for x sufficiently large.

Finally we consider the class $C_{2,2}$. This consists of almost primes

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

for which

$$\delta_i = (n - 1, p_i - 1) < T, \qquad (i = 1, 2, \cdots, k).$$

It is well known that the exponent $g(n)$ divides

$$\phi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1).$$

Also $g(n)$ divides $n-1$ since n is an almost prime. Hence

$$H \leqq g(n) \leqq (n - 1, \phi(n)) \leqq \prod_{i=1}^{k} (n - 1, p_i - 1) \leqq T^k.$$

That is,

$$k \geqq (\log H)/\log T = \log T.$$

Thus, if M denotes the number of members of $C_{2,2}$, we have, as before,

$$2x \log x > \sum_{m\leqq x} d(m) \geqq \sum_{m\leqq x} 2^{v(m)} \geqq 2^k \sum_{m\subset C2,2} 1 \geqq M \cdot 2^{\log T}.$$

Hence, for x sufficiently large, we have

$$\begin{aligned}M &\leqq 2x(\log x) \exp\left\{-(\log 2) \log T\right\} \\ &\leqq x \exp\left\{-\tfrac{1}{2}(\log x)^{1/4}\right\}.\end{aligned}$$

Combining this result with (2) and (3) we have

$$\begin{aligned}f(x) &< x\left\{\exp\left(-\tfrac{1}{30}(\log x)^{1/2}\right) + 2 \exp\left(-\tfrac{1}{2}(\log x)^{1/4}\right)\right\} \\ &< x \exp\left(-\tfrac{1}{3}(\log x)^{1/4}\right),\end{aligned}$$

for x sufficiently large. This is our theorem.

4. Discussion. By a slightly more complicated argument we could prove that, for some positive constant c,

$$f(x) < x \exp\left\{-c(\log x)^{1/2}\right\};$$

but the true order of $f(x)$ seems to be considerably smaller. As far as I know, the only estimate for $f(x)$ from below is

$$f(x) > C \log x,$$

which is due to Lehmer.

Added later. As far as I know the question of the existence of even numbers satisfying $2^n \equiv 2 \pmod{n}$ has not been considered. Except for the trivial case $n=2$, I have not succeeded in finding any such even numbers.* By the method of this paper it is easy to see that their number $\leqq x$ is certainly less than $x \exp\left\{-\frac{1}{3}(\log x)^{1/4}\right\}$.

References

1. D. H. Lehmer, On the Converse of Fermat's Theorem, I, II, this MONTHLY, vol. 43, 1936, pp. 347–354; vol. 56, 1949, pp. 300–309. These papers contain references to other work on almost primes.

2. P. Erdös, On the Converse of Fermat's Theorem, this MONTHLY, vol. 56, 1949, pp. 623–624.

ANGLES WITH RATIONAL TANGENTS**

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1. Introduction. The purpose of this note is to show that the class of angles having rational tangents, and the class of angles which are rational multiples of π, intersect only in the obvious cases.

2. Theorem. We shall establish the following result.

THEOREM. *If x is a rational multiple of π, and* tan x *is rational, then x is an integral multiple of $\pi/4$.*

3. Proof. Let $\tan x = q/p$. The theorem is trivially satisfied if $q=0$ or if $|p| = |q|$. Further, without loss of generality, x may be restricted to the first quadrant, so that p and q may be assumed positive, integral, unequal, and coprime.

If $x = m\pi/n$, then $e^{inx} = e^{-inx} = \pm 1$, or

$$(\cos x + i \sin x)^n = (\cos x - i \sin x)^n,$$

and

$$(p + iq)^n/(p^2 + q^2)^{n/2} = (p - iq)^n/(p^2 + q^2)^{n/2}.$$

* Added still later: Lehmer has just informed me that $2^{161038} \equiv 2 \bmod (161038)$, $(161038 = 2 \cdot 73 \cdot 1103)$.

** Revised by J. D. Swift.

Thus

$$(p - iq)^n = (p + iq)^n = (p - iq + 2iq)^n$$
$$= (p - iq)^n + \binom{n}{1}(p - iq)^{n-1}2iq + \cdots$$
$$+ \binom{n}{n-1}(p - iq)(2iq)^{n-1} + (2iq)^n.$$

Therefore $(p-iq)$ divides $(2qi)^n$, and p^2+q^2 divides $(2q)^{2n}$. Similarly, p^2+q^2 divides $(2p)^{2n}$ and therefore divides $(2^{2n}p^{2n}, 2^{2n}q^{2n}) = 2^{2n}$. Then $p^2+q^2 = 2^k$; but this is possible in positive coprime integers only when $p=q=k=1$, a contradiction.

4. Corollary. By writing tan nx as a rational function in terms of tan x, the reader may verify the following corollary.

COROLLARY: *The equations*

$$\sum_{k=0}^{a} (-1)^k \binom{n}{2k+1} x^{2k} = 0, \qquad a = \left[\frac{n-1}{2}\right], \quad n > 2,\ n \neq 4,$$

and

$$\sum_{k=0}^{b} (-1)^k \binom{n}{2k} x^{2k} = 0, \qquad b = \left[\frac{n}{2}\right], \quad n > 2,$$

have no rational roots.

A GENERALIZATION OF GAUSS' LEMMA*

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1. Introduction. We say that a number *belongs to the first half* modulo a number n if it is congruent mod n to one of the numbers $1, 2, \cdots, [(n-1)/2]$, and that it *belongs to the second half* modulo n if it is congruent mod n to one of the numbers $[n/2]+1, [n/2]+2, \cdots, n-1$. The well-known Gauss' lemma can then be stated as follows:

A number A is a quadratic residue modulo an odd prime p if and only if an even number of the terms

$$A, 2A, 3A, \cdots, (p-1)A/2 \tag{1}$$

belongs to the second half mod p.

2. Theorem. In this note we shall prove the following result.

THEOREM: *If p is an odd prime and A is odd, then the number of terms in the sequence* (1) *which belong to the second half* mod p *is equal to the number of terms which belong to the second half* mod $2p$.

* Translated and revised by E. G. Straus.

We shall call it a *transition* whenever multiplication by A transforms a number of the sequence $1, 2, \cdots, (p-1)/2$ into a number of the second half, while we shall call it a *retention* if multiplication by A leaves such a number in the first half.

3. Proof of the theorem. We perform the following one-to-one mapping of the sequence $1, 2, \cdots (p-1)/2$ on itself:

$$\text{For} \quad 0 < x < p/4, \quad \text{we let} \quad x \to 2x.$$
$$\text{For} \quad p/4 < x < p/2, \quad \text{we let} \quad x \to p - 2x.$$

For the numbers $0<x<p/4$ we see that

$$Ax \equiv r \pmod{p}, \qquad 0 < r < p,$$

implies

$$2Ax \equiv 2r \pmod{p};$$

and hence the number of transitions mod p in the sequence $0<x<p/4$ equals the number of transitions mod $2p$ in the image of that sequence. Considering the sequence $p/4<x<p/2$, we see that

$$Ax \equiv r \pmod{p}, \qquad 0 < r < p,$$

implies

$$-2Ax \equiv 2p - 2r \pmod{2p};$$

and since $Ap \equiv p \pmod{2p}$ (it is here that the oddness of A is essential), we have

$$A(p - 2x) \equiv \begin{cases} p - 2r \pmod{2p} & \text{for} \quad 2r < p, \\ 3p - 2r \pmod{2p} & \text{for} \quad 2r > p. \end{cases}$$

Hence the number of transitions mod p in the sequence $p/4<x<p/2$ equals the number of transitions mod $2p$ in the image of that sequence. This completes the proof of our theorem.

4. The quadratic reciprocity theorem. With the help of this generalization of Gauss' lemma we can give a very simple variant to Gauss' third proof of the quadratic reciprocity theorem. We make the following preliminary remarks:

(1) Just as the evenness or oddness of the number of *transitions* determines whether or not A is a quadratic residue, so the evenness or oddness of the number of *retentions* determines whether or not $-A$ is a quadratic residue.

(2) The number $[n/k]$ is even if and only if n belongs to the first half mod $2k$.

In order to count the number of transitions and retentions mod $2p$ among the numbers $1, 2, \cdots, (p-1)/2$, we subdivide the interval $(0, p/2)$ by insert-

ing the points p/A, $2p/A$, $\cdots$, $(A-1)p/(2A)$. The numbers in the odd numbered subintervals yield retentions while those in the even numbered subintervals yield transitions. The last subinterval $((A-1)p/(2A), p/2)$ is an odd or even numbered one according as A is of the form $4n+1$ or $4n+3$. We shall do our counting so as to avoid counting this last interval. That is to say we count the transitions if A is of the form $4n+1$, and the retentions if A is of the form $4n+3$. The numbers obtained are

$$-[p/A]+[2p/A]-[3p/A]+\cdots+[(A-1)p/(2A)], \tag{2}$$

and

$$[p/A]-[2p/A]+[3p/A]-\cdots+[(A-1)p/(2A)], \tag{3}$$

respectively.

In deciding the evenness or oddness of these sums the signs are immaterial (they could all be replaced by $+$ signs); in both cases it depends on whether an even or odd number of the terms

$$[p/A], [2p/A], \cdots, [(A-1)p/(2A)]$$

is odd. As remarked above, one of the terms $[kp/A]$ is odd if and only if kp is in the second half mod $2A$.

Hence we have the following result.

If $A=4n+1$, then A is a quadratic residue modulo the odd prime p if and only if (2) is even; that is, if and only if the number of transitions mod $2A$ in the multiplication of $1, 2, \cdots, (A-1)/2$ by p is even.

If $A=4n+3$, then $-A$ is a quadratic residue modulo the odd prime p if and only if (3) is even; that is, if and only if the number of transitions mod $2A$ in the multiplication of $1, 2, \cdots, (A-1)/2$ by p is even.

In particular, if A is an odd prime q, then the latter condition is equivalent to the statement that p is a quadratic residue mod q. Introducing the Legendre symbol (p/q), we can state this as follows:

If p, q are odd primes, then

$$\left(\frac{q}{p}\right)=\left(\frac{p}{q}\right) \quad \textit{if} \quad q=4n+1,$$

and

$$\left(\frac{-q}{p}\right)=\left(\frac{p}{q}\right) \quad \textit{if} \quad q=4n+3.$$

This latter statement is easily seen to be equivalent to the quadratic reciprocity theorem.

CLASSROOM NOTES

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A THEOREM ON THE REMAINDER OF A TAYLOR SERIES*

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Let $f(x)$ be a function for which at least the first n derivatives exist. Then,

$$(1)\quad f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + r_n$$

represents the Taylor expansion of $f(x)$ at the point $x=a$ and for an increment h. The remainder r_n may be expressed as

$$r_n = \frac{h^n}{n!}f^{(n)}(a+\alpha h) \quad \text{with} \quad 0 \leqq \alpha \leqq 1.$$

Therefore, r_n is a function of both a and h, and it may be important to know how it varies with a change of either one of these parameters. In reviewing the literature, an expression was found† for the derivative of r_n with respect to a only. However, in some recent work,‡ the sign of the derivative of r_n with respect to h was required under the condition that $f^{(n)}(x)$ always had the same sign in some interval. A theorem on this derivative was proven and is given below since there may also be other applications for it.

Since the sign of the derivative of r_n with respect to h may also depend on the sign of h, a more compact formulation of the theorem was obtained by finding the sign of the derivative of r_n with respect to h^n.

THEOREM. *If the nth derivative of a function $f(x)$ always has the same sign in some interval which includes the point $x=a$, then for any increment h within this interval, the derivative of r_n with respect to h^n is of the same sign as $f^{(n)}(x)$.*

PROOF. Equation (1) may be written as

$$(2)\quad r_n = f(a+h) - \sum_{i=0}^{n-1}\frac{h^i}{i!}f^{(i)}(a).$$

Differentiating this with respect to h

$$(3)\quad \frac{dr_n}{dh} = f'(a+h) - \sum_{i=1}^{n-1}\frac{h^{i-1}}{(i-1)!}f^{(i)}(a).$$

* The results presented here were obtained in connection with research sponsored by the Office of Naval Research.

† R. Courant, Differential and Integral Calculus, New Revised Edition, Vol. 1, p. 322, Nordeman Publishing Company, Inc., New York, 1942.

‡ R. Weatherston and J. V. Foa, Considerations on the selection of the mode of heat addition to a gas flowing in a pipe, Cornell Aeronautical Laboratory Report No. DB-605-A-1, May 1949.

However,

$$f'(a+h) = f'(a) + \frac{h}{1!}f''(a) + \cdots + \frac{h^{n-2}}{(n-2)!}f^{(n-1)}(a) + \frac{h^{n-1}}{(n-1)!}f^{(n)}(a+\beta h)$$

or

$$f'(a+h) = \sum_{i=1}^{n-1} \frac{h^{i-1}}{(i-1)!}f^{(i)}(a) + \frac{h^{n-1}}{(n-1)!}f^{(n)}(a+\beta h) \quad \text{with} \quad 0 \leqq \beta \leqq 1. \tag{4}$$

When equation (4) is substituted in equation (3), one obtains

$$\frac{dr_n}{dh} = \frac{h^{n-1}}{(n-1)!}f^{(n)}(a+\beta h) \tag{5}$$

and dividing equation (5) by nh^{n-1} yields

$$\frac{dr_n}{d(h^n)} = \frac{1}{n!}f^{(n)}(a+\beta h) \quad \text{with} \quad 0 \leqq \beta \leqq 1. \tag{6}$$

This proves the theorem since $f^{(n)}(x)$ was assumed to be always of the same sign in some interval.

THE PROBABILITY INTEGRAL

C. P. NICHOLAS and R. C. YATES, United States Military Academy

The following evaluation requires only a knowledge of elementary calculus and thus with reasonable assumptions may be presented at the sophomore level.

Assuming its existence, let

$$\int_0^\infty e^{-x^2}dx = A, \quad \text{where} \quad \int_0^\infty = \lim_{h\to\infty}\int_0^h.$$

If the curve $z=e^{-y^2}$ be revolved about the Z-axis, the surface generated is $z=e^{-(x^2+y^2)}$ with volume:

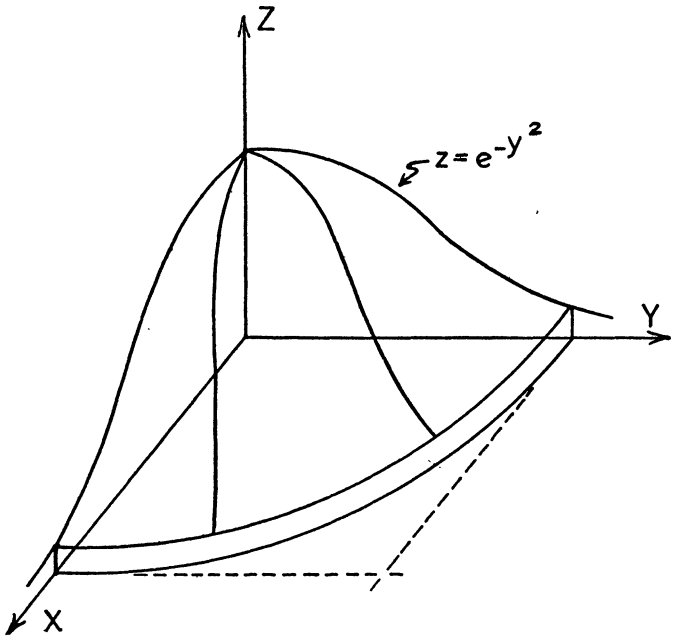
$$\begin{aligned} V &= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy \\ &= 4\int_0^\infty\left[\int_0^\infty e^{-x^2}dx\right]\cdot e^{-y^2}dy \\ &= 4\int_0^\infty A\cdot e^{-y^2}dy = 4A^2. \end{aligned}$$

But, using the method of hollow cylinders:

$$V = 2\pi \int_0^\infty ye^{-y^2}dy = \pi.$$

Thus

$$4A^2 = \pi, \qquad A = \frac{\sqrt{\pi}}{2}.$$



A DIFFERENT APPROACH TO A KNOWN INTEGRAL

H. L. LEE, University of Tennessee

The purpose of this paper is to give a new method of determining a known integral formula and to show that some integrals are special cases of this more general form. This result is obtained without the use of partial fractions or the method of completing the square as is generally the case.

Consider the integral

$$\int \frac{kdu}{(a_1u + b_1)(a_2u + b_2)} \tag{1}$$

where $u=f(x)$ and a_1, a_2, b_1, b_2 and k are constants. It is assumed that $(a_1u+b_1)(a_2u+b_2)\neq 0$. This condition excludes the possibility that $a_i=b_i=0$ for $i=1$ or 2. There are two cases to be considered.

CASE 1. Let

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

(a) If $a_1=a_2=0$, but $b_1\neq 0$, $b_2\neq 0$, the integral (1) is of the simple form $c\int dv$.

(b) If $b_1=b_2=0$ and $a_1\neq 0$, $a_2\neq 0$, or if $a_1\neq 0$, $a_2\neq 0$, $b_1\neq 0$ and $b_2\neq 0$, (1) assumes the form $c\int v^{-2}dv$.

CASE 2. Let

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

(a) If $a_1=0$ or $a_2=0$, then (1) is of the form $c\int dv/v$.

(b) Suppose $a_1\neq 0$, $a_2\neq 0$. If $(a_1u+b_1)(a_2u+b_2)<0$, let $a_3=-a_1$ and $b_3=-b_1$, so that (1) becomes

$$-\int \frac{kdu}{(a_3u+b_3)(a_2u+b_2)}, \tag{2}$$

where $(a_3u+b_3)(a_2u+b_2)>0$. There is therefore no loss in generality in assuming that $(a_1u+b_1)(a_2u+b_2)>0$.

With this assumption, (1) may be written

$$\frac{k}{a_1a_2}\int \frac{du}{(u+c_1)(u+c_2)}, \tag{3}$$

where $c_1=(b_1/a_1)$ and $c_2=(b_2/a_2)$. If the numerator and the denominator of (3) are divided by $(u+c_1)^2$ and the resulting integral multiplied by $-(c_2-c_1)\div -(c_2-c_1)$, (3) becomes

$$-\frac{k}{a_1a_2(c_2-c_1)}\int \frac{-\dfrac{c_2-c_1}{(u+c_1)^2}du}{1+\dfrac{c_2-c_1}{u+c_1}}. \tag{4}$$

When the integration of (4) is performed, the result is

$$-\frac{k}{a_1a_2(c_2-c_1)}\ln\left(1+\frac{c_2-c_1}{u+c_1}\right)+C', \tag{5}$$

which may be simplified, by replacing c_1 and c_2 by their values, into

$$\frac{k}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}\ln\frac{a_1u+b_1}{a_2u+b_2}+C, \tag{6}$$

where

$$C = \frac{k}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \ln \frac{a_2}{a_1} + C'.$$

Thus the following formula is obtained

$$\int \frac{k\,du}{(a_1u + b_1)(a_2u + b_2)} = \frac{k}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \ln \frac{a_1u + b_1}{a_2u + b_2} + C. \tag{7}$$

The integral

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \frac{u - a}{u + a} + C \qquad u^2 > a^2$$

is a special case of (7) with $k = 1$, $a_1 = 1$, $a_2 = 1$, $b_1 = -a$ and $b_2 = a$. Similarly

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \frac{a + u}{a - u} + C \qquad a^2 > u^2$$

is a special case of (7).

ON THE EQUATION OF A LINE

CAROL S. SCOTT, St. Petersburg Junior College

Given a point (x_1, y_1) and a slope m one usually writes the equation of the corresponding line in the form:

$$y - y_1 = m(x - x_1).$$

Similarly given two points (x_1, y_1) and (x_2, y_2) one finds the slope m and applies the above formula. Simplification of the resulting expression involves several algebraic steps which can be eliminated by the following method of procedure.

It is well known that for the line

$$Ax + By + C = 0$$

the slope $m = (A/-B)$. So to find the desired equation, write m in fractional form and call its numerator "A" and its denominator "$-B$." The value of C can be determined mentally from:

$$Ax_1 + By_1 + C = 0.$$

Examples: Find the equation of the line through $(4, -2)$ with slope $3/5$. Then $A = 3$, $B = -5$, $C = -22$. The answer is: $3x - 5y - 22 = 0$.

Find the equation of the line through $(2, 6)$ and $(-1, 4)$. Since $m = 2/3$, $A = 2$ and $B = -3$. Using $(2, 6)$ we find $C = 14$. As an easy check we observe that $(-1, 4)$ also satisfies the resulting equation: $2x - 3y + 14 = 0$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 921. *Proposed by R. T. Hood, Beloit College*

Devise a multiple-pan balance, with minimum number of pans, by which one can weigh any whole number of pounds using weights which are powers of n, n a given positive integer. What other amounts can be weighed with such a balance?

E 922. *Proposed by Michael Goldberg, Washington, D. C.*

Show that any given triangle can be dissected by straight cuts into four pieces which can be arranged to form two triangles similar to the given triangle.

E 923. *Proposed by R. C. Buck, University of Wisconsin*

Show that no polyhedron in three-space can have exactly seven edges, while any other integer greater than five is admissible.

E 924. *Proposed by D. J. Newman, New York University*

Find $\lim_{n\to\infty} n \sin(2\pi en!)$.

E 925. *Proposed by J. B. Kelly, University of Wisconsin*

Is it possible so to weight a pair of dice that the probability of occurrence of every sum from 2 to 12 shall be the same?

SOLUTIONS

A Cryptarithmic Greeting

E 891 [1949, 691]. *Proposed by Richard and Josephine Andree, University of Oklahoma*

Given that $E^2=H$ and that each symbol represents a unique digit, find, and prove unique, the numerical solution of the following addition problem.

$$\begin{array}{rrrrrr} & & & D & \& & J \\ A & N & D & R & E & E \\ & & S & E & N & D \\ \hline C & H & E & E & R & \end{array}$$

Solution by Esther T. Hall, College Park, Maryland. Evidently $A=0$. Since

$E^2=H$, either $E=2$ and $H=4$, or $E=3$ and $H=9$. In the latter case $D+S+1=19$, which is impossible as D and S are different. Therefore $E=2$, $H=4$, $D+S=13$, $D+R=9$. Thus $D\neq 1, 2, 3, 4, 5, 7, 9$. From $J\neq R$ it follows that $D\neq 8$. Thus $D=6$, $R=3$, $S=7$. As $C=N+1$, $N=8$, $C=9$. The last two columns give $J=5$ and the ampersand $=1$. Consequently, the problem has at most one solution. It has at least one since the one we found checks as follows:

$$\begin{array}{cccccc} & & & 6 & 1 & 5 \\ 0 & 8 & 6 & 3 & 2 & 2 \\ & & 7 & 2 & 8 & 6 \\ \hline & 9 & 4 & 2 & 2 & 3 \end{array}$$

Also solved by Aaron Buchman, W. E. Buker, J. E. Darraugh, B. V. Dean, Monte Dernham, William Douglas, Ragnar Dybrik, J. L. Ericksen, B. K. Gold, H. E. Gould, M. S. Klamkin, F. T. Kocher, Sydney Kravitz, Elmer Latshaw, H. R. Leifer, J. J. Livers, J. B. Mertie, Jr., C. H. Murphy, J. A. Nickel, J. S. Nodvik, C. S. Ogilvy, Ingram Olkin, P. W. A. Raine, L. A. Ringenberg, Azriel Rosenfeld, C. M. Sandwick, Nathan Schwid, Mary Seybold, Thom Sumner, C. W. Trigg, Maud Willey, and Fred Young.

In reply Dernham sent

$$MANY + HAPPY = DAYS!,$$

with $P^2=M$ and no zero, and from Trigg came

$$TRIGG + SAYS = MERCI$$

in seventeen ways, or uniquely if either $T>I>R$ or TR is an even square.

Area Property of Isogonic Configuration

E 892 [1949, 691]. *Proposed by J. P. Ballantine and G. E. Ulrich, University of Washington*

Let T be a given triangle, U the triangle whose vertices are the centroids of equilateral triangles described externally on the sides of T, and V the triangle whose vertices are the centroids of equilateral triangles described internally on the sides of T. Show that area $T=$area $U-$area V.

Solution by Roger Lessard, École Polytechnique, Montreal. It is known that triangles U and V are equilateral (*cf.*, Johnson, Modern Geometry, p. 221). Let A' and B' be the centroids of the equilateral triangles described externally on BC and CA, and A'' and B'' the centroids of those described internally. Then, if $BC=a$ and $CA=b$,

$$(A'B')^2 = a^2/3 + b^2/3 - (2ab/3)\cos(C + 60°),$$
$$(A''B'')^2 = a^2/3 + b^2/3 - (2ab/3)\cos(C - 60°),$$

and

$$\begin{aligned}\text{area } U - \text{area } V &= (\sqrt{3}/4)(A'B')^2 - (\sqrt{3}/4)(A''B'')^2 \\ &= (ab/2\sqrt{3})[\cos(C - 60°) - \cos(C + 60°)] \\ &= (ab/2\sqrt{3})(2 \sin C \sin 60°) \\ &= (ab \sin C)/2 = \text{area } T.\end{aligned}$$

Also solved by Joseph Carlson, F. M. Carpenter, W. B. Clarke, A. L. Epstein, D. H. Erkiletian, Jr., Albert Furman, Vern Hoggatt, L. M. Kelly, Joseph Langr, J. S. Nodvik, J. H. Simester, A. Sisk, C. W. Trigg, and the proposers.

Trigg pointed out that this problem has previously appeared as problem 358, *School Science and Mathematics*, Jan. 1914, p. 64, and problem 478, *National Mathematics Magazine*, April 1943, p. 317. For a solution by complex coordinates see problem 3367 [1930, 97].

Limit of a Series

E 893 [1949, 691]. *Proposed by N. J. Fine, University of Pennsylvania*

Find

$$\lim_{x\to 1} \sum_{n=0}^{\infty} (-1)^n \tan^{-1} x^{2n+1}.$$

Solution by W. B. Fulks, University of Minnesota. Since the series diverges for $x>1$, the limit can exist only as a left-hand one. Then for $x<1$ we may write

$$\begin{aligned}\sum_{n=0}^{\infty} (-1)^n \tan^{-1} x^{2n+1} &= \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{\infty} (-1)^k x^{(2n+1)(2k+1)}/(2k+1) \\ &= \sum_{k=0}^{\infty} (-1)^k/(2k+1) \sum_{n=0}^{\infty} (-1)^n x^{(2k+1)(2n+1)},\end{aligned}$$

for the double integral converges absolutely. The inner sum is easily evaluated, and our double sum becomes

$$\sum_{k=0}^{\infty} (-1)^k x^{2k+1}/(2k+1)(1 + x^{2(2k+1)}). \tag{1}$$

Designate the sum of series (1) by $S(x)$, and the kth partial sum by $S_k(x)$. Then, since the series is an alternating series with numerically decreasing terms, we have

$$|S(x) - S_k(x)| \leqq |x|^{2k+1}/(2k+1)(1 + x^{2(2k+1)}) \leqq 1/2(2k+1).$$

This establishes uniform convergence of series (1) for all real x. Hence

$$\begin{aligned}\lim_{x\to 1-0} \sum_{n=0}^{\infty} (-1)^n \tan^{-1} x^{2n+1} = S(1) &= (1/2) \sum_{k=0}^{\infty} (-1)^k/(2k+1) \\ &= (1/2) \tan^{-1} 1 = \pi/8.\end{aligned}$$

Note. For $|z|<1$, $S(z)$ represents an analytic function whose natural boundary is the unit circle. For $|z|>1$, $S(z)=S(1/z)$, and it represents another *different* analytic function whose natural boundary is also the unit circle. These are not analytic continuations of each other but they do have the same limiting values at $z=1$ and at $z=-1$. Their infinities are dense on the boundary of the unit circle.

Also solved by P. G. Kirmser, M. S. Klamkin, Roger Lessard, C. S. Ogilvy, and L. A. Ringenberg.

A Minimum Property of the Cube

E 895 [1949, 692]. *Proposed by L. Fejes Tóth, Budapest, Hungary*

Let the *incircle* of a convex polygon be defined as the largest circle whose interior may lie in the interior of the polygon. Show that the sum of the squares of the edges of a convex polyhedron is at least twelve times the square of the diameter of the least incircle of the faces.

Solution by the Proposer. Let f be the number of faces, e the number of edges, and r the radius of the least incircle of the faces of the polyhedron. Consider an arbitrary face and an arbitrary side of length s of this face, and let this side subtend an angle 2α at the center of one of the permissible positions of the incircle of the face. Then $s\geqq 2r\tan\alpha$, and the sum to be estimated is

$$S = (1/2)\sum^{2e} s^2 \geqq 2r^2\sum^{2e}\tan^2\alpha.$$

Applying Jensen's inequality to the function $\tan^2\alpha$, $0\leqq\alpha<\pi/2$, which is convex from below, we have

$$\tan^2(\pi f/2e) \leqq \Big(\sum^{2e}\tan^2\alpha\Big)/2e,$$

whence

$$S \geqq (2r)^2 e\tan^2(\pi f/2e).$$

Since $e\leqq 3(f-2)$, this leads to

$$S \geqq (2r)^2 3(f-2)\tan^2\{\pi f/6(f-2)\} = 3(2r)^2 T(f).$$

Numerical calculation shows that for $f=4, 5, \cdots, 13$ we have $T(f)\geqq T(6)=4$, while for $f\geqq 14$, $T(f)>12\tan^2(\pi/6)=4$. Hence for any number of faces $S\geqq 12(2r)^2$.

Editorial Note. It is easily seen that in the relation $S\geqq 12(2r)^2$, equality holds only for the cube.

In a manner similar to the above it may be shown that if L denotes the sum of the lengths of the edges of a convex polyhedron, then $L\geqq 12\sqrt{3}r$, equality holding only for the regular tetrahedron. Fejes Tóth has established another lower bound for L. In his paper *On the total length of the edges of a polyhedron*,

Det Kongelige Norske Videnskabers Selskab Forhandlinger, Bd. XXI, Nr. 8, pp. 32–34, he has shown that $L>10D$, where D is the diameter of any sphere containing the convex polyhedron. The conjecture here is that the greatest lower bound is $12D$, the bound being attained only by the cube, but proof of this seems difficult. The conjecture has been established for convex polyhedra of equiareal faces. In the above paper Fejes Tóth has also shown that if the convex polyhedron has only triangular faces, then $L>14D$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTIONS

4370. Corrected. *Proposed by H. F. Sandham, Trinity College, Ireland*

A', B', C' are points on the opposite sides of a triangle ABC. The circles through $B'C'A$, $C'A'B$, $A'B'C$ intersect in M, the Miquel point, whose isogonal conjugate is M'. Prove that M', M are corresponding points under the direct circular transformation set up by A, A'; B, B'; C, C'.

4400. (*Originally proposed as* 3290 [1927, 491] *by J. V. Uspensky.*) *Reproposed by C. D. Olds, San Jose State College, California*

Every positive root of the equation $\tan x = x$ can be expressed as follows: $x=(p+\frac{1}{2})\pi-\theta$ where p is an arbitrary integer $\geqq 0$ and

$$\theta = C_0\,\xi + C_1\xi^3 + C_2\xi^5 + \cdots, \quad \xi = 1/(p+\tfrac{1}{2})\pi.$$

The coefficients C_0, C_1, C_2, $\cdots$, are positive rational numbers. Show that, for n very large,

$$C_n = 12^{-1/6}\Gamma(1/3)\pi^{-2/3}\cdot(\pi/2)^{2n+1}(2n+1)^{-4/3}(1+\omega_n)$$

where $\omega_n \to 0$ when $n\to\infty$, so that the above series remains convergent even for $p=0$.

4401. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In the triangle ABC for which the cotangent of the Brocard angle is equal to 2, show that the sides satisfy the relation

$$5(a^4 + b^4 + c^4) = 6(b^2c^2 + c^2a^2 + a^2b^2).$$

Show also that it is impossible for the lengths of the sides to be all integers.

4402. *Proposed by Carl Cohen, Harvard University*

Let the series of functions $\mathrm{dl}^{(n)}(x)$ be defined as

(1) $$\mathrm{dl}^{(n)}(x) = \int_0^x \mathrm{dl}^{(n-1)}(t)\,dt/t$$

(2) $$\mathrm{dl}^{(1)}(x) = -\int_0^x \log(1-t)\,dt/t \qquad \text{[Euler].}$$

Show that for $|x| < \pi/2$,

(i) $$\tan x = \sum_{n=1}^{\infty} (2/\pi)^{2n}\left[\mathrm{dl}^{(2n-1)}(1) - \mathrm{dl}^{(2n-1)}(-1)\right]x^{2n-1},$$

(ii) $$\sec x = 1 + i\sum_{n=1}^{\infty} (2/\pi)^{2n+1}\left[\mathrm{dl}^{(2n)}(-i) - \mathrm{dl}^{(2n)}(i)\right]x^{2n}.$$

4403. *Proposed by J. H. Butchart, Arizona State College*

A necessary and sufficient condition that a hyperboloid of one sheet be an altitude quadric (*i.e.* any four rulings of one regulus may be altitudes of a tetrahedron) is that any plane section perpendicular to a ruling is an equilateral hyperbola. (Compare the Proposer's note, *The altitude quadric of a tetrahedron* [1940, 383].)

4404. *Proposed by Ky Fan, University of Notre Dame*

Determine all positive solutions x of the infinite system of equations

(1) $$[[n/x]x - x] = n - 1, \qquad n = 1, 2, 3, \cdots.$$

The brackets denote, as usual, the greatest integer function.

SOLUTIONS

Infinite Series and Product

4314 [1948, 505]. *Proposed by N. J. Fine, University of Pennsylvania*

Prove the identity:

$$1 + \frac{x}{(1-x)^2} + \frac{x^2}{(1-x)^2(1-x^2)^2} + \frac{x^3}{(1-x)^2(1-x^2)^2(1-x^3)^2} + \cdots = \frac{1 - x + x^3 - x^6 + x^{10} - \cdots}{[(1-x)(1-x^2)(1-x^3)\cdots]^2}.$$

Solution by the Proposer. Let $F(z)$ be defined by

$$(1)\quad F(z) = 1 + \frac{x}{(1-x)(1-xz)} + \frac{x^2}{(1-x)(1-x^2)(1-xz)(1-x^2z)} + \cdots,$$

where x is a fixed complex number inside the unit circle, and z is a variable $\neq x^n$, $n = 1, 2, \cdots$. Upon replacing z by zx in (1) and subtracting corresponding terms it is easy to see that

$$(2)\quad F(z) - F(zx) = \frac{zx^2}{(1-zx)(1-zx^2)} F(zx^2).$$

Now we define the function $G(z)$,

$$(3)\quad G(z) = F(z) \prod_{n \geq 1} (1 - zx^n).$$

$G(z)$ is an entire function of z, and

$$(4)\quad G(0) = F(0) = 1 + \frac{x}{1-x} + \frac{x^2}{(1-x)(1-x^2)} + \cdots = \prod_{n \geq 1} \frac{1}{1-x^n}.$$

Furthermore, from (2) and (3), we see that

$$(5)\quad G(z) - (1 - zx)G(zx) = zx^2 G(zx^2).$$

Assuming, as we may, the expansion

$$(6)\quad G(z) = a_0 + a_1 z + a_2 z^2 + \cdots,$$

where the a_n depend only on x, and equating coefficients of z^n in (5), we obtain easily

$$(7)\quad a_n = -a_{n-1}x^n \qquad n > 0,$$

the solution of which is

$$(8)\quad a_n = (-1)^n x^{(n^2+n)/2} a_0 = (-1)^n x^{(n^2+n)/2} \prod_{k \geq 1} \frac{1}{1-x^k},$$

by (4). Hence

$$(9)\quad G(z) = \prod_{k \geq 1} \frac{1}{1-x^k} \sum_{n \geq 0} (-1)^n x^{(n^2+n)/2} z^n.$$

Now, recalling (3), we have

$$(10)\quad F(z) = \prod_{k \geq 1} \frac{1}{(1-x^k)(1-x^k z)} \sum_{n \geq 0} (-1)^n x^{(n^2+n)/2} z^n.$$

Setting $z = 1$, we obtain the required result.

It should be remarked that the series in this problem completes a set of well known formulas:
Write $[0]=1$, $[n]=(1-x)(1-x^2)\cdots(1-x^n)$; then

$$(i) \qquad \sum_{n\geqq 0}\frac{x^n}{[n]}=\prod_{k\geqq 1}\frac{1}{1-x^k},$$

$$(ii) \qquad \sum_{n\geqq 0}\frac{x^{n^2}}{[n]^2}=\prod_{k\geqq 1}\frac{1}{1-x^k},$$

$$(iii) \qquad \sum_{n\geqq 0}\frac{x^{n^2}}{[n]}=\prod_{k\geqq 1}\frac{1}{(1-x^{5k-4})(1-x^{5k-1})},$$

$$(iv) \qquad \sum_{n\geqq 0}\frac{x^n}{[n]^2}=\prod_{k\geqq 1}\frac{1}{(1-x^k)^2}\sum_{m\geqq 0}(-1)^m x^{(m^2+m)/2}.$$

The first two are very well known, they probably go back to Euler; (*iii*) is one of the famous Rogers-Ramanujan identities; while (*iv*) is the present problem.

In much the same way as (*iv*) we may derive

$$(v) \qquad \sum_{n\geqq 0}\frac{x^{2n}}{[2n]}=\prod_{k\geqq 1}\frac{1}{1-x^k}\sum_{n\geqq 0}(-1)^n x^{n^2}$$

which has an interesting interpretation in terms of partition functions. If $p(m)$ is the unrestricted partition function, the right member of (*v*) can be written

$$\sum_{N\geqq 0}x^N\sum_{n\geqq 0}(-1)^n p(N-n^2).$$

On the other hand the left member of (*v*) is easily seen to generate the number of partitions of an integer into parts the largest of which is even, or, by reading the conjugate graph, into an even number of parts. Denoting this partition function by $p_E(N)$, we have proved

$$(vi) \qquad p_E(N)=p(N)-p(N-1)+p(N-4)-p(N-9)+\cdots.$$

It would be interesting to see if there is a combinatorial proof of (*vi*).

Square Inscribed in Arbitrary Simple Closed Curve

4325 [1949, 39]. *Proposed by Orrin Frink, Pennsylvania State College*

Show that on every simple closed plane curve there are four points which are the vertices of a square.

Solution by C. S. Ogilvy, Trinity College, Hartford. Let m_{x_i}, or more simply, m_x, be a system of horizontal straight lines in the plane, some of which cut the curve at X_i and X_i', or more simply X and X'. Let m_{x_1} avoid the curve by passing wholly below it, and m_{x_2} avoid the curve by passing wholly above it (Jordan.) If m_x is now moved continuously from position x_1 to position x_2, it will pass across the curve, producing two continuous sequences of points of intersection

X and X'. In the event of concavities, choose those intersections which allow X and X' to move continuously along the curve. Thus there will be times when the motion of m_x and, say, X will be retrograde while accommodating the wanderings of X'; but at all times the motion of all three, and in particular the length XX', will be continuous functions.

Now let YY' be the perpendicular bisector of XX', meeting XX' in P and terminated by its intersections with the curve. There will be an initial (first contact) position of m_x where $Y'P > YP = 0$, and a final (last contact) position where $0 = Y'P < YP$. But because of the continuous behavior of XX', YY' of m_y moves continuously also, and there must be some position of YY' where $YP = Y'P$ (Bolzano). This means that there is always an inscribed rhombus whose diagonals are horizontal and vertical. It follows that there is an inscribed rhombus for any slope of the diagonals (replace the word horizontal by parallel in the first sentence).

To continue: either $XX' = YY'$, in which case the rhombus is a square and the theorem is proved; or $XX' \neq YY'$. Suppose $XX' > YY'$. Then rotate m_x continuously through 90° until it reaches the position m_y; now $XX' < YY'$. Therefore there must have been an intermediate position of the rhombus (Bolzano again) where $XX' = YY'$. This is the position of the required square.

Editorial Note. V. L. Klee refers to a paper (in Russian) by Snirelman, L. G., *On certain geometrical properties of closed curves*, Uspehi Matem. Nauk 10, 34–44 (1944). See also abstract in Math. Rev., 7, 35 (1946). The case where the curve is convex is given by Emch, *Some properties of closed convex curves in a plane*, American Journal of Mathematics, vol. 35, 1913, pp. 407–412.

Function with Natural Boundary

4329 [1949, 40]. *Proposed by A. W. Goodman, University of Kentucky*

Let θ be an irrational number, $a = e^{i\theta\pi}$. Prove that

$$f(z) = \sum_{n=0}^{\infty} a^{n^2} z^n$$

has the unit circle as a natural boundary.

Solution by R. C. Buck, Brown University. The function $f(z)$ obeys the equation

$$f(z) - 1 = azf(a^2z). \tag{1}$$

$f(z)$ has at least one singularity, z_0, on the unit circle. From (1) a^2z_0 is also a singularity; iterating this, $a^{2m}z_0$ is a singularity for $m = 0, 1, \cdots$. Since θ is irrational, these points are everywhere dense on the unit circle, which is consequently a natural boundary for $f(z)$. If θ is rational, then $f(z)$ is a rational function.

Also solved by George Piranian.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Rank Correlation Methods. By M. G. Kendall, Charles Griffin and Co. Ltd., London, 1948. 8+160 pp. 18s.

In writing this small volume the author has set a pattern that points the way towards the solution of a difficulty that is frequently encountered by persons using statistical methods in analyzing experimental data. In most publications an attempt is made to cover the whole field of statistical methods in one or two volumes. As a result many statistical methods receive such limited attention that a research worker using such a source either is unaware of the full power of the technique being considered or is not adequately informed as to the limitations and restrictions that must be considered when applying the technique. By limiting the material covered in the text to a single concept the author has been able to include most of the available theory relating to rank correlation as well as to include numerous and varied illustrative examples. Among the topics considered by the author are: the general theory of rank correlation, tied ranks, tests of significance, the problem of m rankings, partial rank correlation, and paired comparisons.

Each phase of the rank correlation problem is treated in two chapters. The first chapter is expository in nature and considers the subject from a nontechnical approach, while the second chapter contains the derivations of the formulas and the proofs of the propositions that were used in the expository chapter. Since the nontechnical chapters are entirely self-contained, the book should be of value to the research workers in applied fields especially in psychology and the social sciences. Students of statistical theory should find the book of interest since the technical chapters contain many interesting applications of the mathematical tools and devices one so often finds of use in developing the theory associated with statistical concepts.

Although the book on the whole is very well written, there are a few minor weaknesses that might be noted. The text contains an adequate number of illustrative examples; however the lack of any sets of exercises for the student affects the usefulness of the publication as a text book. One at times encounters difficulty in following the argument used in the development of the theory because of the complexity of the notation. It may be that no better notation can be developed, but one cannot help but feel that there is a lot to be gained through an improved notation. The author has confused his terminology with regard to the size of a probability value. In fact, after defining his term "small" and "very small" on page 40, he misuses the expressions in two illustrations that follow the definition.

The book is written in a style that makes it interesting reading and should find wide acceptance by workers in both the applied and theoretical field of statistics.

C. F. KOSSACK

Analytic Geometry. By A. L. Nelson, K. W. Folley, and W. M. Borgman. New York, The Ronald Press Company, 1949. 8+215 pages. $3.00.

The authors suggest that this book is to be used by undergraduates as a "preparation for the calculus rather than as a study of geometry." While one cannot separate the two concepts entirely yet it can be said that this book does satisfy the goal of the authors.

Unusual attention is given to the idea of directed line segments, which naturally make the x_2-x_1 concept more reasonable to the student. Very wise use of time and effort is devoted to the geometric meaning of the abscissa and ordinate before such topics as division of line segments, slope of a line, and the angle between two lines is discussed in the first chapter. On page 19, the figures explaining the fact that perpendicular lines have negative reciprocal slopes seem more involved than necessary even though the effort is commendable.

Throughout the book no work has been spared in making the geometric exhibits as attractive and meaningful as possible. In fact, those near the end of the book showing the ellipsoid, as well as those of the hyperboloids of both one and two sheets are excellent. Prospective calculus students will gain so much from these drawings that they should feel at home when the mechanics of the calculus is studied.

The definition of a graph (page 49) is stated with elegance. A chapter on graphs of single valued algebraic functions followed by one on single valued transcendental functions precede the study of the chapter on the circle, after which follows the conic section chapter. The reviewer is impressed with the inclusion of the topic concerning the graph of the general conic section by means of the method of the addition of two ordinates, before consideration of the transformation of coördinates is met. The latter is relegated to the last chapter just before solid analytics is reached. Since students often lose themselves in the graphing of conic sections by use of coördinate transformation methods, many instructors will welcome the arrangement of this text.

In Chapter VIII one finds appropriate emphasis given to loci and their equations. This topic is closely tied up with the usual proofs of the theorems on plane geometry by use of analytic methods.

While tables of natural and exponential functions, as well as those of natural and common logarithms appear, it would please some students and instructors to have a short résumé of algebraic and trigonometric formulae available somewhere in the text. The chapters are concise, well written, and sprinkled with stimulating graphs. Answers to odd problems are given with the problems.

The publisher seems to have done as an acceptable job as have the authors.

W. R. HUTCHERSON

Calculus. By L. L. Smail, New York, Appleton-Century-Crofts, 1949. 16+592 pages. $4.50.

It is a stated aim of the present book to pay greater attention than usual to the careful formulation of the fundamental definitions and to the meanings of the basic concepts of calculus. The author has indeed attempted to write a superior book by introducing more preciseness and rigor than is customary. To a considerable degree the author has succeeded in carrying out his laudable purpose.

One of the features of the book is the introduction of a chapter on derivatives and one on the definite integral immediately after the differentiation of algebraic functions and before the differentiation of transcendental functions. Another feature is the inclusion of carefully chosen examples illustrating concepts, definitions, and theorems as well as methods. The exercise lists, which occur at the end of almost every article, are outstanding. Answers are given in the back of the book for the odd-numbered exercises.

Professor Smail's book appears to be very teachable and should prove a welcome addition to the small class of acceptable calculus texts.

R. G. HELSEL

The Life and Works of Herbert Ellsworth Slaught. By H. J. Dark. Bureau of Publications, George Peabody College for Teachers, Nashville, Tenn. 1948. 4+152 pages, and frontispiece. $2.25.

This volume is a thoroughly documented account of the many and varied activities of Professor Slaught in educational affairs and in mathematical organizations in the United States. We find in Chapter II a brief outline of the main facts of his life. The 40 pages of Chapter III are devoted to a description of his activities and methods as a teacher, both at Peddie Institute and at the University of Chicago, including his extra-classroom activities. The mathematical texts he wrote or edited are listed in Chapter IV, together with a number of comments on their purpose and value. This chapter also includes a brief account of Slaught's work as an editor of this MONTHLY from 1907 to 1937. In Chapter V we find an extensive account of the role Slaught played in the Central Association of Science and Mathematics Teachers, the American Federation of Teachers of the Mathematical and the Natural Sciences, the American Mathematical Society, the National Committee on Mathematical Requirements, the National Council of Teachers of Mathematics, and especially in the Mathematical Association of America. This chapter is really an amazing commentary on the extraordinary vigor and vision of the man. The last two chapters are brief summaries, and there is an extensive bibliography. The many friends and admirers of Professor Slaught will find much of interest in this volume. It is written in the spirit of an enthusiastic tribute. The author naturally does not claim to have covered his subject completely, but the array of accomplishments described here is certainly inspiring, and surprising even to one who knew Professor Slaught over a number of years.

L. M. GRAVES

Freshman Mathematics. By Slobin and Wilbur. Third Edition Revised by C. V. Newsom 15+559 pages. Rinehart and Co., 1949. $5.00.

This book is a second revision of the book bearing the same title by H. L. Slobin and W. E. Wilbur. Its aim is to impart to the student "a real understanding of the fundamental principles" and appreciation of "the values of these subjects vocationally and culturally."

That this is an admirable objective is axiomatic and if a book does not achieve complete success in this respect it may still be an excellent textbook. In the humble opinion of the reviewer—humble because of his own shortcomings and doubts as a teacher—the book falls quite short of its avowed aim, as far as the first section is concerned. It is probably true that there is no unanimity among teachers as to what constitutes "fundamental concepts and principles" of freshman mathematics. Some of the most fundamental ideas are either not mentioned at all or stated very briefly or misstated. I recall hearing the late J. H. M. Wedderburn make the statement that the division transformation for polynomials is the fundamental theorem of elementary algebra; without it the work with rational algebraic expressions is hollow. It is not mentioned in the book under review. In the definition of polynomials it is stated that the exponents must be positive integers (p. 24). This is obviously a bad slip, but there is no mention of the zero polynomial and of the important fact that no degree is attached to it. It seems that the authors were trying to simplify and clarify the ideas they wanted to present and thus introduced "restricted" definitions which in some instances only confuse the issue. This is—again in the reviewer's humble opinion—regrettable particularly in case the general and correct formulation is just as simple and economical of mental effort.

The second and third sections dealing with Trigonometry and Analytic Geometry are exceptionally good. It would almost seem that they were written by a different author or authors. The material on Trigonometry is well selected and clearly presented; the solution of triangles is not over emphasized and the analytical part of the subject is given clearly and accurately so that a student can see the real value of Trigonometry.

The part dealing with Analytic Geometry is equally well done. The reviewer would like to see more time and space devoted to parametric equations, but that of course is a matter of taste. This is one of the few elementary books on the subject that discuss the problem of finding all points of intersection of two curves given in polar coördinates. The part dealing with Solid Analytic Geometry is of necessity rather brief but it is clear and very much to the point. The chapter on curve fitting is a very welcome addition to an elementary textbook and will no doubt be stimulating, both to the student and to the teacher. There is a number of tables and a bibliography that enhance the value of the book; the type is clear and I noticed only one misprint (page 71, line 8: + instead of =).

The reviewer can only hope that some day soon the first section of this book will be rewritten with the same care, good taste and discrimination that characterize the last two sections; for then it will be an outstanding book on "Freshman Mathematics."

M. S. KNEBELMAN

Klassische Funktionentheorie. Teil II. By H. Behnke and F. Sommer. Aschendorffsche Verlagsbuchhandlung, Munster Westf. 1948. 217+16 pp. DM 6.

The material of this book was prepared from lectures of H. Behnke with the editorial help of F. Sommers. The entire volume is devoted to the study of Riemann surfaces and the uniformization theory. Although the material is neither new nor complete, the presentation is in a very readable style—a style which prefers clearness to conciseness.

The book begins with a general introduction to Riemann surfaces. The postulates are made understandable by numerous examples. An analytic function having a compact Riemann surface is said to be an algebraic function. It is shown that such functions satisfy an irreducible algebraic equation and conversely. A function meromorphic on the Riemann surface belonging to an algebraic function $A(z)$ is said to belong to the function field of $A(z)$. It is shown $F(z)$ is a function of the field of $A(z)$ if and only if $F(z)$ can be expressed as a rational function of z and $A(z)$. In fact if the Riemann surface of $A(z)$ has n-sheets then the functions $1, A, \cdots, A^{n-1}$ form an algebraic basis over the field of rational functions of z for the function field of $A(z)$. The results of this theory are applied to Abelian integrals and algebraic curves.

The second half of the book is devoted to uniformization theory. A function $t=h(P)$ is called a uniformization mapping for a Riemann surface R if it is a schlicht mapping of the universal covering surface of R onto a simply connected domain D of the t-plane. The necessary machinery is first developed concerning normal families of meromorphic and schlicht functions, the Koebe distortion theorem, and universal covering surfaces. The existence of a uniformization mapping is then established and the Riemann surfaces are classified according as the normalized domain D is the unit circle, the open plane, or the closed plane. To each automorphism of the covering surface which leaves R invariant, there corresponds a bilinear transformation in the t-plane which leaves $z=h^{-1}(t)$ invariant (where z is the point of R underlying P). By definition h^{-1} is an automorphic function on D under the above group of automorphisms. A final section is devoted to automorphic functions. Here the author discusses the problem of finding a function which has a given Riemann surface for its associated Riemann surface.

R. S. PHILLIPS

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1948–1949

Mathematics Club, Adelphi College

The members of the *Mathematics Club* of Adelphi College heard three guest speakers during 1948–49. The speakers and their topics were:

Methods and models, by Col. Robert Beard of Mount Vernon, New York

Methods and models, by Lt. Col. R. C. Yates of West Point

The number e and the law of continuing growth, by Dr. Howard Fehr of Columbia University

The Hofstra College chapter of *Kappa Mu Epsilon* attended the meetings at which Colonel Beard and Dr. Fehr spoke.

Field trips were taken to the home of Rutherford Boyd at Leonia, New Jersey to see some of his artistic work and to the Brookhaven National Laboratory at Upton, New York.

The officers for 1949–50 are: President, Katherine Engler; Secretary-Treasurer, Sylvia Krasner.

The Hall Mathematics Society, Lafayette College

During the academic year 1948–49, the *Hall Mathematics Society* held six meetings. The programs for these meetings were:

The game of nim, by George Veronis

An appraisal of the life and work of Descartes, by William McLean, Professor of Mechanics

Applications of mathematics to sailing, by Karl Larsen, Professor of Physics

Electrical integrator, by Hugh Davidson, research physicist at the General Analine Corporation and founder of the idea for the integrator

Symposium on various earth maps, by George Veronis, Donald McIntyre, Robert Koch and Alexander McKittrick

The relationship of logic and mathematics, by George Clark, Professor of Philosophy.

Pi Mu Epsilon, Hunter College

The topic of the Hunter College Chapter of *Pi Mu Epsilon* for the year 1948–49 was *Highlights in the history of mathematics as reflected in some of the monumental books*. Papers presented during the first semester were:

The mathematics of ancient Babylonia, by Elaine Traub

The mathematics of ancient Egypt, by Estelle Rosen

Pythagoras and the Pythagorean School, by Helen Sarfaty

Euclid, by Sarah Kaplan

Appollonius, by Gloria Hoffman
Archimedes, by Marian Boykan
Diophantus, by Ruth Roberts
Ptolemy, by Shirley Kurman.

Papers delivered during the second semester were:

Number notations and arithmetics of ancient Babylonia and Egypt, by Arlene Withrow

Number notations and arithmetics of ancient Greece, Rome, and India, by Miriam Rosenberg

The algebra of the Hindus, by Janis Resnick
Algebra in the Arab world, by Caryl Klein
Galileo Galilei, by John Keiper
Renê Descartes, by Angeliki Demetriades
Gottfried Wilhelm Leibnitz, by Rita Challenger
Isaac Newton, by Josephine Guaragna.

Guest speaker at the fall initiation dinner was Prof. Carl Boyer of Brooklyn College who delivered an address entitled *The greatest analytic geometer*. At the spring initiation dinner the speaker was Prof. Richard Courant of New York University whose topic was *Existence and construction in mathematics*.

Officers for the year were: President, Josephine Guaragna; Corresponding Secretary, Estelle Rosen (1st term) and Ruth Roberts (2nd term); Recording Secretary, Rita Challenger; Treasurer, Arlene Withrow; Faculty Director, Miss Carolyn Eisele.

Mathematics Club, University of British Columbia

Papers were presented to the *Mathematics Club* of the University of British Columbia by undergraduate members at eight meetings held on the campus in the evening. The titles and authors of the papers are in the order of presentation:

Linear difference equations of the first degree, by P. C. Gilmore
Cardinal numbers, by J. L. McGregor
Theory of quaternions, by H. Harris
A proof of Minkowsky's theorem, by R. A. Macauley
Introduction to the theory of Lebesque integration, by P. Cuttle
Group theory, by G. B. Kyle
A qualitative method for some differential equations of mechanics, by P. C. Gilmore
The solution of cubic and quartic equations, by K. D. Hage.

A special speaker to the club was Professor Maslow of the department of philosophy of U.B.C. The topic of his lecture was *Mathematical logic*. Each meeting was followed by an informal discussion period while refreshments were served.

An interesting year for the club is being planned by the new president, Robin Thornton. Included in his plans are informal meetings to be held at the homes of members, adding social interest to the academic nature of the club.

Mathematics Club, Hunter College

Student papers presented at meetings of the *Mathematics Club* of Hunter College were:

Euclid, by Evelyn Margoe
Hypocycloid and epicycloid, by Patricia Cohen
Babylonian mathematics, by Elaine Traub
The birth of algebra, by Helen Smith
Center of gravity, by Marian Boykan
Stellar motions, by Flora Johns.

Faculty talks were:

Fermat's last theorem, by Prof. Marie Whelan
Mathematical induction, by Prof. Marie Whelan
The slide rule, by Prof. Hobart Bushey
The concept of curve, by Prof. Jewell H. Bushey.

The club had two supper parties during the year at the Roosevelt House, and in addition held a theatre party and a boat ride.

The officers for the year were: President, Dorothy Carli; Vice-President, Gloria Bullock; Treasurer, Helen Smith; Secretary, Doris Cohen.

Mathematics Club, United States Naval Academy

During the year 1948–49, the *Mathematics Club* of the United States Naval Academy arranged for two extra-curricular series of lectures and conferences given by members of the department of mathematics, as follows:

Modern algebra, by Prof. J. C. Abbott and Prof. T. J. Benac
Advanced calculus, by Prof. H. K. Sohl.

The club also held a series of informal evening meetings, the feature of which was a group of discussions and lectures on *The mathematical foundations of potential theory.*

For the coming year the club plans to have a larger number of guest lecturers and thus vary and supplement its program.

Pi Mu Epsilon, University of Alabama

Two university-wide competitive mathematics examinations featured the activities of the *Alabama Alpha* chapter of *Pi Mu Epsilon* this year. In the advanced section R. C. Asquith was judged the winner while W. D. Driggers was second. In the elementary division Walter Wilson won first prize and D. K. Gleason was second. In both classifications the winners received twenty dollar prizes while the runners-up won ten dollars.

Prizes were also awarded for the best student papers presented before the chapter during the year. Twenty dollars was won by Ann Lutz who spoke on *A refinement for linear interpolation* and Jack Caldwell won ten dollars for his paper entitled *The abacus*. Other papers included:

The life of Fourier, by Mrs. Charlotte Speight

Fourier Series, by Mrs. Martha Weeks

Mathematical mayhem, by Frank Butler

Plane areas by complex integration, by Prof. J. D. Mancill

Some mathematical applications of the cathode ray oscilloscope, by Robert Whitehurst.

The social activities of the chapter were high-lighted by a Christmas party and a spring picnic.

The officers for 1949–50 are: Director, Mr. Neal Rowell; Vice-Director, Frank Butler; Treasurer, Bean Green; Secretary, Dr. H. S. Thurston; Publicity Chairman, Clyde Wiley; Librarian, Dr. J. D. Mancill; Social Chairman, Miss Susie Lee Ward.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

INTERNATIONAL CONGRESS OF MATHEMATICIANS

The Association is sponsoring an exhibit of all types of mathematical models at the International Congress of Mathematicians which will be held at Harvard University on August 30–September 6, 1950. Mathematicians are encouraged to send lists of items for exhibit and requests for approximate space allotments to Lieutenant Colonel R. C. Yates, Department of Mathematics, West Point, New York.

SUMMER MEETING OF THE NATIONAL COUNCIL

The Tenth Summer Meeting of The National Council of Teachers of Mathematics will be held at The University of Wisconsin, August 21, 22, and 23. Members of the Association are welcome at all sessions, and registration fees for Association members are the same as for members of The National Council. Papers on problems of teaching mathematics in the first and second college years and on teacher education programs will be presented by H. G. Ayre, Western Illinois State College; Aaron Bakst, New York University; J. G. Bowker, Middlebury College; Henriette Brudos, North Dakota State Teachers College; C. H. Butler, Western Michigan College of Education; F. N. Fisch, Colorado College of Education; Elinor B. Flagg, Illinois State Normal University; M. L. Hartung, The University of Chicago; E. R. Heineman, Texas Technological College; J. S. McNair, Canal Zone Junior College; H. V. Price, State University

of Iowa; Vera Sanford, New York State Teachers College at Oneonta; J. H. Zant, Oklahoma A & M College.

Requests for rooms in University Residence Halls should be addressed to J. R. Mayor, North Hall, Madison 6, Wisconsin.

INSTITUTE FOR TEACHERS OF MATHEMATICS

The Institute for Teachers of Mathematics which is sponsored by the Association of Teachers of Mathematics in New England will be held at Tufts College, Medford, Massachusetts on August 22–29, 1950.

The program will include speakers on the latest developments in pure mathematics and others on applications of mathematics. There will also be discussion groups on methods of teaching and an abundance of recreation.

For more information write to Janet S. Height, Wakefield High School, Wakefield, Massachusetts or Albert Norris, Milton Academy, Milton, Massachusetts.

PERSONAL ITEMS

Dean Gillie A. Larew of Randolph-Macon Woman's College represented the Association at the inauguration of President O. W. Wake of Lynchburg College at Lynchburg, Virginia, on April 25, 1950.

Professor C. N. Moore, University of Cincinnati, served as the representative of the Association at the Convocation commemorating the Centenary of the Founding of the University of Dayton.

Professor J. R. Musselman of Western Reserve University has been appointed as the representative of the Association on the Council of the American Association for the Advancement of Science for 1950–51.

Professor H. S. M. Coxeter of the University of Toronto has been awarded the Tory Medal by the Royal Society of Canada.

Professor Henry Wallman of Massachusetts Institute of Technology who is now on leave at Chalmers Institute of Technology in Gothenburg, Sweden, has been made a foreign member of the Royal Swedish Academy.

Professor K. P. Williams of Indiana University has received a special mention award from the publishing trade for his book *Lincoln Finds a General;* also, he has been awarded a gold medal by New York University's Society of Libraries.

Dr. Iacopo Barsotti of the University of Rome and Princeton University has been appointed to an associate professorship at the University of Pittsburgh.

Professor P. F. Byrd of Fisk University has accepted a position as research scientist with the National Advisory Committee on Aeronautics.

Professor S. S. Chern of the Institute for Advanced Study has been appointed to a professorship at the University of Chicago.

Dr. Peter Chiarulli of Brown University has been appointed to an assistant professorship at Carnegie Institute of Technology.

Associate Professor Lester Dawson of Adams State College has accepted an assistant professorship at Colorado Agricultural and Mechanical College.

Dr. Werner Fenchel of the Technical University of Denmark has been appointed to a visiting professorship at the University of Southern California.

Mr. A. G. Hansen has been appointed to an instructorship at the University of Maryland.

Associate Professor R. G. Helsel who is now Acting Head of the Department of Mathematics of Ohio State University has been promoted to a professorship and named Head of the Department, effective July 1, 1950.

Dr. P. J. Kelly of the University of Southern California has received an appointment to an assistant professorship at the Santa Barbara College of the University of California.

Dr. M. S. Kramer of the University of Illinois has been appointed to an assistant professorship at the New Mexico College of Agriculture and Mechanic Arts.

Associate Professor M. Z. Krzywoblocki of the University of Illinois has been promoted to a professorship.

Dr. J. P. Nash of the Kimberly-Clark Corporation is now Research Assistant Professor at the University of Illinois.

Professor B. C. Patterson, head of the Department of Mathematics of Hamilton College, has been appointed President of Washington and Jefferson College.

Miss Ruth E. Porter of Berea College has been appointed to an assistant professorship at John Brown University, Siloam Springs, Arkansas.

Associate Professor E. L. Post, City College of the City of New York, has been promoted to a professorship.

Professor C. J. Rees who has been serving as director of graduate studies of the University of Delaware has been appointed to the first deanship of the School of Graduate Studies, effective July 1, 1950.

Assistant Professor R. W. Shephard of New York University has accepted a position with the Rand Corporation, Santa Monica, California.

Professor J. G. Van der Corput, University of Amsterdam, will spend the academic year 1950–51 at Stanford University.

Professor R. J. Walker of Cornell University has been appointed Chairman of the Department of Mathematics, effective July 1, 1950.

Assistant Professor W. R. Wasow of Swarthmore College is on leave and is serving as mathematician at the Institute for Numerical Analysis of the National Bureau of Standards.

Mr. R. T. Duffner died on March 22, 1950.

Professor Emeritus J. A. Eiesland of West Virginia University died March 11, 1950. He was a charter member of the Association.

Dr. E. D. Hellinger, visiting professor of mathematics at Illinois Institute of Technology, died on March 28, 1950.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 115 persons have been elected to membership by the Board of Governors on applications duly certified:

H. L. ALDER, Ph.D. (California) Asst. Professor, University of California, Davis, Calif.

H. W. ALEXANDER, Ph.D. (Princeton) Professor, Adrian College, Mich.

P. N. ARMSTRONG, Student, University of Nebraska, Lincoln, Nebr.

E. L. ARNOFF, M.S. (Case) Grad. Asst., California Institute of Technology, Pasadena, Calif.

LUCIEN BAGNETTO, M.S. (Louisiana) Asst. Professor, Harding College, Searcy, Ark.

JEAN M. BALDWIN, Student, Carleton College, Northfield, Minn.

JOHN BENDER, Student, Rutgers University, New Brunswick, N. J.

K. S. BERGMAN, Student, Gonzaga University, Spokane, Wash.

S. D. BERKOWITZ, Student, Columbia University, New York, N. Y.

H. F. BLASCH, Student, Hofstra College, Hempstead, N. Y.

ROBERT BREUSCH, Ph.D. (Freiburg) Asso. Professor, Amherst College, Mass.

W. E. BRIGGS, M.A. (Colorado) Part-time Instructor, University of Colordao, Boulder, Colo.

RICHARD CROWLEY CAMPBELL, A.M. (Pennsylvania) Asst. Professor, U. S. Naval Postgraduate School, Annapolis, Md.

LEONARD CARLITZ, Ph.D. (Pennsylvania) Professor, Duke University, Durham, N. C.

C. R. CARR, A.M. (Michigan) Teaching Fellow, University of Michigan, Ann Arbor, Mich.

Mona Cascio, 171 Rogers Avenue, Brooklyn, N. Y.

W. R. CASHEN, M.A. (U. of Washington) Asst. Professor, University of Alaska, College, Alaska

GEORGE CASSEB, B.S. (Chicago) Instructor, St. Mary's University, San Antonio, Texas.

T. E. CHEATHAM, JR., Student, Purdue University, Lafayette, Ind.

S. S. CHERN, D.Sc. (Hamburg) Professor, University of Chicago, Ill.

A. J. COLEMAN, Ph.D. (Toronto) Lecturer, University of Toronto, Ont.

ESTHER COMEGYS, Ph.D. (Radcliffe) Asst. Professor, University of Maine, Orono, Maine.

JOANN M. CUMMING, B.S. (Mt. St. Vincent) Grad. Asst., Marquette University, Milwaukee, Wis.

K. B. CUMMINS, M.A. (Bowling Green State) Teacher, New Washington High School, Ohio.

J. F. DARLING, A.B. (Cornell) 11 W. Grant St., Woodstown, N. J.

L. C. DAWSON, M.A. (Wichita) Asst. Professor, Colorado A. and M. College, Fort Collins, Colo.

A. S. DAY, Ph.D. (Yale) Asst. Professor, Wesleyan University, Middletown, Conn.

LUIS DE GREIFF B., Civ.Eng. (Universidad Nacional de Colombia) Professor, Facultad de Minas, Medellin, Colombia.

ANNIE L. DICKS, A.M. (South Carolina) Professor, Columbia College, S. C.

R. J. DICKSON, JR., B.S. (Stanford) Grad. Assistant, California Institute of Technology, Pasadena, Calif.

A. R. DIDONATO, Student, Duquesne University, Pittsburgh, Pa.

RAGNAR DYBVIK, Cand.Real (Oslo) Lektor, Levanger kommunale hogre almenskole, Norway.

R. D. EDWARDS, M.A. (Pittsburgh) Instructor, University of Pittsburgh, Pa.

P. G. ENGSTROM, Student, Bethany College, Lindsborg, Kansas.

H. D. FELDMAN, Student, Rutgers University, New Brunswick, N. J.

E. R. FINCK, JR., Student, St. Mary's University, San Antonio, Texas.

R. M. FREEBORN, A.B. (Whittier Coll.) Instructor, Mark Keppel High School, Alhambra, Calif.

Leona Freeman, B.A. (Hunter Coll.) Teacher, Stitt Jr. High School, New York, N. Y.

C. B. GERMAIN, Student, College of St. Thomas St. Paul, Minn.

LEON GOLDSTEIN, Staff Member, Division of Industrial Cooperation, Massachusetts Institute of Technology, Cambridge, Mass.

W. A. GOLOMSKI, M.S. (Marquette) Grad. Asst., Marquette University, Milwaukee, Wis.

CLEM GRABNER, JR., M.A. (Indiana) Asst. Professor, Madison College, Harrisonburg, Va.

EMIL GROSSWALD, M.S. (Bucharest) Asst. Instructor, University of Pennsylvania, Philadelphia, Pa.

L. C. HILL, B.S. in Ed. (Miami U.) Grad. Student, Miami University, Oxford, Ohio.

HILDEGARDE HORENI, M.A. (Minnesota) Instructor, State Teachers College, Mankato, Minn.

R. J. JAEGER, JR., Student, Hofstra College, Hempstead, N. Y.

P. B. JOHNSON, Ph.D. (C.I.T.) Asst. Professor, Occidental College, Los Angeles, Calif.

L. G. JONES, M.A. (Oregon) Grad. Asst., University of Oregon, Eugene, Ore.

JOHN KAISER, M.A. (Ohio U.) Asso. Professor, Kent State University, Ohio.

RAYMOND KASSLER, Student, Brooklyn College, N. Y.

J. B. KELLY, Ph.D. (M.I.T.) Instructor, University of Wisconsin, Madison, Wis.

T. W. KELLY, B.S. (C.I.T.) Project Engineer, Holloman Air Force Base, Alamogordo, N. M.

MRS. EVELYN K. KINNEY, M.A. (Illinois) Head of Math. Dept., Ferrum Junior College, Va.

J. N. P. LAWRENCE, Student, Johns Hopkins University, Baltimore, Md.

L. H. N. LEE, M.S. (Minnesota) Instructor, University of Minnesota, Minneapolis Minn.

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DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The regular fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at American University, Washington, D. C., on Saturday, December 10, 1949. Professor L. M. Kells, the Chairman of the Section, presided at both the morning and afternoon sessions.

Sixty-five persons registered their attendance at the meeting, including the following forty-six members of the association: R. P. Bailey, R. H. Bing, W. E. Bleick, R. S. Burington, H. H. Campaigne, Randolph Church, G. F. Cramer, C. H. Frick, B. C. Getchell, Michael Goldberg, R. A. Good, E. C. Gras, D. W.

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The spring meeting of the section will be held during May, 1950. The place and exact date will be announced at a later time.

The following papers were presented:

1. *A Sturmian separation theorem*, by Professor C. Gras, United States Naval Academy.

Consider $\Phi=\phi_1 u+\phi_2 Ku'$ and $\Psi=\psi_1 u+\psi_2 Ku'$, where u is a solution of the differential equation $d(Ku')/dx=Gu$. The properties of the zeros of Φ and Ψ are readily obtained by use of vector algebra, rather than resorting to the Ricatti differential equation.

2. *Nonlinear transformations of divergent and slowly convergent sequences*, by Mr. Daniel Shanks, Naval Ordnance Laboratory, introduced by the Secretary.

An analogy between mathematical sequences and the transients of linear systems is developed. Through each $2K+1$ consecutive values of the sequence A_n one passes a continuous function of the form $B+\sum_{i=1}^{k} a_i e^{\alpha_i n}$. The series of exponentials either converges to, diverges from, or is asymptotic to, the constant B. An explicit formula for B in terms of the A_n is given, and this forms the basis of several nonlinear sequence-to-sequence transforms $A_n \rightarrow B_n$. The first order transform ($K=1$) is

$$B_n = \frac{A_n^2 - A_{n+1}A_{n-1}}{2A_n - (A_{n+1} - A_{n-1})}.$$

If iterated, this transform is very useful for obtaining numerical results. The transforms are applied to a variety of convergent and divergent sequences. Some theorems concerning these transforms, and some relations to the Padé Table, Thiele's Reciprocal Differences, and Gauss' Numerical Integration, are indicated.

3. *Mathematical requirements for the personnel of a computing laboratory*, by Dr. R. F. Clippinger, Aberdeen Proving Ground, introduced by the Secretary.

In order to operate the modern computing machine for maximum output, a staff of perhaps twenty mathematicians of varying degrees of training is required. There is currently such a shortage of persons trained for this work, that machines are not working full time. With the number of machines building or projected it is probable that within ten years, two thousand persons will be required in this work. Since this is a substantial fraction of all professional mathematicians in the country, Dr. Clippinger urged increased attention to the phases of mathematics which are essential for such work. This would include, among other topics, emphasis on differential equations, particularly general existence theorems, numerical analysis, and such algebraic topics as matrices and the inversion of matrices of high order. It was also urged that work in coding, *i.e.* specific discussion of techniques of preparation of problems for machines, be introduced. A course of this nature is currently being conducted at the Aberdeen Proving Ground.

4. *On the minimum in a sequence of random variables*, by Dr. M. L. Juncosa, The Johns Hopkins University, introduced by the Secretary.

Let $\xi_n=\min_{k\leqq n} X_k$, $n=1, 2, \cdots$, where $\{X_k\}$ are mutually independent random variables

whose respective distribution functions are $\{F_k(x)\}$. Previous work of Gnedenko (*Annals of Math.*, 1943) on identically distributed X_k's is extended to X_k's with different distributions. With some natural restrictions on $\{F_k(x)\}$ some necessary and sufficient conditions for the stability and also relative stability of $\{\xi_n\}$ are given. The limiting distribution of $(\xi_n - a_n)/b_n$ where $\{a_n > 0\}$ and $\{b_n\}$ are suitable sequences of constants, is investigated. The chief result is that limiting distributions much more general than Gnedenko's three classes now appear. The limiting distribution in case $F_k(x) = F(\alpha_k x + \beta_k)$ is also examined. Finally, in the case of positive random variables X_k, conditions on $\{F_k(x)\}$ are given in order for the limiting distributions of ξ/a_n to be of the form $1 - e^{-x}$. (This work appears in *The Duke Mathematical Journal*, Dec. 1949.)

5. *Roulettes in gas dynamics*, by Prof. M. H. Martin, University of Maryland. This was an invited address.

The classical construction due to Prandtl and Meyer for supersonic flow of a polytropic gas around a corner is extended from isentropic flows to anisentropic flows. For isentropic corner flow, it is well known that the hodograph of the flow is an epicycloid. This epicycloid is generated by rolling a circle of radius s externally on the circle $q = r$ where $r = c$, $s = (1-c)/2$, $c = \sqrt{\gamma-1}/\sqrt{\gamma+1}$ = critical speed.

For anisentropic flow the hodographs are generated by rolling a circle of radius $|s|$ on a circle of radius r, and are described by a point fixed on a radius of the rolling circle at a distance t from its center. Here

$$r = \frac{1 + q_0}{q_0 + c^2} c^2, \qquad s = \frac{(1 + q_0)(q_0 - c^2)}{2(q_0 + c^2)}, \qquad t = \frac{1 - q_0}{2},$$

where q_0 is a parameter restricted to the unit interval $0 < q_0 < 1$. Depending on the value selected for q_0, the hodograph will be a hypotrochod, circle, epitrochoid, epicycloid, or limaçon. In particular, for $0 < q_0 < c$, the hodograph curve crosses the sonic circle $q = c$, and the corner flow in the physical plane affords an example of transonic flow.

S. B. JACKSON, *Secretary*

MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirtieth regular meeting of the Southern California Section of the Mathematical Association of America was held at Immaculate Heart College, Hollywood, California on Saturday, March 11, 1950. Professor H. R. Pyle, Chairman of the Section, presided.

The attendance was one hundred and six, including the following seventy-one members of the Association: L. J. Adams, O. W. Albert, E. F. Beckenbach, Lulu Bechtolsheim, May M. Beenken, Clifford Bell, C. M. Bell, Jr., H. F. Bohnenblust, Herbert Busemann, W. D. Cairns, L. M. Coffin, Myrtie Collier, F. E. Cothran, P. H. Daus, R. P. Dilworth, D. C. Duncan, Arthur Erdélyi, Ruth A. Fish, W. B. Fulks, S. I. Gass, B. K. Gold, Jr., J. W. Green, C. J. A. Halberg, Jr., A. F. Herbst, R. B. Herrera, M. R. Hestenes, R. E. Horton, J. M. Howell, D. H. Hyers, C. G. Jaeger, P. B. Johnson, G. R. Kaelin, Samuel Karlin, P. J. Kelly, A. J. Kempner, Cornelius Lanczos, Ella E. Lausman, L. C. Lay, Margaret B. Lehman, Fred Marer, D. W. Matlack, G. F. McEwen, F. R. Morris, W. H. Myers, J. A. S. Neilson, J. B. Nelson, D. J. Peterson, W. T. Puckett, H. R. Pyle, R. L. Rauch, E. C. Rex, J. M. Robb, R. M. Robinson, R. W. Shephard, Sister Rose Gertrude, Samuel Skolnik, I. S. Sokolnikoff, D. V. Steed, Robert Steinberg, J. D. Swift, T. E. Sydnor, W. I. Thompson, V. C. Throck-

morton, C. W. Trigg, D. D. Wall, Jean B. Walton, R. L. White, A. L. Whiteman, B. R. Wicker, R. L. Wilder, G. M. Wing.

At the business meeting the following officers were elected for the next academic year: Chairman, Herbert Busemann, University of Southern California; Vice Chairman, W. H. Glenn, Jr., John Muir College; Program Committee, R. E. Horton (Chairman), P. J. Kelly, P. A. White, and the Acting-Secretary W. T. Puckett, ex-officio; Acting-Secretary, W. T. Puckett, University of California at Los Angeles. The next meeting was scheduled for March 10, 1951 at Whittier College, Whittier, California.

The following program was presented:

1. *Models useful for mathematical instruction*, by Mrs. Margaret B. Lehman, University of California at Los Angeles, Mr. D. W. Matlack, North American Aviation, Inc., Mr. C. J. A. Halberg, Pomona College, and Mr. L. C. Lay, John Muir College.

The models exhibited were left on display during the social hour which preceded the program, and during the noon intermission. Mrs. Lehman stressed the desirability of making models with visibility in mind. Mr. Matlack exhibited twenty models representing various stellated forms of the rhombic triacontahedron, and thread models of the tangent and binormal surfaces of the "baseball curve" lying on a sphere. Mr. Halberg stressed certain aspects of the construction of models using the newer plastics. Mr. Lay exhibited a number of slides constructed by students in classes in analytic geometry.

2. *Problems sections in American mathematical magazines*, by Mr. C. W. Trigg, Los Angeles City College.

The speaker discussed the problem departments of this MONTHLY, *School Science and Mathematics*, and *Mathematics Magazine*, with particular reference to their composition, contributors, recurrent problems, and unsolved problems. He distributed mimeographed copies of unsolved problems from this MONTHLY and *Mathematics Magazine*, and urged the listeners to join the ranks of "problem solvers."

3. *Convexity and related notions*, by Professor J. W. Green, University of California at Los Angeles.

In this invited hour address, the definitions and some fundamental properties of convex sets and convex functions were outlined, and a number of generalizations of these notions were discussed, together with some of the problems arising under these definitions. Considerable interest was shown in the subject, and the discussion lasted for some time after the talk was given.

4. *Fixed points*, by Professor R. L. Wilder, University of Michigan.

Professor Wilder gave an exposition of theorems and problems concerning fixed points, and their applications, which presupposes no knowledge of topology.

5. *Quadratic congruences and quadratic reciprocity*, by Professor A. L. Whiteman, University of Southern California.

Making use of Gauss sums, the speaker presented a simple derivation of the classical formula for the number of solutions of the congruence

$$x_1^2 + x_2^2 + \cdots + x_n^2 \equiv a \pmod{p}.$$

Lebesgue's proof of the law of quadratic reciprocity was given as an application.

6. *Asymptotic series*, by Professor Arthur Erdélyi, California Institute of Technology.

In this expository paper Euler's series $1-1!x+2!x^2-3!x^3+\ldots$ was taken as a point of departure for a brief introduction to semiconvergent and asymptotic series. The speaker contrasted the operational and arithmetic points of view, and showed how such series can be made useful.

7. *The M.A. degree in mathematics as offered at San Jose State College*, by Dr. W. H. Myers, San Jose State College.

The speaker announced an extension of the graduate program at San Jose State College to include the granting of the M.A. degree in mathematics. This accredited college, whose enrollment exceeds 8000, has been on the approved list of the Association of American Universities since before the war. The fifth year of work leading to the California General Secondary Teaching Credential is already available here.

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P. H. DAUS, *Secretary*

CALENDAR OF FUTURE MEETINGS

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December 30, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
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IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, Duluth Branch of University of Minnesota, October 7, 1950.
MISSOURI, Central College, Fayette, Spring, 1951.
NEBRASKA
NORTHERN CALIFORNIA, University of San Francisco, January 27, 1951.
OHIO
OKLAHOMA
PACIFIC NORTHWEST, University of Washington, Seattle, June 16, 1950.
PHILADELPHIA, Lehigh University, Bethlehem, Pennsylvania, November 25, 1950.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Whittier College, Whittier, March 10, 1951.
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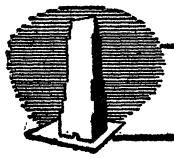
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THE FIRST TRANSLATION OF EUCLID'S ELEMENTS INTO ENGLISH AND ITS SOURCE

R. C. ARCHIBALD, Brown University

1. Introductory. Note 74 of my recently published *Outline of the History of Mathematics*, sixth edition, January, 1949, was as follows: "The first English edition of EUCLID'S *Elements* was brought out at London in 1570, under the name of Sir HENRY BILLINGSLEY, later sheriff and Lord Mayor of London. (For the small part which BILLINGSLEY may have had to do with the translation see A. DEMORGAN, *British Almanac and Companion for 1837*, pp. 38–39 of *Companion.*) It contained 'a very fruitfull Praeface' by JOHN DEE, a man of great erudition; see W. F. SHENTON, 'The first English Euclid,' AM. MATH. MONTHLY, v. 35, 1928, pp. 505–512."

Not long after my *Outline* was published I discovered that this Note was mortifyingly inadequate in a work professing to present an up-to-date scholarly presentation of known facts in the field treated.

Happening to handle the *Mathematical Gazette*, v. 31, I found in the first number, Feb., 1947, that the frontispiece was a reproduction of the title-page of Billingsley's translation and in the note, "The first English Euclid," on the opposite page, the following sentences occur: "Perhaps because of Dee's notoriety, a good deal of credit for this translation has often been given to him, as for instance by DeMorgan, and also to an Augustinian friar, Whytehead, who appears to have been resident with Billingsley for a time. But, according to Rouse Ball, the copy of the Greek text of Theon's Euclid which Billingsley used, contains notes, comments and emendations in his own hand which make it clear that whatever may have been contributed by Dee and Whytehead the main work was done by Billingsley himself."

I then turned to W. W. R. BALL, *A History of the Study of Mathematics at Cambridge*, Cambridge, 1889, pp. 22–23, where I found the following: "In preparing this [English translation] he had the assistance both of Whytehead and of John Dee. In spite of their somewhat qualified disclaimers, it was formerly supposed that the credit of it was due to them rather than to him, especially as Whytehead, who had fallen into want, seems at the time when it was published to have been living in Billingsley's house. The copy of the Greek text of Theon's Euclid used by Billingsley has however been recently discovered, and is now in Princetown College, America;[1] and it would appear from this that the credit of the work is wholly due to Billingsley himself. The marginal notes are all in his writing, and contain comments on the edition of Adelhard and Campanus from the Arabic and conjectural emendations which shew that his classical scholarship was of a high order."

2. DeMorgan statements. Before indicating what are some of the principal statements of the Halsted Note let us quote portions of DeMorgan statements

[1] In the footnote there is a reference to G. B. HALSTED, "Note on the first English Euclid," *Amer. Jn. Math.*, v. 2, 1878, pp. 46–48.

in the *Companion*. "The principal work of Dee is the translation of Euclid, which appeared with his preface in 1573 [*sic*]. We call it his, though it is universally stated (and by himself* among others) to have been made by Sir Henry Billingsley. Considering that he wrote the preface, the notes, and the translation of the book of Mohammed of Badgad [*sic*] on the division of surfaces, which was published with the second edition; that he had lectured on Euclid in various places, and left behind him MSS. on the subject, and in particular 'Instructions and Annotations upon Euclid's Elements'; considering, also, that the name of Sir H. Billingsley is not mentioned in the preface to his own Euclid; we imagine that the translator was a pupil of Dee, who worked under general, if not special, instructions, and executed the more mechanical part of the undertaking. . . . This translation of Euclid was either made from the Greek, or corrected by the Greek, as is evident from comparing the early Latin versions and the Greek text with it. As there may be some dispute about the degree in which Greek was studied in England at the period of which we write, we shall annex one sentence of comparison, namely, the Greek text of the enunciation of the fourth proposition of the first book, the Latin of Campanus, and the English of Billingsley . . . Latin of Campanus (Editions of 1482, 1491, and 1516)."

Apart from the incorrect date given as the year of publication of Billingsley's Euclid, other inaccuracies may be pointed out.

DeMorgan states that Dee wrote "the Notes" for Billingsley's translation, whereas Dee's only notes are a few in connection with Books X–XIII of the *Elements*. Indeed, Dee in his autobiographical tracts distinctly states that, besides the introduction, he only contributed 'divers and many Annotations and Inventions mathematicall added in sundry places of the aforesaid English Euclide after the tenth booke of the same' (*Miscellanies of Chetham Soc.*, v. 1, no. 5, 1851, p. 73; also *Remains*, v. 24). The complete list of references for Dee's notes is folios 255 *verso*, 306, 311 *recto*, book X; 325 *verso*–326, 329 *verso*, 337 *recto*, 346–347, 348 *verso*, 349, 352, book XI; 356 *verso*–357, 359–362, 371, 376, 380, 381 *verso*, book XII; 391–394 *recto*, 395, 397 *verso*, book XIII.

DeMorgan states that Dee wrote "the translation of the book of Mohammed of Bagdad," whereas he merely published a Latin manuscript which he had in his library; see R. C. ARCHIBALD, *Euclid's Book on Division of Figures*, Cambridge, 1915.

DeMorgan writes "considering . . . that the name of Sir H. Billingsley is not mentioned in the preface to his own Euclid." Why should it be when his name as translator is on the title page and each of the immediately following pages of the preface is headed: "The Translator to the Reader"?

DeMorgan wrote: "As there may be some dispute about the degree in which Greek was studied in England at the period of which we write . . . , " and he then quotes from a Greek edition of Euclid of 1703, 133 years later!

DeMorgan refers to a second edition of Billingsley's work; it will presently appear that there was no such edition.

* In the list of his works, in his apologetical letter to the Archbishop of Canterbury.

DeMorgan states: "This translation [Billingsley's] was made from the Greek, or corrected by the Greek." It will presently appear that the translation was made from the Latin.

DeMorgan cites Campanus Latin editions of Euclid to fortify his argument. Since Billingsley used an entirely different Latin edition, based on translations from both the Greek and Arabic, the whole argument of DeMorgan falls to pieces.

3. Halsted. It was Professor Halsted who, 71 years ago, apparently first made known in print (at the reference given above) that the volume or volumes of which Billingsley made use, in preparing his English Euclid, were located in the library of Princeton University. Halsted concludes his article as follows: "By reading what he has done, it immediately appears that . . . the corrections he has actually made sufficiently prove his scholarship and render entirely unnecessary DeMorgan's suppositious aid from Dr. Dee, while, on the other hand, they establish the conclusion about the translation to which DeMorgan's sagacity had led him, that 'It was certainly made from the Greek, and not from any of the Arabico-Latin versions'."

"To the one sentence of comparison in proof of this published by DeMorgan, Billingsley's autograph indications would enable me to add as many as any one desired, but suffice it to say, that the definitions of the Eleventh Book are alone entirely decisive."

That these statements are wholly unreliable shall presently appear.

4. Sidney Lee. In Lee's biography of Billingsley in the *Dictionary of National Biography* (*DNB*) the following erroneous statement occurs: "His original was the Latin version attributed to Campanus, which had been first printed in 1482, and again in 1509."

5. John Aubrey. In Aubrey's famous *Brief Lives, edited from the Original Manuscripts and with an Introduction* by O. L. DICK, London, 1949, p. 89, we find in the biographical notes on John Dee, "His Picture in a wooden cutt is at the end of Billingsley's *Euclid*." This erroneous statement is also made in the *DNB* biography of Dee, 1888. Professor Shenton has well pointed out that this portrait is of John Day, the printer of Billingsley's volume. See J. JOHNSON, *Typographia*, v. 1, London, 1824, p. 540, where it is stated that, this woodcut bearing the date 1562, is a portrait of Day and "perhaps the earliest of an ancient printer, which can be depended upon as genuine." See also J. AMES, *Typographical Antiquities*, v. 1, London, 1785, p. 647.

6. The Billingsley volumes. The Librarian of Princeton University most graciously allowed these precious volumes to be sent to Brown University for my use.

Before describing one of these volumes it seems desirable to correct many prominent writers concerning the first Latin translation of Euclid's Elements, which was made from the Arabic, about 1120, by an Englishman, ADELHARD (or

Athelhard or Aethelhard) of Bath.† Many historians state that CAMPANO (usually called CAMPANUS) of Novarra, a mathematician, astronomer and medical man, made the first translation of Euclid's Elements from the Arabic and that his manuscript was published at Venice in 1482. This translation, made 150 years after that of Adelhard, was not independent of the first translation. The definitions, postulates and axioms, and the 364 enunciations are word for word identical in Adelhard and Campanus. . . . Campanus may have used Adelhard's translation and only developed the proofs by means of another redaction of the Arabian Euclid. . . . It seems most probable that Campanus stood to Adelhard somewhat in the relation of a commentator, altering and improving his translation by means of other Arabic originals. (Quoted from T. L. Heath, *A History of Greek Mathematics*, v. 1, Oxford, 1921, p. 362–364.) It is interesting that this same point of view is expressed in *Enciclopedia Italiana*, v. 1, 1929, article "Adelardo."

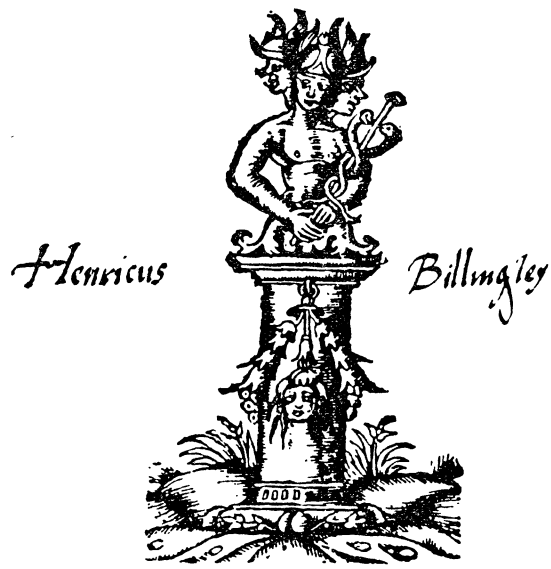
The first translation of Euclid's Elements from the Greek (a Theonine ms.) into Latin was made (after 7 years of labor) by BARTOLOMEO ZAMBERTI (b. 1473), and published at Venice in 1505. His animus against Adelhard-Campanus appears in a number of notes. The first printed edition of the Greek text was brought out at Basle (I. Hervagius, printer) by SIMON GRYNAEUS (d. 1541) in 1533. Combined with this 268-page work is a separately paged (115 pp.) PROCLUS commentary, in Greek, on the first book of Euclid's *Elements*.

The first of the editions giving Adelhard-Campanus's and Zamberti's translations in conjunction was brought out at Paris in 1516. (I now quote from Heath's edition of Euclid.) "The idea that only the enunciations were Euclid's, and that Campanus was the author of the proofs in his edition, while Theon was the author of the proofs in the Greek text, reappears in the title of this edition; and the enunciations of the added Books XIV, XV, are also attributed to Euclid, HYPSICLES being credited with the proofs. . . . The arrangement of the propositions is as follows: first the enunciation with the heading *Euclides ex Campano*, then the proof with the note *Campanus*, and after that *Campani additio*, any passage found in the edition of Athelhard-Campanus's translation but not in the Greek text; then follows the text of the enunciations translated from the Greek with the heading *Euclides ex Zamberto*, and lastly the proof headed *Theo ex Zamberto*. There are separate figures for the two proofs. This edition was reissued with few changes in 1537 and 1546 at Basel (*apud Iohannem Hervagium*), but with the addition of the *Phaenomena*, *Optica*, *Catoptrica*, *etc.*"

Of this 1546 edition there was (according to RICCARDI) an "unauthorized" edition of 1558. It is a copy of this volume, owned by Billingsley and filled with manuscript notes, which was basic in preparing the manuscript for his English Euclid.

† There is a brief sketch of Adelhard by R. ADAMSON in *Dict. Nat. Biog.*, v. 1, 1885. There is also the comparatively recent full-length biography: F. BLIEMETZRIEDER, *Adelhard von Bath. Blätter aus dem Leben eines englischen Naturphilosophen des 12. Jahrhunderts und Bahnbrechers einer Wiedereweckung* der griechischen Antike. *Eine kulturgeschichtliche Studie.* Munich, 1935, viii, 395 p.

The title of this volume is as follows: *Evclidis Mega- | rensis mathematici clarissimi | Elementorum geometrico- | rum libri XV. | Cum expositione Theonis in priores XIII à Bartholomaeo | Zamberto Veneto latinitate donata, Campani in | omnes, & Hypsiclis Alexandrini in du- | os postremos. | His adiecta sunt Phaenomena, Catoptrica & Optica, | deinde Protheoria Marini, & Data. | Postremùm uerò, Opusculum de Leui & Ponderoso, | hactenus non uisum, eiusdem autoris.* | [design reproduced in this article, in the illustration with the autograph, "Henricus Billingsley."] Cum priuilegio Caesareo. | Basileae, Per Ioannem Her- | uagium, & Bernhardum Brand, Anno | M.D. LVIII. It contains 537 folio pages. The six-page preface by Philip Melanchthon is dated Wittenberg, August, 1537.



Cum priuilegio Cæſareo.

BASILEAE, PER IOANNEM HER-
uagium, & Bernhardum Brand, Anno
M. D. LVIII.

There are many notes in margins evidently written by the person whose autograph is on the title page. At some time since the volume was originally used by Billingsley it has been rudely trimmed, probably when bound with another volume; in this way many of the manuscript notes are badly mutilated.

The other volume bound with the 1558 work is the 1533 Greek *editio princeps* to which we referred above. That this volume was also briefly used by Billingsley at some time is shown by slight manuscript notes in his handwriting, on pages 5–9, in connection with propositions 6–8, 13, 17, of the first book of the Elements. All the other 374 pages of the volume seem to be as clean as when they first came from the printer. The manuscript notes are of no importance. Hence further consideration of Billingsley's Greek edition of Euclid's *Elements* may be omitted. We shall presently report in some detail on his Latin edition.

7. Princeton and the 1558 volume. When or whence this most interesting volume became the property of Princeton University or Princeton College or the New Jersey College is unknown, but it would seem as if the Greek volume at least might have been in the College library for 135 years or more; for, on the fly leaf is: "Editio Princeps. Rare & Dear. P.L.," written doubtless by Dr. Philip Lindsley, Librarian in the College from 1812 to 1824. Hence since this Librarian knew the cost of the volume it may well have been acquired during his period of office. Lindsley's ms. Catalogue of books in the College Library (made in 1821) includes this volume. Assuming that the two Billingsley volumes were bound together at this time, and assuming further that the volume had been acquired even in 1812, Halsted's statement in 1879 that Princeton had the volume "for nearly a century, perhaps longer" is not by any means checked. The librarian of today has no such information. The Lindsley Catalogue makes no reference to the 1558 volume.

The present binding of the volume was put on in 1923. There is reason to surmise first, that the binding (of the two volumes in one, bound together previous to this date), had become so unsatisfactory, it was then replaced; secondly, that no trimming of the edges then occurred. The present accession number in the volume was given in 1892.

Whether or not Halsted was the first to discover (in print) the special interest of the volume because of Billingsley's signature is uncertain. On the inside cover of the new binding is pasted an anonymous undated, and unidentified periodical clipping, entitled "An archaeological treasure," which reveals the facts. The initials "J.T.D." are added in ink at the end of the clipping. It has been surmised that these initials probably refer to JOHN T. DUFFIELD (d. 1901) who was at Princeton a tutor in Greek (1845–47), and a member of the mathematics department (1847–98). One sentence of the article is: "It has been in the Library for at least half a century, but the fact that it at one time belonged to the first translator of Euclid into English does not seem to have been known until a few years ago."

8. The manuscript notes in the 1558 volume. The notes commence with the correction of a single Greek letter in Zamberti's discussion of book I, proposition 9 of the *Elements*, on page 14, and end with more than a dozen corrections of the text in proposition 9 of the *Phaenomena* on page 491. In the *Elements*, books

I–XV, there are notes on 256 pages, 21 of these being on the Adelhard-Campanus text, and 235 on material in Zamberti's Latin translation from the Greek. Hence there was a slight show of reason for scholars surmising that Billingsley's translation was from a Greek original, although in reality it was from a Latin translation of a Greek source.

The notes consist, practically wholly, of corrected misprints in letters, words, phrases, of deletions, phrase and word substitutions, additions of letter to figures,—such as every careful student would ordinarily make. All of these notes occur in connection with propositions. The only commentary on a definition is where, in book XI, the omission of a definition of a tetrahedron is noted; this makes the last sentence quoted above from Halsted appear as a decided mental lapse. Not a single note is concerned with setting forth original supplementary ideas. All of these were reserved for record in the so-called "translation," consideration of which, with reference to the Latin original, we shall now take up.

9. The 1558 and 1570 volumes. The title-page of Billingsley's work is reproduced in facsimile in Professor Shenton's interesting article referred to above [as well as in the *Mathematical Gazette*], and the Billingsley preface is quoted in full. The plan "to translate, and set abroad some other good authors, both pertaining to religion (as partly I have already done) and also pertaining to the Mathematicall Artes," referred to in the concluding lines of the preface, was never carried out, according to Lee, in *DNB*.

The preface is immediately followed by a large John Dee folding plate, of size 34.7×41.6 cm. in the Brown Univ. copy, "the Groundplat of my Mathematicall Preface"; this Preface occupies the next 46 pages. Then follow 464 folios 928 pages of a finely printed English edition of the *Elements* with beautifully designed initial letters throughout. The large D, at the beginning of Dee's "fruitfull Praeface" seems to contain his coat of arms.

The Billingsley volume is by no means merely a translation of parts of the 1558, but as the title page states: "Whereunto are annexed certaine Scholies, Annotations, and Inuentions, of the best Mathematiciens, both of time past, and in this our age." For example, the writings of the following authors are quoted: Apollonius, Archimedes, Barlaam, Boetius, Campanus, Dee, Eudoxus, Flussates, Hypsicles, Jordanus Nemorarius, Montaureus, Oenopides, Orontius, Pappus, Pelitarius, Plato, Proclus, Pythagoras, Regiomontanus, Sceubelius, Thales, Theodosius, Timeus. Thus in this single volume was brought together all the earlier and current commentary of importance. The print and appearance of the work was also worthy of its contents, and no pains were spared to represent everything in the clearest and most perfect form—a truly monumental work.

It is noticeable that the names of Theon and Zamberti are not included in the list of authors given above. But the explanation of this is that the Theon-Zamberti text was the one translated as fundamental—the one to which other

discussions were added. Thus the true Greek phrases, which DeMorgan noted, were preserved. Just this procedure would also be expected from the mass of manuscript corrections of the Theon-Zamberti text in the 1558 volume.

10. Authenticity of the Billingsley volumes. In order to check that the autograph and notes were actually in the handwriting of the former Lord Mayor of London, I procured from the British Museum photostat copies of two letters written by the Lord Mayor in 1587 and 1591 [Landsdowne Mss. 62, f. 41 and 67, f. 213]. The comparison of the notes with the writing of these letters was conclusive in showing that they were written by the same person.

11. Euclid Editions by Billingsley 1570, and Leeke & Serle 1661. In the passage by DeMorgan which we quoted above from the *Companion* he refers by indirection to the following work as a "second edition" of that by Billingsley: Euclid's | Elements | of | Geometry. | In XV. Books: | With a supplement of divers *Propositions* | and Corollaries. | To which is added, a *Treatise* of Regular Solids, | By *Campane* and *Flussas.* | Likewise | Euclid's Data: | And Marinus his Preface | thereto annexed. | *Also a* Treatise *of the* Divisions *of* Superficies *ascribed to* | Machomet Bagdedine, *but published* by Commandine, *at the* | *request of* John Dee *of* London; whose Preface *to the said* Treatise | *declares it to be the* Worke of Euclide, | *the* Author *of these* Elements. | Published by the Care and Industry of | John Leeke and George Serle, students | in the *Mathematicks.* | *London:* | Printed by R. & W. Leybourn, for George | Sawbridge at the *Bible* upon *Ludgate-hill, MDCLXI.* [44 p.+folding plate+6 p.]+650 p.+1 p., Errata.

Although this title-page refers to "XV Books" of the *Elements* the authors give "The sixteenth Element of Euclide," pages 490–522. This last page concludes with "*The End of the Sixteenth Element of* Euclide, *added by* Flussas." These books are practically identical in the two volumes. In each case this sixteenth book is followed by "A brief treatise added by Flussas" (folio verso 458–folio recto 463 in Billingsley, with which the volume concludes except for errata and colophon), (pp. 523–532 in Leeke & Serle).

The brief treatise of Flussas here quoted was from the edition of the *Elements* published at Paris in 1566, by Franciscus Flussates Candalla (François de Foix, Comte de Candale, 1502–1594). Two of the figures given here, showing how two Archimedean semiregular solids $P_2 \equiv (8_3, 6_4)$, $P_7 \equiv (20_3, 12_5)$, may be obtained by folding two pieces of appropriately cut paper are of special interest. On folios 320 *verso*–322 *verso* of Billingsley, similar figures are indicated for the five regular solids, but these do not appear in Leeke & Searle. Are these the earliest presentations of such figures? The answer to this question is in the negative. All of these figures and more of the Archimedean semi-regular solids were pictured by ALBRECHT DÜRER in his *Anderweysung der Messung mit dem Zirckel und richtscheyt in Linien, ebenen und gantzen Corporen*, Nürnberg, 1525, in the fourth book on regular bodies. See MAX STECK, *Dürers Gestaltlehre der Mathematik und der bildenden Künste.* Halle, 1948, pp. 64–74.

The first part of the Leeke & Serle volume (L. & S.) contains a dedication (2 p.); Dee's preface as in Billingsley (42 p.+folding plate); "To the Reader" (3 p.); "An account of the author according to ancient philosophers" (3 p.). Then begins the presentation of the Elements.

In the Billingsley volume (B.) each of the 16 books was preceded by an introduction and statement and discussion of definitions. In Books 2, 8, 9, L. & S. has no introductions or definitions and in the case of most of the other books these parts were appreciably shorter; in every case but for Book 12 the forms of statements were quite different. In exceptional manner L. & S. discussion of definitions in Book 1 extends to 17 p. as against 12 in B.

In Book 2, B. gave in connection with each proposition the numerical commentary of Barlaam the 14th century monk (later an abbot and a bishop) edited with Latin translation by Dasypodius in 1564. L. & S. put all of this Barlaam exemplification at the end of their Book 2.

The three kinds of lines in figures throughout L. & S. are explained as follows: "the greater black lines stand for, and denote the Data, or things given, the lesser black lines the things required, and the pricked lines serve for the Constructions and Demonstration; the like method is also observed in the Circular lines given and required." It is further stated in "To the Reader": "that the explications of the things given and required, are comprehended in the words of the Proposition, by means of the correspondent letters relating to the several parts of the Schemes, in such sort, as the Propositions may be read with them, or without them, and so by this means the Demonstrations are much abbreviated." Such editing is of course wholly foreign to Greek form.

The wording of proofs in L. & S. and B. are usually entirely different. L. & S. practically never give any secondary discussions of proofs.

Summing up, we may say that it is wholly improper to state, as DeMorgan does, that the L. & S. volume is a second edition of B. It is simply an independent, and inferior, edition of the *Elements* in the preparation of which L. &. S.—the obscure "students in Mathematicks"—perhaps consulted B. constantly.

The remaining two works of L. & S., not in B. are the first English editions of: (a) Euclid's *Data* (pages 533–600, pages 533–539 being "A commentary or Preface written by the Philosopher Marinus); and (b) *A Book of Divisions of Superficies*, ascribed to Machomet Bagdedine. This book, pp. 601–650, has a special title-page dated 1660. This work was translated from the Latin edition of 1570 published by Dee & Commandinus.

Two other English editions of the *Data* were in editions of Euclid's Elements published by ROBERT SIMSON (1687–1768), Glasgow, 1756, and by THOMAS HASELDEN, in an edition of ISAAC BARROW (1630–1677), London, 1732. Of each of these works there were many other editions.

To the list of Corrigenda for my *Outline of the History of Mathematics*, sixth edition, noted in this MONTHLY, v. 56, Aug.–Sept. 1949, the following may now be added:

P. 13, line 22, *for* 2β as, *read* an

P. 13, line 23, *for* Diophantine equation, *read* Diophantine quadratic equation

P. 33, line 7, for CAMPANUS, read ADELHARD-CAMPANUS

P. 38, line 28, *add:* And in his *Traité du Triangle Arithmetique*, 1665, is a good statement of the method of mathematical induction

P. 59, line 28, *for* 303, *read* 302

P. 61, note 46b, line 3, *for* 1932, *read* 1944, and *for* TAER, *read* THAER

P. 64, note 75, line 2, *for* earliest mathematical, *read* earliest Greek mathematical

P. 71, note 123, line 4, *interchange the values of x and y*

P. 82, note 201, line 9, *for* Levi S. Smith, *read* Levi B. Smith

P. 88, line 4, for MALMKE, *read* MAHNKE

P. 90, note 238, line 3, *delete* Z.d.

P. 94, note 259, *for* 1948, *read* 1918

P. 98, line 1, *for Suppl. Revue, read Suppl. Elemente d. Mathematik*, or *Revue*

P. 104, *after* Adelard of Bath (12th cent.), *add* 33,

P. 107, *for* Erastosthenes, *read* Eratosthenes

P. 110, *for* Malmke, *read* Mahnke, *and put after* Magowan

P. 113, *for* Smith, L. S., *read* Smith, L. B., and *for* TAER [**46b**], *read* THAER [**46, b**] *and put after* Terquem.

In the Index the following dates of death may be added: Engebach, 1946; Nunn, 1944; Ruska, 1949; Simons, 1949; J. W. A. Young, 1948.

To note 208 the following paragraph may be added:

In June 1915 Dr. GEORGE A. PLIMPTON presented to, and installed in, the Delta Kappa Epsilon House at Amherst College a portrait of Sir Isaac Newton, and a fireplace made of wood and bricks taken from the Newton residence referred to above as supplying the Babson Institute "foreparlor." The Amherst Newtoniana are now in the library of the Fraternity House, and a view of them is given on the last page of *Memorial Rooms at Amherst College*, 1937.

SOME NUMERICAL METHODS FOR SOLVING SYSTEMS OF LINEAR EQUATIONS

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Recent papers have exhibited the basic identity of some apparently distinct numerical methods of solving linear systems (see the list of papers at the end). It seems worth-while to note such basic identities since the single general principle under which the methods can be subsumed may well suggest others, and while one may agree with von Neumann and Goldstine [10], Bodewig [1], and others that the Gaussian method of elimination is the best direct method in general, special circumstances will often indicate special methods. We shall therefore attempt to summarize and unify some of these methods.

1. Direct methods. Laderman [7] and Dwyer [3] recommend the use of the "Square-rooting" method of Choleski for symmetric matrices. Bodewig [1] states, von Neumann and Goldstine [10] imply, that this is in reality identical with the Gaussian method of elimination. If the equations are, in matrix form,

$$Ax = y, \tag{1}$$

and A is symmetric, one seeks a triangular matrix L such that

$$A = L'L, \tag{2}$$

the prime denoting transpose. The columns of L are determined sequentially so that in the ith column at most the first i components are non-null. These are chosen so as to give the correct scalar products with the previous i-1 vectors and the correct length, as fixed by the first i components of the ith column of A. In this form square roots occur, but by an obvious modification one can replace this decomposition by

$$A = M'D^2M, \tag{3}$$

where M is triangular with units along the diagonal, while D^2 is a diagonal matrix. This is the form achieved by von Neumann and Goldstine by Gaussian elimination. The point of the decomposition is, of course, the ready invertibility of triangular and of diagonal matrices.

Fox, Huskey and Wilkinson [4] describe a method of orthogonal vectors which they assert to be identical with the "escalator" method of Morris. Their method is the following: One seeks a set of vectors x_i, which are mutually orthogonal with respect to A as a metric tensor. This is to say that if X is the matrix whose columns are the x_i, then

$$X'AX = D^2, \tag{4}$$

where D^2 is a diagonal matrix. One then expresses the required vector x as a linear combination of the columns of X; that is,

$$x = Xw. \tag{5}$$

This is readily done once X is found, since the Equations (1) have clearly become

$$AXw = y, \tag{6}$$

and by (4)

$$D^2w = X'y. \tag{7}$$

In fact,

$$w = D^{-2}X'y$$

and therefore, by (5),

$$x = XD^{-2}X'y. \tag{8}$$

Now the orthogonal vectors x_i can be chosen by orthogonalizing any set of independent vectors, but a natural choice is the set of coordinate vectors. If these vectors are orthogonalized in sequence, then the matrix X is triangular, and, since it is clear that the matrix product in (8) multiplying y is A^{-1}, the relation of this method to those of Gauss and Choleski is exhibited.

Bodewig [1] attributes to Schmidt what amounts to an adaptation of Choleski's method to equations with a non-symmetric matrix, and Turing [9] has also made such an adaptation. The method of orthogonal vectors can be extended immediately if one orthogonalizes, in the ordinary sense, the column vectors of the matrix A. Thus if these column vectors are $a_1, a_2, \cdots$, one chooses a_1 to be the first vector, one replaces a_2 by a combination of a_1 and a_2 that is orthogonal to a_1, and so on. These multipliers form a triangular matrix X whose diagonal elements are all units, and such that

$$(AX)'(AX) = D^2. \tag{9}$$

One uses (5) again so that the equations become

$$AXw = y, \tag{10}$$

whence

$$\begin{aligned} D^2w &= X'A'y, \\ w &= D^{-2}X'A'y, \\ x &= XD^{-2}X'A'y. \end{aligned} \tag{11}$$

Note that X is triangular, and that $X'A'$ is the transpose of the matrix of orthogonalized vectors.

This is, of course, the result one would arrive at from solving

$$A'Ax = A'y \tag{12}$$

by the method of orthogonal vectors, except that the actual formation of the matrix product $A'A$ is unnecessary. Moreover, the normal equations in correla-

tion theory are of the form (12) in which generally A is a rectangular matrix having more rows than columns. Approached in this fashion the equations can be solved without having to be written explicitly.

The method of orthogonal vectors is, in a sense, the inverse of the method of Choleski, since one factorizes A^{-1} rather than A, and the foregoing extension to non-symmetric matrices is the extension of the method of orthogonal vectors. In the Schmidt extension of Choleski's method as described by Bodewig, one orthogonalizes the rows, rather than the columns of A, so that the triangular matrix X is such that

$$XAA'X' = D^2. \tag{13}$$

If

$$A'X'w = x$$

so that

$$AA'X'w = y$$

then

$$D^2w = Xy$$

and

$$x = A'X'D^{-2}Xy. \tag{14}$$

2. Successive approximations. If the matrix A is positive definite we may adopt it as the metric tensor. The equations (1) may be interpreted as relating the contravariant representation x to the covariant representation y of the same vector, the latter being given, the former required. Suppose $x^{(p)}$ is some approximation to x. One seeks an approximation $x^{(p+1)}$ which is to be in some sense better and so that the sequence approaches x as a limit. The vector whose contravariant representation is $x^{(p)}$ has $Ax^{(p)}$ as its covariant representation, and it differs from the required covariant representation by an amount

$$r^{(p)} = y - Ax^{(p)}. \tag{15}$$

This we may call the covariant representation of the pth residual. If u is the contravariant representation of any vector whatever that is not parallel to the residual, then we may project the residual orthogonally on this arbitrary vector, adjoin the projection to $x^{(p)}$ to get $x^{(p+1)}$, and have remaining a residual $r^{(p+1)}$ that is surely smaller than $r^{(p)}$.

To do this one forms $x^{(p)}+\lambda u$ and asks that λ be such that u is orthogonal to the difference between the vectors whose representations are λu and r:

$$u'(\lambda Au - r^{(p)}) = 0$$

$$\lambda = u'r^{(p)}/u'Au.$$

For uniformity we shall write, then,

$$\lambda^{(p)} = u^{(p)\prime} r^{(p)} / u^{(p)\prime} A u^{(p)}, \tag{16}$$

$$x^{(p+1)} = x^{(p)} + \lambda^{(p)} u^{(p)}, \tag{17}$$

and the sequence so defined will converge to x unless, perhaps, one happened to make some peculiar choice of the sequence $u^{(p)}$.

In principle any prescription for choosing a sequence of vectors u would lead to a method of successive approximations. In the method of steepest descent (8) one chooses

$$u^{(p)} = r^{(p)}. \tag{18}$$

In a sense this choice is optional. One would like, of course, to take the vector whose covariant representation is $r^{(p)}$, but one needs the contravariant representation of that vector, and finding this is equivalent to solving the equation. Hence one uses instead the vector whose contravariant representation is $r^{(p)}$. The more nearly orthogonal the base vectors, the more perfect is the choice. But of course if the vectors were orthogonal the matrix A would be diagonal and the equations no trouble in the first place.

In the Gauss-Seidel iteration, one chooses for the vectors u the coordinate vectors taken in rotation, from the first through the last and back to the first. In the method of relaxation one again chooses the coordinate vectors but not in any specified order. One examines at each stage the components of $r^{(p)}$, and if the kth component is largest, one chooses the kth coordinate vector for $u^{(p)}$. One could think of many other possibilities, but probably none so easy to apply.

Adaptation of this basic idea to non-symmetric matrices is almost immediate. One now uses the identity matrix as the metric tensor and considers the space of row vectors of A, or the space of column vectors. Bodewig describes two methods, one due to Kaczmarz and one to Cimmino, both using the space of row vectors. The equations then represent hyperplanes, and one seeks their intersection.

By the method of Kaczmarz, one passes from any approximation $x^{(p)}$ to $x^{(p+1)}$ by projecting the point $x^{(p)}$ orthogonally on one of the hyperplanes. Actually the projection can be made on any hyperplane through x, *i.e.* on a plane

$$u'Ax = u'y \tag{19}$$

where u is an arbitrary column vector. Since $u'A$ is normal to the plane, the projection is

$$x^{(p)} + \lambda A'u = x^{(p+1)}, \tag{20}$$

where λ is so chosen that $x^{(p+1)}$ lies in the plane (19):

$$u'A(x^{(p)} + \lambda A'u) = u'y$$

$$\lambda u'AA'u = u'y - u'Ax^{(p)} = u'r^{(p)}.$$

Hence

$$\lambda = u'r^{(p)}/u'AA'u. \tag{21}$$

If u is one of the coordinate vectors, λ is one component of the residual divided by the sum of squares of the coefficients of the corresponding equation. These sums of squares could all be calculated in advance and each step in the sequence should be fairly simple.

The method of Cimmino consists in reflecting $x^{(p)}$ into each hyperplane. Then $x^{(p)}$ and its n reflections all terminate on a single hypersphere about x, and the centroid of these points, with any associated set of non-negative weights, lies within this hypersphere and defines $x^{(p+1)}$. The method of Kaczmarz (as Bodewig remarks) becomes a special case of this when two weights only are taken non-null and equal.

A. de la Garza has described a method based upon the space of column vectors of A. In this picture, the equations to be solved required the resolution of the vector y along the n column vectors of A, and the components of x are the required multipliers. Any approximate set of multipliers $x^{(p)}$ will yield a vector $Ax^{(p)}$ deviating from y by

$$r^{(p)} = y - Ax^{(p)}.$$

Then if u is an arbitrary set of multipliers, one seeks an optimal improvement of the form λu:

$$x^{(p+1)} = x^{(p)} + \lambda u.$$

The optimal improvement is, of course, the orthogonal projection of $r^{(p)}$ on the vector Au:

$$u'A'(r^{(p)} - \lambda Au) = 0,$$

$$\lambda = r^{(p)\prime}Au/u'A'Au. \tag{22}$$

Again the simplest choice for u is a coordinate vector, in which case the denominator is now a sum of squares of a column. This method has the disadvantage of bringing the matrix A into the numerator for λ. It has, however, the decided advantage of requiring the adjustment of only one component of $x^{(p)}$, when u is a coordinate vector. In Kaczmarz's method, it is to be noted, every component of $x^{(p)}$ is adjusted by an amount proportional to its coefficient in one of the equations. The multiplier λ provides the constant of proportionality.

3. Inversion of matrices. In principle y could be a matrix of any number of columns and x a matrix of an equal number of columns, and the foregoing methods could be applied to the simultaneous solution of all the sets represented. In equation (16), for example, $\lambda^{(p)}$ would be a row vector to multiply on the right of $u^{(p)}$. Hence y could be the identity matrix, in particular, and x is then A^{-1}. In practice, however, it is probably better to observe the following: In the Gauss-Seidel method, the adjustment at any stage consists in solving one of the equa-

tions exactly for the corresponding x component, using the current estimates for the other components. The result of a complete cycle of such operations, passing from the first to the last of the equations in order, can be expressed in the form

$$A_1 x^{(p+n)} = y - A_2 x^{(p)}, \tag{23}$$

where

$$A = A_1 + A_2, \tag{24}$$

A_2 being the triangular matrix of supra-diagonal elements, A_1 the matrix of diagonal and subdiagonal elements. Then A_1 is triangular, non-singular, and readily inverted. Hence it is probably advantageous to write

$$x^{(p+n)} = A_1^{-1} y - A_1^{-1} A_2 x^{(p)} \tag{25}$$

whenever y is a matrix of several columns, and, more particularly, whenever $y = I$ and A^{-1} is required. Applied to this case, if $C^{(p)}$ is any approximation to A^{-1}, one can now write the sequence

$$C^{(r+1)} = A_1^{-1}(I - A_2 C^{(r)}). \tag{26}$$

It is immediately suggested that other decompositions such as (24) might be made, where one would require only that A_1 be easily inverted. In fact, one method that is used sometimes, to advantage when IBM equipment is available, is to take for A_1 the matrix of diagonal elements of A_1, A_2 containing the off-diagonal elements. This amounts to making a complete cycle as in Gauss-Seidel except that the corrections are not incorporated until the cycle is complete.

We shall not follow this line any farther since the geometry tends to get lost in the analysis. Reference should be made to Hotelling [5, 6], and to Bodewig [1] who rediscovers some of Hotelling's results.

4. Conclusion. This does not exhaust the list of either direct methods or methods of successive approximation. In particular Bodewig [1] describes several methods of the latter type for which the geometry is not apparent. Moreover, the fact that two methods may be demonstrably identical in principle does not necessitate that the sequence of numerical operations will be the same, and von Neumann and Goldstine [10] have emphasized the non-commutativity of some of the "pseudo-operations." Nevertheless, an appreciation of the significance of the arithmetic operations should be of assistance in adapting a method to the individuality of the problem in hand.

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THE GENERALIZED LEIBNIZ FORMULA

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1. Introduction. In 1695 and 1696 Leibniz and Johannes Bernoulli carried on a correspondence in the course of which they agreed on the symbol $\int$ for the integral and on d^n for the differential of the nth order. This latter symbol appeared in the famous letter of Leibniz to Bernoulli dated October, 1695, in which Leibniz pointed out the analogy between powers of a sum and differentials of a product, and stated the formula now known as the Leibniz formula for the nth derivative of the product of two functions. He also pointed out that integrals of the nth order can be regarded as differentials of the $(-n)$th order [1].

A little later in a letter to Wallis [2] dated May 28, 1697 Leibniz again briefly mentioned this "marvelous analogy" (*Et notavi mirabilem analogiam relationis inter differentias et summas*) but did not there carry the idea further. It was not until several years later that the formula was first printed in the publications of the Berlin Academy [3].

In the present note the formula is extended to differential operators with constant coefficients, both direct and inverse, and given a noninductive proof which in a sense explains Leibniz' analogy. Various applications are developed, including a systematic foundation of the theory of linear differential equations with constant coefficients.

2. Direct operators. Let $D^n u$ indicate the nth derivative with respect to x of any sufficiently differentiable function, $u(x)$. If then an operator of the form,

$$a_i D + b_i, \qquad i = 1, 2, \cdots, n, \ a_i, b_i \text{ independent of } x, \tag{1}$$

is applied to u and then another operator of this form is applied to the result, and so on, it is easily established that the final result is the same as if these operators had been formally multiplied and the result applied to u. In fact, these operators obey the associative and distributive and exponential laws of algebra, and may be added and multiplied exactly as if they were algebraic quantities.

Similarly, if $u(s)$ and $v(t)$ are respectively sufficiently differentiable functions of the two independent variables s and t, we may indicate partial derivatives of the product uv by $D_1^n uv = \partial^n uv/\partial s^n$, and $D_2^n uv = \partial^n uv/\partial t^n$. Then if operators of the form,

$$a_i D_1 + b_i D_2 + c_i, \qquad i = 1, 2, \cdots, n, \ a_i, \ b_i, \ c_i \text{ independent of } s \text{ and } t, \tag{2}$$

are, as above, successively applied to uv, they obey the same laws of algebra and may be added and multiplied exactly as if they were algebraic quantities. In fact, since u and its derivatives are independent of t, while v and its derivatives are independent of s, this statement is a consequence of the former.

3. Generalized Leibniz formula for direct operators. Let $f(D)$ be an integral rational function of the above operator D of the degree n, which is to be applied to the product $u(x)v(x)$ of two functions of x. We shall do this by first regarding u and v as respective functions $u(s)$ and $v(t)$ of two independent variables, s and t, and then later make the change of variables identifying s and t with x; $s = t = x$. With the above notation and this change of variables, we have

$$Duv = vDu + uDv = (D_1 + D_2)uv$$

and with the aid of the principles of §2 we may write

$$f(D)uv = f(D_1 + D_2)uv = f\{(D_1 - a) + (D_2 + a)\}uv, \ a \text{ independent of } x. \tag{3}$$

We now apply the algebraic identity,

$$f(h + k) = f(k) + \frac{h}{1!}f'(k) + \frac{h^2}{2!}f''(k) + \cdots + \frac{h^n}{n!}f^{(n)}(k),$$

to the operator $f\{(D_1 - a) + (D_2 + a)\}$, obtaining

$$f(D)uv = \left\{f(D_2 + a) + \frac{D_1 - a}{1!}f'(D_2 + a) + \frac{(D_1 - a)^2}{2!}f''(D_2 + a) + \cdots + \frac{(D_1 - a)^n}{n!}f^{(n)}(D_2 + a)\right\}uv, \tag{4}$$

and on the setting $s = t = x$ in the right member, we find that

$$f(D)uv = u \cdot f(D + a)v + \frac{D - a}{1!}u \cdot f'(D + a)v + \frac{(D - a)^2}{2!}u \cdot f''(D + a)v + \cdots + \frac{(D - a)^n}{n!}u \cdot f^{(n)}(D + a)v, \tag{5}$$

which is the Generalized Leibniz Formula for Direct Operators.

For $f(D)=D^n$ and $a=0$ this becomes the usual form of the Leibniz formula for the nth derivative of a product, that is,

$$D^n uv = u\cdot D^n v + nDu\cdot D^{n-1}v + \frac{n(n-1)}{2!}D^2u\cdot D^{n-2}v + \cdots + D^n u\cdot v,$$

the coefficients being those of the binomial theorem.

4. Special properties of direct operators. Equation (5) is especially useful in certain cases. Thus if $u=e^{ax}$, then $(D-a)u=0$, and the entire second member of equation (5) vanishes except the first term. We thus obtain for direct operators the important Formula for Exponential Shift, namely,

$$f(D)e^{ax}v = e^{ax}f(D+a)v. \tag{6}$$

By expressing $\cos \omega x$ and $\sin \omega x$ in exponential form and applying the above formula we may derive the analogous Formula for Trigonometric Rotation,

$$\begin{aligned} f(D)\cos \omega x\cdot v &= \cos \omega x\cdot f_1(D)v - \sin \omega x\cdot f_2(D)v, \\ f(D)\sin \omega x\cdot v &= \sin \omega x\cdot f_1(D)v + \cos \omega x\cdot f_2(D)v, \end{aligned} \tag{7}$$

where f_1 and f_2 are defined by the identity, $f(D+\omega i)=f_1(D)+if_2(D)$.

If $u=e^{ax}P(x)$, in which $P(x)$ is an integral rational function of x of degree p, by application of formula (6) and then formula (5) we have,

$$\begin{aligned} f(D)e^{ax}P(x)v(x) &= e^{ax}f(D+a)P(x)v(x), \\ f(D)e^{ax}P(x)v(x) &= e^{ax}\left\{Pf(D+a)v + \frac{P'}{1!}f'(D+a)v + \cdots \right. \\ &\qquad \left. + \frac{P^{(p)}}{p!}f^{(p)}(D+a)v\right\}. \end{aligned} \tag{8}$$

For $v=1$ this reduces to,

$$f(D)e^{ax}P(x) = e^{ax}\left\{f(a)P + \frac{f'(a)}{1!}P' + \frac{f''(a)}{2!}P'' + \cdots + \frac{f^{(p)}(a)}{p!}P^{(p)}\right\}$$

and for $a=0$ and $P(x)=x^p$, we find

$$f(D)x^p v(x) = x^p f(D)v + px^{p-1}f'(D)v + \frac{p(p-1)}{2!}x^{p-2}f''(D)v + \cdots + f^{(p)}(D)v,$$

the coefficients being those of the binomial theorem.

Evidently formula (8) may be extended to $f(D)\cos \omega x\ P(x)v(x)$ and to $f(D)\sin \omega x\ P(x)v(x)$ by the procedure used in developing equations (7).

5. Application to homogeneous linear equations with constant coefficients. The general solution of the homogeneous linear differential equation with constant coefficients,

$$f(D)y = 0, \tag{9}$$

is easily deduced from equation (5). Let r be a root of multiplicity m of the characteristic equation, $f(r)=0$. Then $f(D), f'(D), \cdots, f^{(m-1)}(D)$ all have the factor $D-r$ and yield zero when operating on e^{rx}. Also if $P(x)$ is an arbitrary integral rational function of x of degree $m-1$,

$$P(x) = C_0 + C_1x + C_2x^2 + \cdots + C_{m-1}x^{m-1},$$

then $D^m, D^{m+1}, \cdots, D^n$ all yield zero when operationg on $P(x)$. Thus in equation (5) with $a=0$, if we take $v=e^{rx}$, the first m terms on the right vanish; and if we take $u=P(x)$, the remaining terms likewise vanish and, hence,

$$y = P(x)e^{rx} \tag{10}$$

is a solution of equation (9) involving m independent arbitrary constants. The sum of the solutions thus obtained from the various roots of the characteristic equation will thus involve a number of independent arbitrary constants equal to the sum of the multiplicities of the roots of the characteristic equation and hence equal to the order of the given equation (9). It is thus the general solution.

6. Inverse operators. If $f(D)$ is an integral rational function of D, then we agree to indicate by,

$$y(x) = \frac{1}{f(D)}u(x) \tag{11}$$

any function $y(x)$ satisfying the equation,

$$f(D)y(x) = u(x). \tag{12}$$

The function $y(x)$ indicated by the inverse operation $\{1/f(D)\}\ u(x)$ is thus not unique, but the difference between any two possible determinations is a solution of the corresponding homogeneous equation, $f(D)y(x)=0$. A statement that two inverse operators are equal is admissable in solving equation (12) if the statement is reducible to an identity by the application of $f(D)$ to both members. It follows that inverse operators in D may be transformed exactly as if they were algebraic quantities, since the direct operators are known to possess this property.

However, when an inverse operator, not a factor of $1/f(D)$, occurs in an algebraic transformation of $1/f(D)$, it is necessary that the same determination of this inverse operation be employed wherever it occurs, as otherwise there is no assurance that these terms would cancel properly if $f(D)$ were applied. This obvious requirement is often overlooked [4].

With the definitions and notation of §2 we may, as above, define

$$y(s, t) = \frac{1}{f(D_1, D_2)} u(s)v(t) \tag{13}$$

as any solution of the differential equation,

$$f(D_1, D_2)y(s, t) = u(s)v(t) \tag{14}$$

in which $f(D_1, D_2)$ is an integral rational function of D_1 and D_2, and s and t are two independent variables. Since $u(s)$ and its derivatives and integrals play the role of constant factors in operations with D_2, while $v(t)$ and its derivatives and integrals play a like role in operations with D_1, it follows that the conclusions of the above paragraphs also apply here, and the operator $1/f(D_1, D_2)$ may be modified in form by algebraic operations subject to the precaution mentioned above.

7. Generalized Leibniz theorem for inverse operators. Now let $1/f(D)$ be an inverse operator which is to be applied to the product $u(x)v(x)$ of two functions of x. As in §3, we first regard u and v as respective functions $u(s)$ and $v(t)$ of two independent variables s and t, and then after any desired algebraic transformations, set $s=t=x$. For to prove that

$$\frac{1}{f(D)} u(x)v(x) = \frac{1}{f(D_1 + D_2)} u(s)v(t), \tag{15}$$

we have merely to observe that the two members yield the same result when operated upon by the respective direct operators, $f(D)$ and $f(D_1+D_2)$, which are equivalent under this change of variables.

For greater generality let us consider the operation,

$$F(D)u(x)v(x) = \frac{P(D)}{Q(D)} u(x)v(x),$$

in which $P(D)$ and $Q(D)$ are integral rational functions of D of degrees p and q, respectively. We introduce the variables s and t, as above, and write

$$F(D)u(x)v(x) = \frac{P(D_1 + D_2)}{Q(D_1 + D_2)} u(s)v(t).$$

We expand P and Q in successive powers of D_1-a, a constant, as in equation (4),

$$P(D_1 + D_2) = P(D_2 + a) + \frac{D_1 - a}{1!} P'(D_2 + a) + \frac{(D_1 - a)^2}{2!} P''(D_2 + a) + \cdots + \frac{(D_1 - a)^p}{p!} P^{(p)}(D_2 + a)$$

$$Q(D_1 + D_2) = Q(D_2 + a) + \frac{D_1 - a}{1!} Q'(D_2 + a) + \frac{(D_1 - a)^2}{2!} Q''(D_2 + a) + \cdots + \frac{(D_1 - a)^q}{q!} Q^{(q)}(D_2 + a)$$

and perform the indicated division, obtaining the algebraic identity

$$F(D) = \frac{P(D_1 + D_2)}{Q(D_1 + D_2)} = A_0(D_2 + a) + \frac{D_1 - a}{1!} A_1(D_2 + a) + \frac{(D_1 - a)^2}{2!} A_2(D_2 + a) + \cdots + \frac{(D_1 - a)^{n-1}}{(n-1)!} A_{n-1}(D_2 + a) + \frac{(D_1 - a)^n}{n!} \frac{R_n(D_1, D_2)}{Q(D_1 + D_2)},$$

where

$$A_0 = \frac{P}{Q}, \qquad A_1 = \frac{P'Q - PQ'}{Q^2}, \cdots,$$

and in which R_n involves $D_1 - a$ only in positive integral powers of degree not greater than the larger of the two numbers $p-n$ and $q-1$. Regarded as an algebraic identity in D_1, the last or remainder term on the right vanishes together with its first $n-1$ derivatives for $D_1 = a$, and consequently the polynomial of the $(n-1)$th degree in D_1, represented by the other terms on the right, must possess the same value and first $n-1$ derivatives as $F(D_1+D_2)$ for $D_1 = a$. Thus,

$$A_i(D_2 + a) = F^{(i)}(D_2 + a), \quad i = 0, 1, 2, \cdots, n - 1$$

and on setting $s = t = x$, we have the Generalized Leibniz Theorem for Inverse Operators, that is,

$$F(D)uv = u \cdot F(D + a)v + \frac{D - a}{1!} u \cdot F'(D + a)v + \frac{(D - a)^2}{2!} u \cdot F''(D + a)v + \cdots + \frac{(D - a)^{n-1}}{(n-1)!} u \cdot F^{n-1}(D + a)v + \left\{\frac{(D_1 - a)^n R_n(D_1, D_2)}{n!Q(D)}\right\} uv. \tag{16}$$

8. Special properties of inverse operators. The remainder term in equation (16) generally becomes complicated for large values of n, but in certain important cases it takes a simple form. Thus if $u=e^{ax}$, then $(D-a)u=0$ and the entire second member of equation (16) vanishes with the exception of the first term. Formulas (6), (7) and (8) previously proven for direct operators now follow at once for inverse operators, $F(D)$, by the same argument as before.

In the special case, $F(D)=1/D$, $a=0$, of equation (16) we have, $R_n=n!/(-D_2)^n$ and the equation yields the familiar Formula for Iterated Integration by Parts, namely,

$$\frac{1}{D}uv = u\cdot\frac{1}{D}v - Du\cdot\frac{1}{D^2}v + D^2u\cdot\frac{1}{D^3}v + \cdots + (-D)^{n-1}u\cdot\frac{1}{D^n}v + \frac{1}{D}\left\{(-D)^n u\cdot\frac{1}{D^n}v\right\}. \tag{17}$$

As one more special case we might mention an analogous formula for double iterated integration by parts, similarly obtained,

$$\frac{1}{D^2}uv = u\cdot\frac{1}{D^2}v - 2Du\cdot\frac{1}{D^3}v + 3D^2u\cdot\frac{1}{D^4}v + \cdots + n(-D)^{n-1}u\cdot\frac{1}{D^{n+1}}v + \frac{(-1)^n}{D^2}\left\{(n+1)D^n u\cdot\frac{1}{D^{n+2}}v + nD^{n+1}u\cdot\frac{1}{D^{n+3}}v\right\}.$$

If in equation (17) we take $u=(b-x)^n/n!$, then u and all its derivatives except the nth vanish for $x=b$, and $(-D)^n u=1$. And if we choose for $(1/D^m)v$, $m=1, 2, \cdots, n+1$, that determination vanishing for $x=a$, then all the terms in the second member of equation (17) except the last vanish both for $x=a$ and $x=b$, and we have

$$\frac{1}{D}\frac{(b-x)^n}{n!}v(x)\Big]_a^b = \frac{1}{D^{n+1}}v(x)\Big]_a^b. \tag{18}$$

With a slight change in notation and exchange of members, this is the well known Reduction Formula,

$$\int_a^b\int_a^{x_1}\int_a^{x_2}\cdots\int_a^{x_n} f(x_{n+1})dx_{n+1}dx_n\cdots dx_1 = \int_a^b \frac{(b-x)^n}{n!}f(x)dx.$$

9. Application to nonhomogeneous linear equations with constant coefficients. A particular integral of the special differential equation,

$$f(D)y = e^{ax}P(x),$$

in which $P(x)$ is an integral rational function of x of degree p, is readily obtained from the generalized Leibniz theorem for inverse operators. Assuming first that

$f(a)\neq 0$, we have merely to write equation (8) for the inverse operator $F(D) = 1/f(D)$ in the special case $v\equiv 1$ developed in §4,

$$(19)\quad F(D)e^{ax}P(x) = e^{ax}\left\{F(a)P + \frac{F'(a)}{1!}P' + \frac{F''(a)}{2!}P'' + \cdots + \frac{F^{(p)}(a)}{p!}P^{(p)}\right\},$$

and the desired integral is obtained.

If, however, $f(D)$ is of the form, $f(D) = (D-a)^m\phi(D)$, $\phi(a)\neq 0$, then we have

$$F(D) = \frac{1}{f(D)} = \frac{1}{(D-a)^m}\Phi(D), \quad \text{where} \quad \Phi(D) = \frac{1}{\phi(D)}.$$

Hence by property (6),

$$F(D)e^{ax}P(x) = \Phi(D)e^{ax}\frac{1}{D^m}P(x) = \Phi(D)e^{ax}P^{(-m)},$$

in which negative superscripts on P indicate integration. We have now merely to write equation (19) with Φ for F and $P^{(-m)}$ for P,

$$(20)\quad F(D)e^{ax}P(x) = e^{ax}\left\{\Phi(a)P^{(-m)} + \frac{\Phi'(a)}{1!}P^{(1-m)} + \frac{\Phi''(a)}{2!}P^{(2-m)} + \cdots + \frac{\Phi^{(p)}(a)}{p!}P^{(p-m)}\right\},$$

and the desired integral is obtained. Evidently equation (19) is merely the special case of equation (20) for $m=0$. In applying equation (20), any individual terms of lower degree than m may of course be dropped, as they will appear in the complementary function.

Evidently the above solution may be extended to equations of the type,

$$f(D)y = \cos\omega x\cdot P(x), \qquad f(D)y = \sin\omega x\cdot P(x)$$

by the device employed in developing equation (7).

References

1. C. J. Gerhardt, Leibnizens mathematische Schriften, Vol. III, p. 221.
2. Commercium Epistolicum, No. LXXVI, Edition of Biot and Lefort, p. 165.
3. C. J. Gerhardt, Leibnizens mathematische Schriften, Vol. III, p. 228.
4. W. W. Johnson, On a Point Connected with Symbolic Methods of Integration, The Messenger of Mathematics, Vol. 16, p. 86.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH,* College of St. Thomas

The following results of the tenth annual William Lowell Putnam Mathematical Competition, held March 25, 1950, have been determined in accordance with the constitution of the Competition. This Competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of The Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of California Institute of Technology, Pasadena, California. The members of the team were J. Brody, H. A. Forrester, R. S. Pierce; to each of these a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of Harvard College, Cambridge, Massachusetts. The members of the team were Marvin Minsky, Gerhard Rayna, Ariel Zemach; to each of these a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of New York University, New York, New York. The members of the team were Sol Ciolkowski, Donald Newman, Elaine Weiss; to each of these a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of the University of Toronto, Toronto, Ontario, Canada. The members of the team were J. P. Mayberry, R. J. Semple, R. O. Skinner; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, were J. P. Mayberry, University of Toronto; Z. Alexander Melzak, McGill University; J. W. Milnor, Princeton University; Donald Newman, New York University; R. J. Semple, University of Toronto. Each of these will receive a prize of fifty dollars.

The five succeeding persons ranking highest in the examination, named in alphabetical order, were R. E. Cutkosky, Carnegie Institute of Technology; D. B. Gillies, University of Toronto; R. S. Pierce, California Institute of Technology; André Robert, Laval University; Ariel Zemach, Harvard College. To each of these a prize of twenty dollars is awarded.

The following teams, named in alphabetical order, won honorable mention: Carnegie Institute of Technology, Pittsburgh, Pennsylvania, the members of the team being R. E. Cutkosky, John S. Nodvik, Douglas H. Shaffer; City College of New York, New York, New York, the members of the team being Arnold Benson, Herman Hanisch, Harold Widom; Cornell University, Ithaca, New York, the members of the team being Carl S. Herz, Michael Horowitz, Walter Meyer; McGill University, Montreal, Quebec, Canada, the members of

* Director of the Competition.

the team being Arthur K. Kerman, Earl L. Lomon, Z. Alexander Melzak.

Eight individuals were given honorable mention. The names are listed in alphabetical order: H. A. Forrester, California Institute of Technology; George Gioumousis, Polytechnic Institute of Brooklyn; Michael Horowitz, Cornell University; R. Sherman Lehman, Stanford University; Paul Martin, Harvard College; Gerhard Rayna, Harvard College; William F. Reynolds, College of the Holy Cross, S. A. Zwick, California Institute of Technology.

The following is a list of all colleges and universities which entered teams in the Competition: Amherst College, Brooklyn College, California Institute of Technology, Carleton College, Carnegie Institute of Technology, City College of New York, College of the Holy Cross, College of St. Thomas, Columbia College, Cornell University, Fordham College, Harvard College, Iowa State College, Laval University, Massachusetts Institute of Technology, McGill University, McMaster University, Montana State College, New York University, Occidental College, Polytechnic Institute of Brooklyn, Princeton University, Providence College, Purdue University, Queen's University, Rockford College, St. Francis College, St. John's College of St. John's University, Stanford University, Swarthmore College, University of British Columbia, University of Buffalo, University of California, University of Detroit, University of Minnesota, University of Oregon, University of Puerto Rico, University of Rochester, University of Toronto, Ursinus College, Yale University.

The following additional colleges and universities entered individual contestants only: Antioch College, College of St. Francis, Incarnate Word College, Saint Joseph College, State Teachers College (Mankato, Minnesota), United States Naval Academy, University of Arkansas, University of California, University of Chicago, University of Oklahoma, University of Omaha, University of South Carolina, University of Wyoming, West Virginia Wesleyan College, Xavier University.

A total of 223 undergraduates representing 56 institutions took part in the Competition.

Participants in the Competition were given the following lists of problems.

PART I. THREE HOURS

(Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate your answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.)

1. For what values of the ratio a/b is the limaçon $r=a-b\cos\theta$ a convex curve? $(a>b>0)$
2. Answer both (a) and (b). Test for convergence the series

(a) $$\frac{1}{\log(2!)}+\frac{1}{\log(3!)}+\frac{1}{\log(4!)}+\cdots+\frac{1}{\log(n!)}+\cdots$$

(b) $\dfrac{1}{3}+\dfrac{1}{3\sqrt{3}}+\dfrac{1}{3\sqrt{3}\sqrt[3]{3}}+\cdots+\dfrac{1}{3\sqrt{3}\sqrt[3]{3}\cdots\sqrt[n]{3}}+\cdots$

3. The sequence $x_0, x_1, x_2, \cdots$ is defined by the conditions

$$x_0 = a, \qquad x_1 = b, \qquad x_{n+1} = \frac{x_{n-1} + (2n-1)x_n}{2n} \qquad \text{for } n \geqq 1,$$

where a and b are given numbers. Express $\lim_{n\to\infty} x_n$ concisely in terms of a and b.

4. Answer either (a) or (b).
 (a) In a right prism with triangular base, given the sum of the areas of three mutually adjacent faces (that is, of two lateral faces and one base), show that these faces are of equal area and perpendicular to each other when the volume attains its maximum.
 (b) Show that

$$\frac{\dfrac{x}{1}+\dfrac{x^3}{1\cdot3}+\dfrac{x^5}{1\cdot3\cdot5}+\dfrac{x^7}{1\cdot3\cdot5\cdot7}+\cdots}{1+\dfrac{x^2}{2}+\dfrac{x^4}{2\cdot4}+\dfrac{x^6}{2\cdot4\cdot6}+\cdots} = \int_0^x e^{-t^2/2}dt.$$

5. A function $D(n)$ of the positive integral variable n is defined by the following properties: $D(1)=0$, $D(p)=1$ if p is a prime, $D(uv)=uD(v)+vD(u)$ for any two positive integers u and v. Answer all three parts below.
 (a) Show that these properties are compatible and determine uniquely $D(n)$. (Derive a formula for $D(n)/n$, assuming that $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ where $p_1, p_2, \cdots, p_k$ are different primes.)
 (b) For what values of n is $D(n)=n$?
 (c) Define $D^2(n)=D[D(n)]$, etc. and find the limit of $D^m(63)$ as m tends to ∞.

6. Each coefficient a_n of the power series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = f(x)$$

has either the value 1 or the value 0. Prove the easier of the two assertions:
 (a) If $f(0.5)$ is a rational number, $f(x)$ is a rational function.
 (b) If $f(0.5)$ is not a rational number, $f(x)$ is not a rational function.

PART II. THREE HOURS

(Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate your answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.)

1. In each of n houses on a straight street are one or more boys. At what point

should all the boys meet so that the sum of the distances that they walk is as small as possible?

2. Two obvious approximations to the length of the perimeter of the ellipse with semi-axes a and b are $\pi(a+b)$ and $2\pi(ab)^{1/2}$. Which one comes nearer the truth when the ratio b/a is very close to 1?
3. In the Gregorian calendar:
 (a) years not divisible by 4 are common years;
 (b) years divisible by 4 but not by 100 are leap years;
 (c) years divisible by 100 but not by 400 are common years;
 (d) years divisible by 400 are leap years;
 (e) a leap year contains 366 days; a common year 365 days. Prove that the probability that Christmas falls on a Wednesday is not 1/7.
4. The cross-section of a right cylinder is an ellipse, with semi-axes a and b, where $a>b$. The cylinder is very long, made of very light homogeneous material. The cylinder rests on the horizontal ground which it touches along the straight line joining the lower endpoints of the minor axes of its several cross-sections. Along the upper endpoints of these minor axes lies a very heavy homogeneous wire, straight and just as long as the cylinder. The wire and the cylinder are rigidly connected. We neglect the weight of the cylinder, the breadth of the wire and the friction of the ground.

 The system described is in equilibrium, because of its symmetry. This equilibrium seems to be stable when the ratio b/a is very small, but unstable when this ratio comes close to 1. Examine this assertion and find the value of the ratio b/a which separates the cases of stable and unstable equilibrium.
5. Answer either (a) or (b).
 (a) Given that the sequence whose nth term is (s_n+2s_{n+1}) converges, show that the sequence $\{s_n\}$ converges also.
 (b) A plane varies so that it includes a cone of constant volume equal to $\pi a^3/3$ with the surface the equation of which in rectangular coordinates is $2xy=z^2$. Find the equation of the envelope of the various positions of this plane.

 State the result so that it applies to a general cone (that is, conic surface) of the second order.
6. Consider the closed plane curves C_i and C_o, their respective lengths $|C_i|$ and $|C_o|$, the closed surfaces S_i and S_o, and their respective areas $|S_i|$ and $|S_o|$. Assume that C_i lies inside C_o and S_i inside S_o. (Subscript i stands for "inner," o for "outer.") Prove the correct assertions among the following four, and disprove the others.
 (a) If C_i is convex, $|C_i| \leqq |C_o|$.
 (b) If S_i is convex, $|S_i| \leqq |S_o|$.
 (c) If C_o is the smallest convex curve containing C_i, then $|C_o| \leqq |C_i|$.
 (d) If S_o is the smallest convex surface containing S_i, then $|S_o| \leqq |S_i|$.

 You may assume that C_i and C_o are polygons and S_i and S_o polyhedra. (Why?)

MATHEMATICAL NOTES

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ON THE GREEN FUNCTION OF A STAR-SHAPED THREE DIMENSIONAL REGION

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1. Introduction. In 1931 J. J. Gergen[1] proved the following theorem.

THEOREM. *Suppose the region D in the (x, y, z)-space is star-shaped with regard to the origin O, and that $g(x, y, z)$ is the Green function of D with pole at O. Then at every point in $D-O$ the square of the gradient $(\nabla g)^2 = g_x^2 + g_y^2 + g_z^2$ satisfies the condition*

$$(1) \qquad (\nabla g)^2 \geqq \frac{g^2}{r^2} > 0, \qquad (r = \sqrt{x^2 + y^2 + z^2}),$$

the inequality holding throughout $D-O$, unless $g - 1/r \equiv 0$. Furthermore, for every positive constant c, the region determined by the inequality $g > c$ is star-shaped with respect to O.

The function $g(P) = g(x, y, z)$ is the Green function of first kind of Laplace's equation, if the latter exists. If this is not the case, then $g(P)$ denotes the "*extended Green function*," which is defined as follows:[2] Let D_n $(n = 1, 2, \cdots)$ be a sequence of regions such that: (*i*) $O \in D_n \subset D_{n+1} \subset D$, (*ii*) the D_n exhaust the region D, (*iii*) the Green function $g_n(P)$, with pole at O, exists for each D. Then for every $P \in D$, $\lim_{n\to\infty} g_n(P) = g(P)$ exists, uniformly in any closed bounded subset of D; $g(P)$ is the extended Green function of D. This function always exists, is independent of the choice of the sequence D_n, and it coincides with the Green function in the ordinary sense, if the latter exists.

In the following we give a short, elementary proof of Gergen's theorem.

2. Proof. Let λ be a *constant*, $0 < \lambda < 1$. The function

$$h(P) = \frac{g(P) - \lambda g(\lambda P)}{1 - \lambda} \qquad (\lambda P \text{ is the point } (\lambda x, \lambda y, \lambda z))$$

is harmonic for $P \in D$, including the point O, if properly defined at O, since

$$(1 - \lambda)h(P) = \frac{1}{r} + v(P) - \left[\frac{1}{r} + \lambda v(\lambda P)\right] = v(P) - \lambda v(\lambda P),$$

[1] Note on the Green function of a star-shaped three dimensional region, American Journal of Mathematics, vol. 42 (1931), pp. 746–752.

[2] See, for example, O. D. Kellogg, Recent progress with the Dirichlet Problem, Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 601–625. See in particular p. 604.

where $v(P)$ is harmonic in D.

We assume at first that the Green function of D exists in the proper sense. Then $\lim g(P)=0$ as P approaches any boundary point P_0 of D. Hence

$$\lim_{P\to P_0} h(P) = -\frac{\lambda g(\lambda P_0)}{1-\lambda} \leqq 0.$$

By the principle of the maximum we have, therefore, for any $P\in D$,

$$\frac{g(P)-\lambda g(\lambda P)}{1-\lambda} \leqq 0. \tag{2}$$

Both statements of the theorem follow at once from (2). If $g(P)>c>0$, then by (2) we have

$$g(\lambda P) \geqq \frac{g(P)}{\lambda} > \frac{c}{\lambda} > c;$$

that is, with P, the point λP, $0<\lambda<1$, is also in the region $g>c$; the region $g>c$ is star-shaped with respect to O.

To prove (1) we let, for *fixed* $P\in D$, $(P\neq O)$, $\lambda\to 1$ and find, from (2),

$$\lim_{\lambda\to 1} \frac{rg(P)-r\lambda g(\lambda P)}{r(1-\lambda)} = \frac{\partial(rg)}{\partial r}\bigg|_P \leqq 0,$$

or

$$g + r\frac{\partial g}{\partial r} \leqq 0,$$

whence

$$\frac{g}{r} \leqq -\frac{\partial g}{\partial r} = \left|\frac{\partial g}{\partial r}\right| \leqq |\nabla g|,$$

since

$$\left|\frac{\partial g}{\partial r}\right| = \left|g_x\frac{x}{r} + g_y\frac{y}{r} + g_z\frac{z}{r}\right| \leqq |\nabla g|.$$

If the equality in (1) holds for just one point $P_0\in D$, then $g\equiv 1/r$. For since $\partial(rg)/\partial r$ is $\leqq 0$ and harmonic in D, it must be identically equal to 0, if it vanishes at an interior point P_0 of D. Thus[3]

$$g + r\frac{\partial g}{\partial r} = v + xv_x + yv_y + zv_z \equiv 0, \qquad P\in D.$$

Hence $v(O)=0$. But since $v=g-1/r$ is non-positive in D (this is also true when

[3] This is merely a repetition of Gergen's argument, *loc. cit.*,[1] p. 751 (bottom of page).

g is the *extended* function of Green), the fact that $v(O)=0$ implies $v(P)\equiv 0$, $g(P)\equiv 1/r$.

3. The general case. In order to prove the theorem in the case that only the extended function of Green exists, it is sufficient to show that (2) remains valid when $g(P)$ is the latter function. Assume at present that D is *bounded.* For a *fixed* λ, $0<\lambda<1$, let $D(\lambda)$ denote the region consisting of all points $\lambda P(\lambda x, \lambda y, \lambda z)$, where $P(x, y, z)$ is in D. Since D is star-shaped and *bounded*, the *closure* $\overline{D}(\lambda)$ of $D(\lambda)$ is contained in D. Let D_n be a sequence of regions lying in D used in the definition of the extended Green function $g(P)$ of D. For sufficiently large n, $D_n \supset \overline{D}(\lambda)$, so that, in particular, for $P\in D_n$, we have $\lambda P\in D_n$. Let $g_n(P)$ denote the (proper) Green function of D_n. Then

$$h_n(P) = \frac{g_n(P) - \lambda g_n(\lambda P)}{1-\lambda}$$

is harmonic in D_n, and it follows just as above that $h_n(P)\leqq 0$ for every $P\in D_n$. Now we keep P *fixed* and let $n\to\infty$; since $\lim_{n\to\infty} g_n(P)=g(P)$, we obtain (2), valid for every $P\in D$ and every λ, $0<\lambda<1$.

If D is *not bounded*, let Δ_n be the intersection of D with the sphere $x^2+y^2+z^2<R_n^2$, $n=1, 2, \cdots$, where $R_n\to\infty$ as $n\to\infty$. Denote by $\gamma_n(P)$ the Green function (extended or proper, if it exists) of Δ_n with pole at O. Then it is easily seen[4] that, for $P\in D$, $P\neq O$, we have

$$\lim_{n\to\infty} \gamma_n(P) = g(P),$$

where $g(P)$ is the function of Green of the *unbounded* region D. Since each Δ_n is star-shaped with regard to O and bounded, (2) holds, with γ_n in place of g, for $P\in\Delta_n$, and therefore, as $n\to\infty$, we see that (2) is true also for the case that D is not bounded.

ON ANALYTIC MAPS OF CIRCLES INTO CONVEX REGIONS

H. SHNIAD, University of Southern California

1. Theorem. In this note we shall establish the following result.

THEOREM. *Let a function $w=f(z)$ have the properties that it is analytic in the open unit circle and that the map of the unit circle is contained in a convex region R. Then the map of the circle $|z|\leqq r (r<1)$ is contained in a convex region obtained by deforming R homothetically, with respect to the map of the origin, in the ratio $2r/(r+1)$.*

[4] Let D_n^* be a sequence of regions which is used in the definition of the Green function $g(P)$ of D. We may assume that $D_n^*\subset\Delta_n$ $(n=1, 2, \cdots)$. If $g_n^*(P)$ is the (proper) Green function of D_n^* with pole at O, then $\lim_{n\to\infty} g_n^*(P)=g(P)$. Furthermore, since $D_n^*\subset\Delta_n\subset D$, we have

$$g_n^*(P) \leqq \gamma_n(P) \leqq g(P),$$

so that $\lim_{n\to\infty}\gamma_n(P)=g(P)$.

2. Proof. Without loss of generality we may assume that $f(0)=0$. Let $w=u+iv$. Then the function $h_\phi(x, y)=u\cos\phi+v\sin\phi$ has a maximum $h(\phi)$, the support function for the convex hull H of the map $f(U)$ of the open unit circle U. If the convex hull H contains the half ray from the origin in the direction ϕ_0, then $h(\phi_0)$ is infinite. For each fixed ϕ, we have that $h_\phi(x, y)$ is harmonic and that $h_\phi(0, 0)=0$. We consider the function $g(z)=2z/(z+1)$, which maps the open unit circle onto the half plane $\Re(w)<1$ and maps zero onto zero. Since the maximum of the real part of $g(z)$ on $|z|=r$ is $2r/(r+1)$, it follows that for every value ϕ the inequality $|z|\leqq r$ implies that $h_\phi(x, y)\leqq[2r/(r+1)]h(\phi)$. If we define $S(\phi)$ as the support function for the region R and $h_r(\phi)$ as the support function for the map of the circle $|z|\leqq r$, then we have $h_r(\phi)\leqq[2r/(r+1)]S(\phi)$. For more details concerning the ideas of the proof, see Pólya and Szegö.*

3. The extreme case. That $2r/(r+1)$ is the best possible ratio may be seen directly by taking $f(z)=2z/(z+1)$.

In addition, for the functions $w=(z-\alpha)/(\bar{\alpha}z-1)$, $(|\alpha|<1)$, mapping the unit circle onto the unit circle, the hypotheses of the theorem apply if the region R is taken as the open unit circle. The best deformation ratio for each of these functions is $(|\alpha|+1)r/(|\alpha|r+1)$. Since the least upper bound of these ratios for $|\alpha|<1$ is $2r/(r+1)$, we have obtained the best possible ratio, even if we are restricted to *bounded* convex regions.

If the region R is symmetric with respect to the map of the origin, then the deformation ratio can be improved to $(4/\pi)$ (arc tan r).*

CONFORMAL MAPPING OF THE INTERIOR OF AN ELLIPSE ONTO A CIRCLE

GABOR SZEGÖ, Stanford University

This mapping can be found in the paper of H. A. Schwarz, *Über einige Abbildungsaufgaben*, Journal für die reine und angew. Mathematik, vol. 70 (1869), pp. 105–120; *Gesammelte Abhandlungen*, vol. 2, pp. 65–83. (See, in particular, pp. 77–78.) The mapping function given by Schwarz appears in terms of a Jacobian elliptic function. Our purpose is to deal with this problem in an elementary way. Later we shall compare our result with that of Schwarz.

1. We assume that the given ellipse E is in the complex z-plane, with foci at ±1 and with semiaxes a and b, $a>b$, $a^2-b^2=1$. We introduce a second complex variable w defined by the elementary mapping

$$2z = w + w^{-1}. \tag{1}$$

It is well known that the circle $|w|=R=a+b$, $R>1$, is transformed into E. We denote by

$$z' = f(z) = c_1z + c_2z^2 + \cdots, \qquad c_1 > 0, \tag{2}$$

* Pólya and Szegö, Aufgaben und Lehrsatze aus der Analysis, Berlin, Springer, 1925, vol. 1, pp. 106, 139, and 140.

the mapping function in question, so that $f(z)$ is schlicht in the interior of E and $|f(z)| = 1$ on E. Hence

$$F(z) = \log [f(z)/z] \tag{3}$$

is regular in the interior of E (real at $z=0$). The function $F(z)$ can be expanded in a series of Tchebychev polynomials $T_n(z)$ defined by

$$2T_n(z) = w^n + w^{-n}. \tag{4}$$

For this theorem, see for instance Picard, *Traité d'Analyse*, 3rd edition, vol. 2, p. 336. It follows immediately from the fact that the function $F(\frac{1}{2}(w+w^{-1}))$ is single-valued and analytic in the circular ring

$$R^{-1} < |w| < R$$

and remains unchanged if we replace w by w^{-1}. Hence it can be expanded in a Laurent series

$$F(\tfrac{1}{2}(w + w^{-1})) = \sum_{n=-\infty}^{\infty} d_n w^n,$$

in which $d_{-n}=d_n$.

We write:

$$\log \frac{f(z)}{z} = \sum_{n=0}^{\infty} A_n T_n(z). \tag{5}$$

Obviously A_n is real.

2. Let $|w| = R$, $w = Re^{i\phi}$. Taking real parts on both sides of (5) we have, since $|f(z)| = 1$,

$$\log \frac{1}{|z|} = \log \frac{2}{|Re^{i\phi} + R^{-1}e^{-i\phi}|} = \sum_{n=0}^{\infty} A_n \frac{R^n + R^{-n}}{2} \cos n\phi.$$

Now

$$\begin{aligned} \log \frac{2}{|Re^{i\phi} + R^{-1}e^{-i\phi}|} &= \log \frac{2}{R} + \log \frac{1}{|1 + R^{-2}e^{-2i\phi}|} \\ &= \log \frac{2}{R} + \Re \log \frac{1}{1 + R^{-2}e^{-2i\phi}} \\ &= \log \frac{2}{R} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} R^{-2n} \cos 2n\phi. \end{aligned}$$

Comparing this with (5) we find

$$\begin{aligned} &A_0 = \log \frac{2}{R}, \qquad A_{2n} \frac{R^{2n} + R^{-2n}}{2} = \frac{(-1)^n}{n} R^{-2n}, \\ &A_{2n} = \frac{(-1)^n}{n} \frac{2R^{-2n}}{R^{2n} + R^{-2n}}, \qquad A_{2n-1} = 0; \qquad n = 1, 2, 3, \cdots. \end{aligned} \tag{6}$$

Hence the following representation of the mapping function arises:

$$\log\frac{f(z)}{z} = \log\frac{2}{R} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{2R^{-2n}}{R^{2n}+R^{-2n}}T_{2n}(z). \tag{7}$$

The first coefficient c_1 of $f(z)$ is r_0^{-1}, where r_0 is the "inner radius" of the ellipse with respect to the origin. In view of the relation

$$T_{2n}(0) = (-1)^n,$$

we obtain

$$\log r_0 = \log\frac{R}{2} - \sum_{n=1}^{\infty}\frac{1}{n}\frac{2R^{-2n}}{R^{2n}+R^{-2n}}. \tag{8}$$

Incidentally, the "outer radius" of the ellipse is

$$\bar{r} = \tfrac{1}{2}(a+b) = \frac{R}{2},$$

and from (8) we see that $r_0 < \bar{r}$.

3. The function

$$G = \Re\log\frac{1}{f(z)} = \log\frac{1}{|f(z)|} \tag{9}$$

is the classical Green function of the interior of E with respect to the origin. We can prove in the customary way (forming the conjugate of G) that the mapping $z'=f(z)$ carries over the interior of E in a schlicht manner onto $|z'|<1$. Another, direct procedure would be the following.

We want to show that if z describes the boundary of E, z' describes the unit circle $|z'|=1$ exactly once and in the same direction. (Then we apply the argument principle.) For this purpose we form the imaginary part of (7):

$$\begin{aligned}\Im\log f(z) &= \Im\log z + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{2R^{-2n}}{R^{2n}+R^{-2n}}\Im T_{2n}(z)\\ &= \Im\log\frac{Re^{i\phi}+R^{-1}e^{-i\phi}}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{2R^{-2n}}{R^{2n}+R^{-2n}}\frac{R^{2n}-R^{-2n}}{2}\sin 2n\phi,\\ \frac{d}{d\phi}\Im\log f(z) &= \Im\frac{i(Re^{i\phi}-R^{-1}e^{-i\phi})}{Re^{i\phi}+R^{-1}e^{-i\phi}} + 2\sum_{n=1}^{\infty}(-1)^n\frac{R^{2n}-R^{-2n}}{R^{2n}+R^{-2n}}R^{-2n}\cos 2n\phi.\end{aligned} \tag{10}$$

The first term on the right-hand side can be written as follows:

$$\Re\left(\frac{1-R^{-2}e^{-2i\phi}}{1+R^{-2}e^{-2i\phi}}\right) = 1 + 2\sum_{n=1}^{\infty}(-1)^n R^{-2n}\cos 2n\phi, \tag{11}$$

so that

$$\begin{aligned}\frac{d}{d\phi}\Im\log f(z) &= 1 + 2\sum_{n=1}^{\infty}(-1)^n\frac{2}{R^{2n}+R^{-2n}}\cos 2n\phi\\ &= \lim_{\rho\to 1-0}\left(1 + 2\sum_{n=1}^{\infty}(-1)^n\frac{2\rho^n}{R^{2n}+R^{-2n}}\cos 2n\phi\right).\end{aligned} \tag{12}$$

We define α by $e^{\alpha\pi}=R^2$ and use the formula

$$\frac{\pi}{\operatorname{ch}\alpha\pi}=\int_0^\infty \frac{\cos\alpha x}{\operatorname{ch}(x/2)}\,dx, \tag{13}$$

so that

$$\frac{2}{R^{2n}+R^{-2n}}=\frac{1}{\operatorname{ch}(n\alpha\pi)}=\frac{1}{\pi}\int_0^\infty \frac{\cos n\alpha x}{\operatorname{ch}(x/2)}\,dx, \qquad n=0,1,2,\cdots. \tag{14}$$

Hence

$$\begin{aligned}1+2\sum_{n=1}^{\infty}(-1)^n\frac{2\rho^n}{R^{2n}+R^{-2n}}\cos 2n\phi &= \frac{1}{\pi}\int_0^\infty \frac{dx}{\operatorname{ch}(x/2)}\left(1+2\sum_{n=1}^{\infty}(-\rho)^n\cos n\alpha x\cos 2n\phi\right)\\ &=\frac{1}{2\pi}\int_0^\infty\frac{dx}{\operatorname{ch}(x/2)}\left(\frac{1-\rho^2}{1+2\rho\cos(\alpha x+2\phi)+\rho^2}+\frac{1-\rho^2}{1+2\rho\cos(\alpha x-2\phi)+\rho^2}\right)>0.\end{aligned} \tag{15}$$

4. Finally we compare our main formula (7) with the formula of Schwarz given in the paper quoted above. As to the elliptic and theta functions occurring below, we use the notation of Whittaker & Watson, *Modern Analysis*, American edition, 1943.

Introducing the notation

$$w=e^{i(\pi/2-x)}=ie^{-ix}, \qquad z=\cos(\pi/2-x)=\sin x, \tag{16}$$

we find

$$\log\frac{f(z)}{z}=\log\frac{2}{R}+\sum_{n=1}^{\infty}\frac{1}{n}\frac{2R^{-2n}}{R^{2n}+R^{-2n}}\cos 2nx. \tag{17}$$

But according to a formula of Jacobi (Fundamenta Nova, p. 99; Whittaker & Watson, p. 509, Example 3):

$$\sum_{n=1}^{\infty}\frac{1}{n}\frac{2R^{-4n}}{1+R^{-4n}}\cos 2nx=\log\operatorname{sn}\frac{2Kx}{\pi}-\log(2q^{1/4})+\tfrac{1}{2}\log k-\log\sin x \tag{18}$$

where

$$R^{-4}=q. \tag{19}$$

Consequently

$$\log\frac{f(z)}{\sin x}=\log\frac{2}{R}-\log(2g^{1/4})+\tfrac{1}{2}\log k-\log\sin x+\log\operatorname{sn}\frac{2Kx}{\pi}, \tag{20}$$

$$\boxed{f(z)=k^{1/2}\operatorname{sn}\left(\frac{2K}{\pi}\sin^{-1}z\right).}$$

This is the formula of Schwarz. For the sake of completeness we quote the well-known expressions for the modulus k in terms of q:

$$k = \left(\frac{\theta_2}{\theta_3}\right)^2 = \left(\frac{2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \cdots}{1 + 2q + 2q^4 + 2q^9 + \cdots}\right)^2 \tag{21}$$
$$= 4q^{1/2}\left(\frac{(1+q^2)(1+q^4)(1+q^6)\cdots}{(1+q)(1+q^3)(1+q^5)\cdots}\right)^4.$$

Also

$$K = \tfrac{1}{2}\pi\theta_3^2 = \tfrac{1}{2}\pi(1 + 2q + 2q^4 + 2q^9 + \cdots)^2 \tag{22}$$
$$= \tfrac{1}{2}\pi((1-q^2)(1-q^4)(1-q^6)\cdots)^2((1+q)(1+q^3)(1+q^5)\cdots)^4,$$
$$r_0 = \left(k^{1/2}\frac{2K}{\pi}\right)^{-1}(\theta_2\theta_3)^{-1}$$
$$= (2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \cdots)^{-1}(1 + 2q + 2q^4 + 2q^9 + \cdots)^{-1}$$
$$= \frac{1 - 2q + 2q^4 - 2q^9 + \cdots}{2q^{1/4} - 3\cdot 2q^{9/4} + 5\cdot 2q^{25/4} - \cdots}$$
$$= \frac{1}{2q^{1/4}}\left(\frac{(1-q)(1-q^3)(1-q^5)\cdots}{(1-q^2)(1-q^4)(a-q^6)\cdots}\right)^2.$$

A NOTE ON THE POISSON KERNEL

A. E. Taylor, University of California, Los Angeles

1. Introduction. The Poisson kernel in two dimensions is defined by

$$P(r, \theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}. \tag{1}$$

The purpose of this note is to give an interesting, and perhaps new, proof of the formula

$$\frac{1}{2\pi}\int_0^{2\pi} P(r, \theta)d\theta = 1, \qquad 0 \leqq r < 1. \tag{2}$$

2. Proof. We observe first of all that if we denote the integral on the left in (2) by $F(r)$, the function of r so defined is continuous when $0 \leqq r < 1$. Also, it is immediately clear that $F(0) = 1$.

Now let us write

$$F(r) = \frac{1}{2\pi}\int_0^{\pi} P(r, \theta)d\theta + \frac{1}{2\pi}\int_{\pi}^{2\pi} P(r, x)dx.$$

Making the substitution $x = \theta + \pi$ in the second integral, and then combining the two integrals, we find

$$F(r) = \frac{1}{\pi}\int_0^{\pi} \frac{1 - r^4}{(1 + r^2)^2 - 4r^2\cos^2\theta}\,d\theta.$$

Next we observe that

$$(1 + r^2)^2 - 4r^2 \cos^2 \theta = 1 - 2r^2(2 \cos^2 \theta - 1) + r^4,$$

and that $2 \cos^2 \theta - 1 = \cos 2\theta$. Hence, setting $\phi = 2\theta$, we find

$$F(r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^4}{1 - 2r^2 \cos \phi + r^4} \, d\phi = F(r^2).$$

By repeated applications of this last result we see that $F(r) = F(r^{2^n})$, $n = 1, 2, \cdots$. Therefore, if $0 \leqq r < 1$, we have

$$F(r) = \lim_{n \to \infty} F(r^{2^n}) = F(0) = 1;$$

thus (2) is proved.

Finally, we observe that $F(r) = -F(1/r)$ if $0 < r \neq 1$. From this result and (2) we infer the formula

$$\frac{1}{2\pi} \int_0^{2\pi} P(r, \theta) d\theta = -1, \qquad 1 < r.$$

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

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ON INTEGRATION BY PARTS

T. E. SYDNOR, Pasadena City College

If $\int dx/x \log x$ is integrated by parts using $u = 1/\log x$ and $dv = dx/x$ in $\int u \, dv = uv - \int v \, du + c$ the well-known but maddening identity $0 = 0$ results.

More generally the method of integration by parts of $\int f_1(x) f_2(x) \, dx$ becomes useless if $f_1(x) \int f_2(x) \, dx = K$ (constant). Such integrals, however, can easily be integrated, for this condition implies that:

$$f_2(x) = -Kf_1'(x)/(f_1(x))^2.$$

So

$$\int f_1(x) f_2(x) dx = -K \int f_1'(x) dx / f_1(x) = -K \log f_1(x) + C.$$

SYNTHETIC TECHNIQUE FOR PARTIAL FRACTIONS

J. P. BALLANTINE, University of Washington

In the March 1950 issue of this MONTHLY, M. R. Spiegel showed by a clever device how to reduce the tedium of resolving a particular rational function into partial fractions. I also deviate from the standard procedure. By my method, the computation for the same function appears as follows:

$$\frac{3x+1}{(x-2)(x^2+x+1)^3} = \frac{A}{x-2} + R(x),$$

where R is a rational function of x whose denominator does not contain the factor $x-2$, and A is seen to be $1/49$.

First solve the equation for $R(x)$:

$$R(x) = \frac{49(3x+1) - (x^2+x+1)^3}{49(x-2)(x^2+x+1)^3} = \frac{N}{49 \cdot B \cdot T^3}.$$

The notations N, B, and T, short for numerator, binomial and trinomial, are merely to explain the following synthetic computation.

$T^2 =$			1	+2	+3	+2	+1	
$x \cdot T^2 =$		1	+2	+3	+2	+1	0	
$x^2 \cdot T^2 =$	1	+2	+3	+2	+1	0	0	
$T^3 =$	1	+3	+6	+7	+6	+3	+ 1	
$49(3x+1) =$						147	+49	
$N =$	−1	−3	−6	−7	−6	+144	+48(2	
		−2	−10	−32	−78	−168	−48	
$N/B =$	−1	−5	−16	−39	−84	−24	(−1	−1
		+1	+4	+11	+24			
			+1	+4	+11	+24		
	−1	−4	−11	−24	/ −49	0		

The last step was a quadratic synthetic division, by x^2+x+1. When dividing by $x-r$, the divisor is written as merely r, so when dividing by x^2-rx-s, the divisor is written as merely $+r\ +s$.

The last two numbers, -49 and 0, are the coefficients of the remainder, namely $-49x+0$. The quotient is $-x^3-4x^2-11x-24$. The process continues, without break:

$$\begin{array}{rrrrrr} -1 & -4 & -11 & -24 & (-1 & -1 \\ & 1 & +3 & & & \\ & & 1 & +3 & & \\ \hline -1 & -3\ / & -7 & -21 & & \end{array}$$

$$R(x) = \frac{-x-3}{49T} + \frac{-7x-21}{49T^2} + \frac{-49x}{49T^3}.$$

This is the final answer, except for the term $1/49\ (x-2)$.

ON SPACE FRAMES AND PLANE TRUSSES

H. E. GOHEEN, Syracuse University

The theory of statically determinate trusses and space frames as usually given in textbooks on mechanics* is difficult for students beyond the necessary difficulty of any mathematical discipline. An understanding of the general theory of systems of n linear algebraic equations in m unknowns, and the proper definition of "statically determinate," will, in the opinion of the author, lead to a better understanding of this subject matter. The following discussion is believed an improvement.

For definitions of a statically determinate space frame and statically determinate plane trusses, we use the following:

A space frame is statically determinate if and only if the compression in all members and the reaction forces necessary to support the frame in equilibrium as a rigid body in space are uniquely determined for any arbitrary space loading from the set of space equilibrium conditions, three at each joint of the frame.

A plane truss is statically determinate if and only if the compressions in all members and the reaction forces necessary to support the truss in equilibrium as a rigid body in the plane are uniquely determined for any arbitrary plane loading from the set of plane equilibrium conditions, two at each joint of the truss.

In space the condition that the resultant force is zero at the kth joint yields three linear equations

$$\text{I} \qquad \begin{aligned} \sum \alpha_{ik} F_i &= -X_k, \\ \sum \beta_{ik} F_i &= -Y_k, \\ \sum \gamma_{ik} F_i &= -Z_k, \end{aligned}$$

in which α_{ik}, β_{ik}, and γ_{ik} are the direction cosines of the ith force (either reaction force or bar compression force) at the kth joint if the ith force is concurrent with that joint and are otherwise taken 0, while X_k, Y_k, and Z_k are the X, Y, and Z

* See for example Timoshenko and Young, Statics, first edition, New York, McGraw-Hill (1937), Chapters II, V, and VII.

components of the loading force at the kth joint.

In the plane truss the equilibrium conditions at each joint are two in number. They are

$$\text{II} \qquad \begin{aligned} \sum \alpha_{ik} F_k &= -Y_k \\ \sum \beta_{ik} F_k &= -X_k \end{aligned}$$

in which α_{ik} and β_{ik} are the sine and cosine respectively of the angle which the ith force makes with the X-axis at the kth joint, if it is concurrent with the kth joint and zero otherwise, while X_k and Y_k are the X and Y components of the loading at the kth joint.

To support a rigid body in space three restraints are necessary to locate a point, two more to locate a line, and one more to prevent rotation about that line. Against each restraint, there will be a reaction force if the body is to be in equilibrium. In each member of the frame the bar force at one end is equal and opposite the force at the other end. Hence there are m unknown bar forces and six unknown reaction forces. We assume that these are j joints.

Thus the system (I) is a system of $3j$ equations in $m+6$ unknowns. If we note that the right hand sides of (I) are arbitrary and appeal to the theorem that the system $\sum_{i=1}^{n} a_{ik}x_k = c_1$ $(i=1, 2, \cdots, m)$ has unique solution for $x_1, x_2, \cdots, x_n$ for arbitrary c_i if and only if $m=n$ and the linear forms $\sum_{i=1}^{n} a_{ik}x_k$ are linearly independent, we have the result, proved in most textbooks by considering only simple frames, that a necessary condition for a space frame to be statically determinate is that

$$m = 3j - 6.$$

Likewise since two restraints are necessary to locate a point in a plane and one more to prevent rotation about that point, there will be three unknown reaction forces at the supports of a plane truss and together with the m unknown bar forces there will be $m+3$ unknowns in the system of $2j$ equations (II). Thus in a plane statically determinate truss we must have

$$m = 2j - 3.$$

If the condition $m=3j-6$ is fulfilled for a space frame and if relative motion of any two joints of it is impossible, then it is statically determinate. For since relative motion is impossible there must exist solutions of the $3j$ equilibrium equations for arbitrary loading. Since there is a solution for arbitrary loading the equations must be consistent for arbitrary right hand sides and since the number of equations is equal to the number of unknowns there must be unique solution for each loading. Hence, by definition, the frame is statically determinate. Likewise in the plane, a truss is statically determinate if $m=2j-3$ and relative motion of any two joints is impossible.

Thus in particular any simple frame or simple truss is statically determinate. More complicated structures must be analyzed by more complicated methods.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 926. *Proposed by H. L. Lee, University of Tennessee*

Show that there exist arithmetic progressions with integral terms such that the sum of the squares of four consecutive terms is a 2^kth power of an integer, k a non-negative integer.

E 927. *Proposed by J. H. Braun, Illinois Institute of Technology*

In a regular polygon of $2n+1$ sides all the diagonals are drawn. Find, as a function of n, the total number of triangles of all shapes and sizes thus formed.

E 928. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a tetrahedron $ABCD$ and a point O. Denote by A', B', C', D' the intersections of AO, BO, CO, DO with the corresponding faces of the tetrahedron, and set $x=AO/A'O$, $y=BO/B'O$, $z=CO/C'O$, $t=DO/D'O$. Show that

$$xyzt = 3 - 2(x + y + z + t) + (xy + xz + xt + yz + yt + zt).$$

E 929. *Proposed by H. S. Shapiro, Massachusetts Institute of Technology*

Given three non-concurrent straight lines l_1, l_2, l_3 in the plane. Let T_i denote reflection in l_i and set $T=T_1T_2T_3$. Show that T^2 is a translation.

E 930. *Proposed by Gordon Raisbeck, Bell Telephone Laboratories*

If

$$\prod_{i=1}^{n} (x + r_i) \equiv \sum_{j=0}^{n} a_j x^{n-j},$$

show that

$$\sum_{i=1}^{n} \tan^{-1} r_i = \tan^{-1} \frac{a_1 - a_3 + a_5 - \cdots}{a_0 - a_2 + a_4 - \cdots}$$

and

$$\sum_{i=1}^{n} \tanh^{-1} r_i = \tanh^{-1} \frac{a_1 + a_3 + a_5 + \cdots}{a_0 + a_2 + a_4 + \cdots}.$$

SOLUTIONS

E 884 [1950, 264]. Correction

The last sign in the quintic given in the note to the solution of this problem should be + instead of −.

Necklace Permutations

E 28 [1933, 175]. *Proposed by H. D. Ruderman, Manhattan High School of Aviation Trades, New York, N. Y.*

Given m marbles of which m_1 are of a first color, m_2 of a second color, m_3 of a third color, and so on to m_n of an n-th color, all arranged in a circle. Determine the number of different arrangements possible if we can start from any point and move in either direction around the circle.

Solution by H. S. Grant, Rutgers University. This is essentially the problem of determining the number of "necklace" permutations, the solution to which appeared in *On a formula for circular permutations*, Mathematics Magazine, Jan.–Feb. 1950, pp. 133–136. If P denote the number of circular permutations, x the number of odd $m_i, j, k, l, \cdots$ the subscripts for the odd m_i, and the modified Kronecker delta $\delta^x_{abc\ldots}$ is defined as 1 when x is any one of $a, b, c, \cdots$, and as 0 otherwise (including the case in which there is no subscript), the formula is

$$P/2 + \delta^x_{012}[(m - \delta^x_1 - 2\delta^x_2)/2]!/2 \prod_{i=1}^{n} [(m_i - \delta^i_{jkl\cdots})/2]!.$$

The formula for P was found in the solution to Problem 519, *National Mathematics Magazine*, April 1944, pp. 276–287, and also in the solution to E 678 [1946, 101]. This formula is given below, D denoting the greatest common divisor of the m_i's, $\phi(d)$ Euler's ϕ-function, and the summation is to be taken over all divisors d of D:

$$P = (1/m)\sum_{d \mid D} \phi(d) \left[(m/d)!/ \prod_{i=1}^{n} (m_i/d)! \right].$$

The reader is also referred to a paper by R. E. Allardice, *On a problem in permutations*, Proceedings of the Edinburgh Mathematical Society, vol. 8 (1890), pp. 64–69. This problem had been suggested by Professor Chrystal.

Also solved by Octave Levenspiel.

For a special case of this problem see E 599 [1944, 472].

Lines Cutting Off One-third the Area of a Triangle

E 774 [1947, 281]. *Proposed by Norman Anning, Ann Arbor, Michigan*

Consider points on the median of a triangle. Through the centroid no straight line can be drawn which will cut off one-third of the area. Through a point four-

fifths of the distance from vertex to base, four such lines can be drawn. Find points on the median at which the number of possible lines changes.

Solution by Vern Hoggatt, Oregon State College. The following solution is largely compounded from the partial results sent to the editor by Free Jamison and C. S. Ogilvy. It is a pleasure to acknowledge the assistance furnished by these results.

Using oblique cartesian coordinates let the vertices A, B, C of the triangle be $(0, b)$, $(a, 0)$, $(0, 0)$. Choose the median through A and consider first a line that intersects the side to which the median is drawn and one of the other sides, say the y-axis. The equation of the chosen median is

$$x = (ab - ay)/2b. \tag{1}$$

The equation of the line passing through $(ar, 0)$ and $(0, b/3r)$, and consequently cutting off one-third of the area, is

$$x = (arb - 3ar^2y)/b. \tag{2}$$

The equation of another line cutting off one-third of the area is

$$x = (2arb - 3ar^2y)/2b. \tag{3}$$

Solving (1) and (2) simultaneously we find $y/b = (1-2r)/(1-6r^2)$, which has a maximum of $(3-\sqrt{3})/6$ when $r = (3+\sqrt{3})/6$. As r varies from $1/2$ to $(3+\sqrt{3})/6$ to 1, y/b varies from 0 to $(3-\sqrt{3})/6$ to $1/5$. Thus, if z is the fractional part of the median from the vertex to the intersection of (1) and (2), we have $z = 1-y/b$, and for $z = (3+\sqrt{3})/6$ there is one line, for $(3+\sqrt{3})/6 < z \leqq 4/5$ there are two lines, and for $4/5 < z \leqq 1$ there is one line. The same results may be obtained by letting the line intersect the third side of the triangle, whence the above number of lines under each condition is to be doubled.

Solving (1) and (3) simultaneously we find $y/b = (1-2r)/(1-3r^2)$, which is a single-valued function of r with no turning-point maximum or minimum between $r = 2/3$ and $r = 1$. The end-point minimum at $r = 1$ gives $y/b = 1/2$, and $r = 2/3$ gives $y/b = 1$. Thus for $0 \leqq z \leqq 1/2$ there are two lines.

Now choose another oblique cartesian frame of reference so that the vertices A, B, C are represented by $(0, 0)$, $(c, 0)$, $(0, b)$. The equation of the median through A is now

$$x = cy/b, \tag{4}$$

and we have, for lines analogous to lines (2) and (3),

$$x = (crb - 3cr^2y)/b, \tag{2'}$$

$$x = (2crb - 3cr^2y)/2b. \tag{3'}$$

We now have $z = 2y/b$.

Solving (4) and (2') simultaneously we find $2y/b = 2r/(1+3r^2)$, which has a maximum of $\sqrt{3}/3$ at $r = \sqrt{3}/3$, since r must be non-negative. At $r = 1$ and

$r=1/3$, $2y/b=1/2$, and consequently at $z=\sqrt{3}/3$ there is one line, and for $1/2 \leqq z \leqq \sqrt{3}/3$ there are two lines. The lines corresponding to $z=1/2$ are the same as those found previously.

Solving (4) and (3′) simultaneously we find $2y/b=4r(2+3r^2)$, which has a maximum of $\sqrt{6}/3$ at $r=\sqrt{6}/3$ and has a value of $4/5$ at $r=1$ and $r=2/3$. So we may conclude that for $z=\sqrt{6}/3$ there is one line, and for $4/5 \leqq z \leqq \sqrt{6}/3$ there are two lines. The lines corresponding to $z=4/5$ are the same as those found previously.

Combining results we have:

value of z	number of lines
$0 \leqq z < \sqrt{3}/3$	2
$z = \sqrt{3}/3$	1
$\sqrt{3}/3 < z < (3+\sqrt{3})/6$	0
$z = (3+\sqrt{3})/6$	2
$(3+\sqrt{3})/6 < z < \sqrt{6}/3$	4
$z = \sqrt{6}/3$	3
$\sqrt{6}/3 < z \leqq 1$	2

Also solved by W. E. Brooke, Free Jamison, Roger Lessard, C. S. Ogilvy, and Prasert Na Nagara. Most of these solutions were incomplete.

Jamison pointed out that an interesting allied problem would be to determine through what part of the area of a triangle it is impossible to draw a line which will cut off one-third of the area of the triangle.

The Generalized Steiner-Lehmus Problem

E 863 [1949, 263]. *Proposed by W. O. Pennell, Exeter, N. H.*

If $\alpha>0$, $\beta>0$, $\alpha+\beta<\pi$, $0<k<1$, then $\alpha=\beta$ is a necessary and sufficient condition for

$$\sin\alpha \sin(k\alpha+\beta) = \sin\beta \sin(k\beta+\alpha).$$

III. *Solution by D. J. Price, University of Malaya, Singapore.* Set $\theta=(\alpha+\beta)/2$ and $\phi=(\alpha-\beta)/2$ and substitute in the equation. One obtains

$$\sin(\theta+\phi) \sin[(1+k)\theta-(1-k)\phi] = \sin(\theta-\phi) \sin[(1+k)\theta+(1-k)\phi]$$

which simplifies readily to

$$\sin(1+k)\theta \cos(1-k)\phi \cos\theta \sin\phi = \cos(1+k)\theta \sin(1-k)\phi \sin\theta \cos\phi.$$

One trivial solution, $\theta=0$, is excluded by the condition $0<\theta<\pi/2$. On the other hand $-\pi/2<\phi<\pi/2$, and therefore there is a solution $\phi=0$, or $\alpha=\beta$. Apart from these we must have

$$[\tan(1+k)\theta]/\tan\theta = [\tan(1-k)\phi]/\tan\phi.$$

Now the expression on the right can take only values in the range 0 to $1-k$, whereas that on the left must lie outside the range 0 to $1+k$. These ranges cannot overlap, within the imposed limits, and therefore no non-trivial solution is possible.

Note: The two previous solutions [1950, 37] contain oversights. Solution I is invalidated because the equation is not an identity for all k within the given range. It must be shown that no values of k are possible for there to be any solution other than $\alpha=\beta$.

In solution II the expression $(k+2)(\beta+\alpha)/2$ can exceed π, thus making the left side of the final equation positive, and removing the contradiction used in the proof.

Space Analogue of a Theorem by Reynolds

E 894 [1949, 691]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let perpendiculars through vertex A of tetrahedron $ABCD$ to the faces ABC, ACD, ADB cut the circumsphere of $ABCD$ in B', C', D' respectively. Show that the volumes of the polyhedra A-$B'C'D'$-B, A-$B'C'D'$-C, A-$B'C'D'$-D are equal to that of $ABCD$.

Editorial Note. In the above problem the proposer intended to extend to the tetrahedron a property of the triangle given by B. Reynolds, *Nouvelles Annales de Mathématiques*, 1884, question 1501, p. 494. The analogy, however, is incorrect. A correct analogy, subsequently pointed out by the proposer, is to *draw perpendiculars to the edges AB, AC, AD of the tetrahedron to meet the circumsphere again in B', C', D' respectively. Then the volumes of the polyhedra A-$B'C'D'$-B, A-$B'C'D'$-C, A-$B'C'D'$-D are equal to that of the given tetrahedron.* To establish this it suffices to observe that B', C', D' are diametrically opposite B, C, D, respectively, and triangles BCD, $B'C'D'$ are congruent and in parallel planes.

A generalization of the corrected analogy is suggested by taking B', C', D' as the reflections of B, C, D in an arbitrary given point P. For example, P may be taken as the center of a quadric surface circumscribing $ABCD$, and the above analogy is then the case where the quadric surface is the circumsphere of $ABCD$.

Although the problem as originally stated is not true in general, it is certainly true if the tetrahedron $ABCD$ is *trirectangular* at A.

A Congruence Related to Wilson's Theorem

E 896 [1950, 34]. *Proposed by F. J. Duarte, Caracas, Venezuela*

If p is an odd prime show that

$$[1\cdot 3\cdot 5\cdots(p-2)]^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$

Solution by Kirk Stewart, College of Puget Sound. Since

$$1\cdot 3\cdot 5\cdots(p-2) \equiv (1-p)(3-p)\cdots(-4)(-2), \pmod{p}$$

it follows that

$$[1\cdot 3\cdot 5\cdot\cdots\cdot(p-2)]^2 \equiv (1)(-2)(3)(-4)\cdots(p-2)(1-p)$$
$$\equiv (p-1)!(-1)^{(p-1)/2} \pmod{p}.$$

But, by Wilson's theorem,

$$(p-1)! \equiv -1 \pmod{p},$$

so that

$$[1\cdot 3\cdot 5\cdot\cdots\cdot(p-2)]^2 \equiv (-1)(-1)^{(p-1)/2} \equiv (-1)^{(p+1)/2} \pmod{p}.$$

Also solved by A. W. Boldyreff, J. L. Brenner, D. H. Browne, B. B. Gresham, F. D. Grogan, B. F. Hadnot, H. T. LaBorde, H. E. Lee, Roger Lessard, Prasert Na Nagara, L. C. Mathewson, G. W. Medlin, Ingram Olkin, L. A. Ringenberg, Alex Rosenberg, P. J. Schillo, N. C. Scholomiti, N. T. Seely, Jr., R. E. Shafer, E. P. Starke, Alfred Sylvester, Margaret Willerding, and R. V. Andree's Number Theory Class at the University of Oklahoma.

M. S. Klamkin and J. D. Swift pointed out that this problem expresses a result due to C. F. Arndt, *Archiv Math. Phys.*, 2, 1842, pp. 32, 34–35, and is also a special case of a formula due to M. Vecchi, *Periodico di Mat.*, 16, 1901, pp. 22–24. See pp. 276–277 of Dickson's *History*.

Brenner also found the following related results: Let u be the number of quadratic non-residues of p between 1 and $p/2$ inclusive. If p is a prime of the form $8m-1$, then

$$1\cdot 3\cdot 5\cdot\cdots\cdot(p-2) \equiv (-1)^{u+1} \pmod{p},$$

and if p is a prime of the form $8m-5$, then

$$1\cdot 3\cdot 5\cdot\cdots\cdot(p-2) \equiv (-1)^{u} \pmod{p}.$$

Discarding Cards

E 898 [1950, 34]. *Proposed by N. S. Mendelsohn, University of Manitoba*

A pack of N cards is disposed of as follows. The top card is placed at the bottom of the deck, the next card is discarded, the third card is placed at the bottom, the fourth discarded, *etc.*, this process being carried on until there is only one card left. Let $f(N)$ be the position of this card, from the top, in the original arrangement of the deck. Prove that

$$f(N) = 2N - 2^{[\log_2 N]+1} + 1,$$

where $[x]$ denotes, as usual, the largest integer not greater than x.

Solution by Roger Lessard, École Polytechnique, Montreal. When the selected card is on top there must be 2^n cards in the pack, and conversely. This is easily seen from the fact that if we reverse the process, then for every card brought from bottom to top another card must first be brought in. Thus when the selected card makes a complete cycle the number of cards is doubled.

Now let there be $2^n \leqq N < 2^{n+1}$ cards in the pack. The selected card will first come on top when $N-2^n$ cards have been discarded to bring the pack down to 2^n, and the same number of cards have gone to the bottom. As $2(N-2^n)$ cards have been taken from the top, the position of the selected card at the beginning was then

$$f(N) = 2(N - 2^n) + 1 = 2N - 2^{n+1} + 1.$$

But $n \leqq \log_2 N < n+1$, whence $n = [\log_2 N]$, and the theorem is established.

Note that if $N = a_1 a_2 \cdots a_r$ in the scale of 2, $a_1 = 1$, then $f(N) = a_2 a_3 \cdots a_r a_1$.

Also solved by Prasert Na Nagara, P. J. Schillo, N. C. Scholomiti, N. T. Seely, Jr., and the proposer.

Editorial Note. What is the solution to the corresponding problem where every rth card is discarded? It would seem to depend on the solution of the difference equation

$$f(N) \equiv r + f(N-1), \bmod N,$$

where $f(1) = 1$ and $0 < f(N) \leqq N$. Does any theory exist for such types of difference equations?

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4405. *Proposed by Paul Erdös, Princeton University*

Let $x_1, x_2, \cdots$ be a sequence of real numbers, $x_i > c > 0$. We assume that they approach all their limit points from below. Then, for any A, there is only a finite number of solutions of the equation

$$A = \sum_{k=1}^{\infty} a_k x_k,$$

where the a_k are positive integers.

A special case occurs if $0 < x_1 < x_2 < \cdots$.

The case $x_k = 1 - 2^{-k}$ was communicated to the Proposer by R. A. Rosenbaum.

4406. *Proposed by H. J. Zimmerberg, Rutgers University*

Show that the most general linear differential equation

$$\sum_{i=0}^{n} a_i[p(x)]^i D^i y = f(x),$$

where $p(x)$ is of class C' and the a_i are constants with $a_n \neq 0$, which is reducible to the case of constant coefficients by a transformation of the independent variable is the generalized form of Euler's equation, wherein $p(x)$ is linear in x.

4407. *Proposed by N. A. Court, University of Oklahoma*

If four conics are circumscribed about the faces of a tetrahedron (T) in such a way that at each vertex of (T) the three tangents to the three conics passing through that vertex are coplanar, then the four conics lie on a quadric surface.

4408. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute*

Prove that $\Gamma^{(n)}(1)$ is positive or negative according as the integer n is even or odd.

4409. *Proposed by Fritz Herzog, Michigan State College*

Given a power series $f(x) = a_1x + a_2x^2 + a_3x^3 + \cdots$ with real coefficients. It is assumed that at least one of the coefficients a_n, $n \geqq 2$, is different from zero and that the radius of convergence R is positive or infinite. Show that there exist constants u_n, $0 < u_n < R$, such that $\sum_n (-1)^n u_n$ converges but $\sum_n (-1)^n f(u_n)$ diverges.

SOLUTIONS

A Circle Configuration

4313 [1948, 505]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Two perpendicular chords AB, CD of a circle (O) intersect at a point P inside the circle. There are eight circles (O_1), (O_2), (O_3), (O_4), (O_1'), (O_2'), (O_3'), (O_4') tangent to both chords and also to the circle (O), the first four exteriorly, the last four interiorly, and such that (O_i) and (O_i') lie in opposite quadrants formed by the given chords. (1) The sum of the squares of the distances O_1O_1', O_2O_2', O_3O_3', O_4O_4' is independent of the position of P, and the products $(O_1O_1')(O_3O_3')$ and $(O_2O_2')(O_4O_4')$ are equal. (2) The radical axes of the circles (O_1), (O_1'), (O_3), (O_3') taken in pairs and those of the circles (O_2), (O_2'), (O_4), (O_4') taken in pairs, intersect one another in thirty-six points of which twelve are on the circle (O), four of these coinciding with the vertices of the square whose diagonals are parallel to AB and CD.

*Solution by the Proposer.** We choose a rectangular coördinate system Oxy so that the equation of APB is $y = y_0$, and that of CPD is $x = x_0$, with $0 < x_0^2 + y_0^2 < r^2$, where r is the radius of circle (O). Let also r_i and r_i' be the radii of

* Translated and condensed by W. E. Byrne, Virginia Military Institute.

circles (O_i) and (O_i'), and let their centers be $O_i:(x_i, y_i)$ and $O_i':(x_i', y_i')$. To fix the figure in mind suppose that under the auxiliary coördinate system PXY with

$$x = x_0 + (X - Y)/\sqrt{2}, \qquad y = y_0 + (X + Y)/\sqrt{2},$$

the XY coördinates of O_1, O_2, O_3, O_4, O_1', O_2', O_3', O_4' are:

$$(0, r_1\sqrt{2}), \quad (r_2\sqrt{2}, 0), \quad (0, -r_3\sqrt{2}), \quad (-r_4\sqrt{2}, 0),$$
$$(0, -r_1'\sqrt{2}), \quad (-r_2'\sqrt{2}, 0), \quad (0, r_3'\sqrt{2}), \quad (r_4'\sqrt{2}, 0).$$

To determine, for example, (x_2, y_2) and r_2, we have

$$r_2 = x_2 - x_0, \qquad r_2 = y_2 - y_0, \qquad x_2^2 + y_2^2 = (r + r_2)^2.$$

Elimination of x_2 and y_2 gives a quadratic whose positive root is

$$r_2 = r - x_0 - y_0 + \sqrt{2[r^2 - (x_0 + y_0)r + x_0y_0]}.$$

Similarly,

$$r_1 = x_0 - y_0 + r + \sqrt{2[r^2 + (x_0 - y_0)r - x_0y_0]},$$
$$r_1' = -x_0 + y_0 - r + \sqrt{2[r^2 + (x_0 - y_0)r - x_0y_0]},$$
$$r_2' = x_0 + y_0 - r + \sqrt{2[r^2 - (x_0 + y_0)r + x_0y_0]}.$$

The calculations for the other circles are of the same type. From them we obtain

$$\overline{O_1O_1'}^2 = 16[r^2 + (x_0 - y_0)r - x_0y_0],$$
$$\overline{O_2O_2'}^2 = 16[r^2 - (x_0 + y_0)r + x_0y_0],$$
$$\overline{O_3O_3'}^2 = 16[r^2 - (x_0 - y_0)r - x_0y_0],$$
$$\overline{O_4O_4'}^2 = 16[r^2 + (x_0 + y_0)r + x_0y_0].$$

Hence we have

$$\overline{O_1O_1'}^2 + \overline{O_2O_2'}^2 + \overline{O_3O_3'}^2 + \overline{O_4O_4'}^2 = 64r^2,$$

which is independent of the position of P, and

$$\overline{O_1O_1'}^2 \cdot \overline{O_3O_3'}^2 = 256(r^2 - x_0^2)(r^2 - y_0^2) = \overline{O_2O_2'}^2 \cdot \overline{O_4O_4'}^2.$$

For the second part of the problem it is convenient to designate by $L(1, 1')$, $L(1, 3)$, $L(2, 4)$, etc., the radical axes of the corresponding pairs of circles. There are six radical axes for the circles (O_2), (O_4), (O_2'), (O_4') taken in pairs and six radical axes for the circles (O_1), (O_3), (O_1'), (O_3') taken in pairs. Their intersections give 36 points. A clue as to how to match them to obtain the twelve points on (O) may be had by considering the limiting case where P coin-

cides with O, as the figure is simple to sketch and continuity enables us to reduce the number of calculations.

We find that $L(1, 1')$, $L(3, 3')$, $L(2, 2')$, $L(4, 4')$ meet in the vertices of a square inscribed in (O), which is also true for P coincident with O. The remaining eight points cited as being on (O) are obtained from the two sets:

$$L(2, 4), \quad L(2', 4'), \quad L(1, 3'), \quad L(1', 3);$$
$$L(2', 4), \quad L(2, 4'), \quad L(1, 3), \quad L(1', 3').$$

To save space a process of verification is indicated without giving the details of all the calculations. To find, for example, the intersection of $L(1, 1')$ and $L(2, 2')$ we note the equations

$$L(1, 1'): \qquad Y = (r_1 - r_1')/2\sqrt{2},$$
$$L(2, 2'): \qquad X = (r_2 - r_2')/2\sqrt{2},$$

which, with the values for the radii as previously given, show that the intersection (X, Y) is $(0, r)$.

Also solved by Joseph Langr.

Editorial Note. The Proposer notes that the analogous proposition for three dimensional space is true, using a sphere with three mutually perpendicular planes through an interior point. Langr refers to a previous solution of his, published in *Mathesis*, VIII, 1908, question 1660, p. 173.

Area Within Simple Closed Plane Curve

4320 [1948, 641]. *Proposed by V. F. Ivanoff, San Francisco*

Let $f(s, \phi) = 0$ be the intrinsic equation of a simple closed curve; s is the length of arc of the curve, measured from a fixed point P_0 to any point P, and ϕ is the angle between the tangent lines at P and P_0. Find the area enclosed by the curve.

Solution by Mary Payne, Michigan State College. Construct a Cartesian coordinate system with the origin at P_0, x-axis in the direction of the slope of the curve at P_0, and y-axis directed so that s is measured from P_0 in the direction of the positive x-axis. Then ϕ is numerically equal to the inclination of the tangent. Let us take ϕ, however, to be the angle between the tangent directed in the direction of increasing s and the positive x-axis. Then ϕ will take on all values between 0 and 2π. We regard $f(s, \phi) = 0$ as being solved for $\phi = \phi(s)$.

Now we take a rectangular strip parallel to the x-axis and between the y-axis and a point on the curve at a distance s from the origin (measured along the curve). The base of this strip is $\int_0^s \cos \phi(t)dt$, and the altitude is $\sin \phi(s)ds$. Hence the total area is

$$\int_0^L \sin \phi(s) \left[\int_0^s \cos \phi(t)dt \right] ds,$$

where L is the total length of the curve.

The correctness of this formula is obvious for a curve which is always convex, and nearly obvious if any line parallel to the x-axis intersects the curve in at most two points. If, however, a line parallel to the x-axis intersects the curve in more than two points, the interpretation of ϕ given above is necessary to make the formula true. The area will be positive if $\phi > 0$ for small s and negative if $\phi < 0$ for small s.

Also solved by C. S. Ogilvy and the Proposer.

Editorial Note. Ogilvy obtained the formula

$$\frac{1}{2}\int_S \left\{\sin\phi \int \cos\phi ds - \cos\phi \int \sin\phi ds\right\} ds,$$

which is a direct consequence of the familiar theorem

$$A = \frac{1}{2}\int_S (xdy - ydx).$$

Divisibility in a Sequence of Integers

4330 [1949, 40]. *Proposed by Paul Erdös, Syracuse University*

Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers. Prove that there exists either an infinite subsequence in which no integer divides another or an infinite subsequence where each integer is a multiple of the preceding one.

I. *Solution by R. S. Lehman, Stanford University.* In the given sequence consider all integers each of which divides no other integer in the sequence. If the number of these is infinite, they are the required subsequence having the property that no member divides any other. If they are finite in number, remove them and all of their divisors from the sequence. The remaining integers will then form an infinite sequence in which each integer divides at least one other in the sequence. From this sequence we can select an infinite subsequence in which each integer divides the following. If b_k is chosen to be an integer in the subsequence, we can select any multiple of b_k appearing in the sequence to be b_{k+1}.

II. *Solution by G. A. Hedlund, Yale University.* C. Visser (*Proceedings of the Royal Academy of Amsterdam*, 1937, p. 359) has proved the following theorem: *Let there be given an infinite sequence of arbitrary elements. Suppose that in every infinite subsequence there exists at least one system of n different elements that has a certain property P. Then there exists an infinite subsequence in which every system of n different elements has the property P.*

To solve the proposed problem, suppose that in every infinite subsequence of the given sequence of integers $a_1 < a_2 < \cdots$ some integer divides some other. Apply Visser's Theorem with $n = 2$ and with two integers said to have property P if and only if one divides the other. We infer that there must exist a subsequence such that each b_i divides b_{i+1}.

III. *Solution by R. C. Buck, University of Wisconsin.* The proposed conclusion follows at once from a simple result of lattice theory. The positive integers form a lattice L under divisibility. A subset S of L is called free if no distinct pair of elements of S is ordered. Rephrasing the proposed problem, we must show that any infinite subset A of L contains either an infinite chain or an infinite free subset. This is true of any partially ordered infinite set A in which the set of all elements below any element is finite. The proof is as follows. Suppose that A has no infinite free subset. Choose any maximal free subset A_0 of A. This is a finite set, and every other element of A is above or below some element of A_0. Hence, there is an element α_0 of A_0 such that infinitely many elements of A lie above it. From these, choose a maximal free subset A_1; this is again finite, and contains an element α_1 above which are infinitely many elements of A. Proceeding thus, we generate an infinite chain $\alpha_0 < \alpha_1 < \cdots$.

Also solved by Harley Flanders, John Kelly, Burnett Meyer, and N. L. Rabinovitch.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 115th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Intermediate Algebra for Colleges. By P. R. Rider. The Macmillan Company, New York, 1949. 10+242 pages. $2.75.

In the preface the author states, "This book is designed for college students who have had one year of high school algebra. It is considerably simpler than Rider's *College Algebra*, but the methods of presenting the material are those which have proved successful in that book. The explanations are set forth principally by the use of illustrative examples which bring out the principles involved. The student can follow these examples much more easily and with a better understanding than he can read more abstract discussions. If he works through them carefully he grasps the meaning of the underlying mathematics with greater facility. This is particularly true in an elementary course.

"A summary is provided at the end of each chapter. It is hoped that these summaries will prove helpful to the student.

"There is a liberal supply of carefully chosen exercises. These are not too difficult, yet some of them offer a challenge to the brighter student. Answers to the odd-numbered exercises are to be found at the back of the book, answers to the even-numbered are printed in a separate pamphlet."

The author has carried out his intentions very well. The definitions are carefully stated, duly noting the restrictions. The illustrative examples are numer-

ous and well chosen, but perhaps the outstanding feature of the book is the wealth of well graded exercises, and the excellent and abundant collection of worded problems. The exercises at the end of the chapter on ratio, proportion, and variation are especially good. The summaries of definitions and procedures which appear at the end of each chapter should be very useful to the students.

This reviewer would have welcomed an introductory chapter with a large selection of simple exercises on permutations, combinations, and probability. Such an addition would have enhanced the value of the book for pre-medical students, as well as have provided a use for the included chapter of five pages on the binomial formula.

This carefully written book is undoubtedly one of the best in its field. The excellent printing and the balanced and uncrowded arrangement of the material on each page contributes greatly to the attractiveness of the book.

K. C. SCHRAUT

Techniques of Statistical Analysis. By the Statistical Research Group, Columbia University, Applied Mathematics Panel, Office of Scientific Research and Development. Edited by Churchill Eisenhart, M. W. Hastay, and W. A. Wallis. New York and London, McGraw-Hill Book Company, Inc., 1947. 14+473 pages. $6.00.

According to the preface, written by W. A. Wallis, Director of Research, the Statistical Research Group at Columbia University was organized July 1, 1942, and was disbanded soon after the end of the war. The senior scientific staff consisted of seventeen members and Wallis says that each of these, together with five members of the junior scientific staff, had an influence on the book. Therefore, the general excellence of the work is not surprising and it is difficult for this reviewer to find any points for criticism.

In order to help the Army, Navy, Office of Scientific Research and Development, and others, with their statistical problems, it was necessary for various members of the Statistical Research Group to investigate the merits of existing statistical techniques and, in some cases, to develop new ones. This book serves as a record of a part of that work. Its complete title, *Selected Techniques of Statistical Analysis for Scientific and Industrial Research and Production and Management Engineering*, properly describes the nature of the volume and the areas in which it could be most useful.

The seventeen thought-provoking chapters of the book were prepared by nine different authors. The chapters vary in length from eight to ninety-two pages and each treats a different topic. However, the book is surprisingly well unified because of exceptionally fine work by the editors and others responsible for its form.

Each chapter is preceded by a table of contents and ended by a good list of references. Notation is uniform and care has been taken in recording its meaning. Figures and tables follow a uniform pattern. Early in each chapter there is a

description of the type of practical problem which might have given rise to the investigation under discussion and, in many of the chapters, there are detailed solutions for specific examples. The book closes with an unusually complete index. No effort has been spared to put the content into as usable a form as possible.

The book has a few rather obvious misprints. The proof of one theorem (pp. 409–10) seems faulty. This reviewer does not agree with the author of chapter three regarding the reason for the use of range in statistical quality control. However, these are minor matters and do not alter the reviewer's opinion that this is an excellent book.

E. G. OLDS

Solid Analytic Geometry. By Adrian Albert. New York, McGraw-Hill Book Company, Inc., 1949. 10+162 pages. $3.00.

The author's apparent chief aim in this book, namely, to present the study of analytical geometry of space by means of the theory of vectors and matrices is a laudable one. He begins the book by an exposition of vectors in n-space and obtains as consequences the usual formulas concerning lines and planes in ordinary space. In the third chapter he gives an elementary treatment of curves and surfaces, and this is followed in the fourth chapter by a brief treatment of spheres. The fifth chapter, which covers the usual discussion of quadric surfaces and their classification, is followed by a useful and elegantly written chapter on the theory of matrices which is necessary for the reading of the remainder of the book. This chapter treats solutions of systems of linear equations, the characteristic equation of an n-rowed square matrix, and the orthogonal reduction of a symmetric matrix. With this background an excellent presentation of rotation of axes and the orthogonal reduction of a real quadratic form are afforded in the seventh chapter.

Some of the material of Chapter 8 on spherical coordinates is novel for a textbook on solid geometry. It contains the development of formulas for parallax corrections in the spherical coordinates incident to translation of the rectangular axes. Planar rotations are named by the terms pitch, roll, or yaw according as the x, y, or z axis is left invariant in the rotation. A general rotation matrix is shown to be expressible as the product of a pitch matrix, a roll matrix, and a yaw matrix, and the relations of the elements of the rotation matrix to the angles of the planar rotations are given. Gnomonic projections and charts are treated in the remainder of the chapter.

In the final chapter a brief discussion of projective geometry is given by the use of matrices. Besides consideration of invariant points, a concise and rigorous demonstration of the invariance of the cross ratio of four points under a projective transformation is included.

The reviewer is of the opinion that the chief fault of the volume, as a textbook on solid analytical geometry, lies in the paucity of stimulating problems on geometry. For instance, merely finding a sphere through four points is not much

fun, but arriving at necessary and sufficient conditions under which two circles lie on the same sphere, or working out the locus of the center of a sphere with one degree of freedom would afford more interest and information for the student. The triviality of many sets of exercises could be reduced by the introduction of a few good problems which would require more of the student than mere substitution of numbers into formulas or routine computation. The word locus is not mentioned in the book. At least, it is not in the index! A set of answers to exercises is not supplied.

The printing and format are excellent. The following errors or misprints were noticed. On page 27, the peculiar statement $(a, b, c) = (0, 0, 0)$ appears. Read z for c in $f(x, y, c)$ on page 32. On page 36, reference should be to (22) instead of (21). One should read "consisting *of* all points" on page 43. The x in the denominator in equations (2) on page 54 should be read as a.

This is a useful addition which all teachers of solid analytical geometry should read and assimilate. Whether they should use it as a textbook depends upon their point of view. Certainly, the student can gain much valuable information in algebra by use of the book.

C. E. SPRINGER

On the Theory of Stochastic Processes and Their Application to the Theory of Cosmic Radiation. By N. Arley, New York, Wiley, 1949. 240 pages. $5.00.

Let $x(t)$ be for each time $t \geqq 0$ a random variable which can take on only integral values. Suppose that the (conditional) probability of a transition from m at time r to n at time $s > r$ is the same as the conditional probability of this transition if $x(t)$ is also prescribed at times before r. In physical language we speak of a system which at time t is in some state denoted by an integer $x(t)$; the system jumps from state to state; the distribution of states after a given jump depends on the state just before the jump, but not on previous states. Analytically the process is determined by the probability $p_{mn}(r, s)$ of a jump to state n at time s from the state m at time r. If $P(r, s)$ is the matrix of these transition probabilities it is shown that $P(r, t) = P(r, s)P(s, t)$ if $r < s < t$, and this functional equation dominates the analytical work.

Processes of this type have been applied in many ways in recent years. For example $x(t)$ can be taken to be the number of radioactive disintegrations of atoms of some substance that have occurred between times 0 and t; $x(t)$ can be taken as the number of individuals (say bacteria or elementary physical particles) at time t in a population which grows by fission as modified perhaps by deaths; $x(t)$ can be taken as the fortune (in dollars) at time t of a stock market speculator or the salary (in cents) at time t of a university professor. In each case of course the situation must be analyzed to check the appropriateness of the mathematical model.

The matrix $A(s)$ with $a_{mn}(s) = (\partial/\partial t)\ P(s, t)$ (for $t = s$) has a simple physical interpretation; $a_{mn}(s)ds$ is for $m \neq n$ the probability at time s of a transition from m to n in time ds and $1 + a_{mm}(s)ds$ is the probability of remaining at m. In many

applications the matrix $A(s)$ is known from theoretical considerations or can be estimated from observations, and the transition probabilities $P(s, t)$ can then be determined, for example, by solving the system of differential equations

$$\frac{\partial p(s, t)}{\partial s} = -A(s)P(s, t)$$

in s for fixed t, or the system

$$\frac{\partial p(s, t)}{\partial t} = P(s, t)A(t)$$

in t for fixed s. In both cases the initial conditions are determined by the fact that $A(r, r)$ should be the identity matrix.

The author gives a survey of these stochastic processes, examining in detail the best known special cases: the Poisson process in which only jumps from m to $m+1$ are possible and $a_{mm+1}(s) \equiv$const.; the birth process suggested by individuals dividing independently of each other at random times into pairs which then further divide, and so on, so that the number of individuals at time t can again only jump from m to $m+1$, and $a_{mm+1}=$const. m; the Polya process, which contains both of these as special cases, and for which again only jumps from m to $m+1$ are possible with $a_{mm+1}(s) = \lambda(1+bm/1+b\lambda t)$ where b and λ are preassigned constants. Multi-dimensional processes, in which $x(t)$ is a vector, are also discussed.

The strictly mathematical section occupies only about one third of the book. The rest of the book contains the application of these stochastic processes to cosmic ray cascade showers, in which moving elementary particles generate other particles in the course of their motion, the new particles generate more, and so on, until the process dies out as the particles lose energy. Several stochastic models of this process have been proposed, all of the type described above. The author analyzes these models and proposes another, together with a computational technique. The difficulty of the problem is increased by the fact that only the crudely unrealistic models lend themselves to easy numerical computation.

This is not the place to evaluate the author's contribution to physics, even if the reviewer were qualified to do so. In any event, the book, which is the author's doctor's thesis, contains a useful discussion of special stochastic processes of the type described above (Markov processes with a finite or enumerable state space). These processes arise in many applications, and most of the material on them has not been readily available. The book's coverage of these processes is not exhaustive, and does not claim to be. The stress is on the application to the cascade problem.

J. L. DOOB

Correction. In the May, 1950, issue of this MONTHLY the number of pages in *Plane and Spherical Trigonometry* by M. Richardson was erroneously given as 15+138 pp. It should have read 15+343 pages.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS 1948–1949

Mathematics Club, Oberlin College

The *Mathematics Club* of Oberlin College had a very successful year, the high spots of which were the Christmas Party and the Annual Banquet. Incorporated into the program were several talks by various professors, outside the Mathematics Department, on the importance of mathematics in their respective fields. The speakers and their topics were:

Mathematics and logic, by Joseph Solomon
Magic squares, by Benson Scheff
Paper-folding geometry, by Anne Johnson
Micro-wave optics, by Prof. Wade Ellis
Unicursal curves, by Marjorie Ireland
Poincare's model of hyperbolic geometry, by Robert Keesey
Mathematical molecules, by Prof. J. A. Campbell
Seventeen-sided polygon, by William Cook
Nine-point circle, by Janet Brown
Transformations and topology, by Emery Thomas
Mathematical calculators, by Jeanne Taylor
Theory of finite groups, by William Kossler
The golden section, by Patricia Mott.

Prof. F. W. Reed of Ohio University spoke on the topic of *Linear sines and cosines* at the Annual Banquet.

The officers during the year 1948–49 were: President, Emery Thomas; Vice-president, Joseph Solomon; Secretary-Treasurer, Evelyn Schmidt and Benson Scheff; Social Chairman, Anne Johnson; Publicity Chairman, Jeanne Taylor; Faculty Advisor, Prof. R. W. Wagner.

Kappa Mu Epsilon, Washburn Municipal University

The *Kansas Delta* Chapter of *Kappa Mu Epsilon* meets regularly on the second Thursday of each month. Some of the special topics during the school years were:

Some famous problems in mathematics, by Prof. G. B. Price

The part that Kappa Mu Epsilon may play in the rehabilitation of patients at Winter General Hospital, by Dr. H. H. Wagenheim, of the Menninger Foundation

Probability, by Mr. T. D. McAdam
Logic and finite geometry, by Mr. K. E. Lake
Some inequalities, by Dr. Robert Schatten

An Initiation Dinner and other social meetings were held during the year.

In April, the *Kansas Delta* Chapter was host to the National Biennial Convention of *Kappa Mu Epsilon* held in Topeka.

The officers for the school years 1949–50 are: President, L. L. Johnson; Vice-president, Mary McCullough; Secretary, Donna Lee Simmons; Treasurer, Steve Powell; Sponsor, Miss Laura Z. Greene; Corresponding Secretary, Miss Margaret E. Martinson.

Mathematics Club, Queens College

Papers presented to the Queens College *Mathematics Club* during 1948–49 were:

Quaternions, by Prof. J. M. Feld

Statics in n-space, Prof. Leon Cohen

Surfaces, By Nick Roescher

Fields and differential fields, by Prof. H. W. Raudenbush, Jr.

Words, by Miss Jean Hirsh.

Officers for last term were: President, Nick Rescher; Vice-president, Gertrude Nissenbaum; Secretary, Dolores Lehrman; Sponsor, Mr. A. B. Brown.

The Mathematics and Physics Club, College of St. Thomas

Papers presented to the *Mathematics and Physics Club* of the College of St. Thomas during the year 1949–50 included:

Topology, by Dr. G. K. Kalisch, Professor of Mathematics, University of Minnesota

Energy production in the stars, by Dr. G. Blass, Professor of Physics

Effects of the atom bomb on human beings, by Dr. J. Ryan, Chief of Medicine, St. Joseph's Hospital, St. Paul, Minnesota.

Other activities of the Club included attendance at a meeting of the Mendel Forum Club, of St. Catherine's College, at which Mr. F. A. Gifford, Meteorologist at Northwest Airlines, spoke on *The effect of climate on certain aspects of human life.*

A number of films were also shown during the year.

The officers for the year 1949–50 are: President, Stephen Dickenson; Vice-president, Richard Zemlin; Secretary, Clarence Germain; Treasurer, Marcus McEllistrem; Moderator, Dr. Laurence Sheridan.

Pi Mu Epsilon, Carnegie Institute of Technology

Some of the papers presented at the monthly meetings of the *Pennsylvania Epsilon* Chapter of *Pi Mu Epsilon* during 1948–49 were:

Classical problems in geometry, by Prof. J. H. Neelley.

Complex numbers and quaternions, by Visiting Professor F. D. Murnaghan of Carnegie Institute of Technology and Professor at the Istituto Technico Aeronautica of Brasil.

Science and art in the Italian renaissance, by Visiting Professor Enrico Bom-

piani of the University of Pittsburgh and Professor at the University of Rome.

Mathematical logic, by Visiting Professor E. R. Lorch of Carnegie Institute of Technology and Professor at Columbia University.

At the annual banquet, thirty-seven persons were initiated, of which thirteen were undergraduates, eight were graduates, and sixteen were faculty members.

The officers for 1949–50 are: Director, R. C. DiPrima; Vice-Director, D.H. Shaffer; Secretary; W. H. Warner; Treasurer, F. B. Smith; Faculty Adviser, Prof. J. B. Rosenbach; Executive Committee, Richard Cutkosky, Robert McKelvey, and William Simon.

Mathematics Club, Collinsville, Illinois, Township High School

Titles of talks given before the Collinsville Township High School *Mathematics Club* include:

Transfinite numbers—a discussion of infinity, by Dick Stowe

Ancient methods of multiplication, by Dorothy Keck

Solutions of some problems from the Stanford University Mathematics Competition for high schools, by Don Bitzer

Euler's totient, by Mel Lieberstein

Some mathematical puzzles and jokes, by Fred Malone

The four-color problem and its extension to the torus, by George Rupprecht

The use of mathematics in music, by Earl Cooley.

The Club sponsors a "Problem of the Week" bulletin board and has succeeded in getting the entire school interested in many of the problems proposed there. It also sponsors weekly chess nights.

The officers are: President, Douglas Phillips; Vice-president, Ivan Vurgener; Secretary, Ethel Rowland; Sponsor, Mel Lieberstein.

Mathematics Club, University of Dayton

Biweekly meetings were held at which members of the Club presented the following papers:

The exact value of pi, by Daniel J. Groszewski

The history and computation of pi, by Robert Hennessey

The application of differential equations in chemical kinetics, by George Moon

Interpolation, by Thomas Riney

The solution of a right triangle without the aid of trigonometric tables, by George Oberer

The set of irrational numbers, by Demetrius Zonars

The mechanics of impact, Richard Segers

The paradox and its place, by Donald Moore

Methods of summation by the calculus of finite differences, by Thomas Beckert

Mathematical probability applied to games of chance, by Richard Biedenbender.

Faculty papers presented during the year were:

Points of intersection of concentric circles, by Dr. K. C. Schraut

How much does a can of tomatoes weigh?, by Prof. C. G. Peckham

Continued fractions and matrices, by Prof. J. S. Frame of Michigan State College.

Several excursions were taken to the University of Cincinnati and the club was host to the Annual Colloquium of the National Mathematics Honor Society for Secondary Schools.

Dr. Frame spoke at the annual dinner at which time an award was given to Thomas Beckert for having presented the most interesting paper during the second term.

Officers during the year were: President, George Moon; Vice-President, John Quinlisk; Secretary, Joseph Schell; Treasurer, Jorge Nunez; Publicity Secretary, Richard Segers and Robert Hennessey.

Kappa Mu Epsilon, Kansas (Pittsburg) State Teachers College

Kansas Alpha chapter of *Kappa Mu Epsilon* held four open meetings at which the following talks were presented:

Mathematics and civilization, by Prof. J. A. G. Shirk

Inevitability of scientific discovery, by Norval Phillips

Development of commercial mathematics, by Mary Morrison

Reports of Kappa Mu Epsilon convention, by Everett Ard and James Helmert

Report of National Council meeting, by Helen Kriegsman.

A combined meeting with the *Physical Science Club* was held for viewing industrial films on the *Production and uses of electric power*, and on *Special uses of glass in electric insulation.*

Three Kappa Mu Epsilon keys were awarded to the three seniors making the highest scholastic record in mathematics. They were awarded to Norval Phillips, Mary Dell Morrison, and James Pike.

Officers for 1948–49 were: President, Norval Phillips; Vice-President, Everett Ard; Secretary, Mary Dell Morrison; Treasurer, William Sellers; Corresponding Secretary, Prof. J. A. G. Shirk; Faculty Sponsor, Dr. R. G. Smith.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PRELIMINARY ACTUARIAL EXAMINATIONS, PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the

nine undergraduates ranking highest on the score of Part 2 of the 1950 Preliminary Actuarial Examinations are as follows:

First Prize of $200

Mattuck, Arthur P., Swarthmore College

Additional Prizes of $100

Dempster, Arthur P., University of Toronto
Haslam, M. Brent, The University of Buffalo
Hudek, Paul R., University of Minnesota
Jamieson, J. Rae, University of Toronto
Leff, Milton M., The University of Western Ontario
Milnor, John W., Princeton University
Reynolds, William F., College of the Holy Cross
Walter, John R., University of Toronto

The Society of Actuaries has authorized a similar set of nine prizes for the 1951 examinations on Part 2.

The Preliminary Actuarial Examinations consist of the following three examinations:

Part 1. Language Aptitude Examination.
(Reading comprehension, meaning of words and word relationships, antonyms, and verbal reasoning.)

Part 2. General Mathematics Examination.
(Algebra, trigonometry, coordinate geometry, differential and integral calculus.)

Part 3. Special Mathematics Examination
(Finite differences, probability and statistics.)

The 1951 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service and will be administered by the Society of Actuaries at centers throughout the United States and Canada on May 18, 1951. The closing date for applications is March 15, 1951.

Detailed information concerning the Examinations can be obtained from: The Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

FIRST NATIONAL CONGRESS OF APPLIED MECHANICS

The First National Congress of Applied Mechanics will be held on June 11–16, 1951, at Illinois Institute of Technology, Chicago, Illinois. The general chairman of the Congress is Professor L. H. Donnell of Illinois Institute of Technology.

Papers for presentation at the Congress must be submitted before April 14, 1951, and should constitute original research in applied mechanics, which includes kinematics, dynamics, vibrations, waves, mechanical properties of materials and failure; stress analysis elasticity, plasticity; fluid mechanics; thermodynamics. Papers should not be longer than 5,000 words. They will be grouped by subject and one half hour will be allotted for presentation and discussion of each paper.

PERSONAL ITEMS

The John Simon Guggenheim Memorial Foundation has granted awards in mathematics to the following: Dr. J. H. Bigelow, Institute for Advanced Study; Professor Samuel Eilenberg, Columbia University; Associate Professor N. E. Steenrod, Princeton University; Dr. R. P. Boas, Jr., executive editor of *Mathematical Reviews*, Cambridge, Massachusetts; and Associate Professor Philip Hartman, Johns Hopkins University.

Mr. C. J. A. Halberg, Jr., who has been an instructor in mathematics during the past year at Pomona College, has been awarded the Honnold Fellowship by Pomona College.

Professor S. B. Jackson of the University of Maryland represented the Association at the Conference on Cultural and Educational Relations with Japan held at Washington on May 25, 1950, under the auspices of the Commission on the Occupied Areas.

Alabama Polytechnic Institute announces the following: Assistant Professor J. C. Eaves, University of Alabama, has been appointed to an associate professorship; Dr. Ernest Ikenberry of Louisiana State University, Dr. Nathaniel Macon of the University of North Carolina, Dr. A. J. Owens, University of Florida, and Dr. W. A. Rutledge, University of Tennessee, have been appointed to assistant professorships; Mr. H. W. Burnette and Mr. R. K. Butz of the University of Georgia, Mr. J. G. Cox, Alabama Polytechnic Institute, and Mr. V. E. Dietrich, Purdue University, have been appointed to instructorships.

Colorado College makes the following announcements: Professor W. V. Lovitt, head of the Department of Mathematics has retired; Assistant Professor J. S. Leech of the University of Chicago has been appointed Professor of Mathematics and Head of the Department of Mathematics.

Denison University reports: Professor F. B. Wiley, chairman of the Department of Mathematics, has retired with the title of Professor Emeritus and has accepted the position of Chairman of the Mathematics Department of Ashland College; Associate Professor Chosaburo Kato has been promoted to the position of Professor and Chairman of the Department of Mathematics; Assistant Professor Marion Wetzel has been promoted to an associate professorship.

De Paul University announces: Dr. John DeCicco of Illinois Institute of Technology has been appointed Professor and Chairman of the Department of Mathematics; Dr. W. B. Caton, Washington State College, has been appointed to an assistant professorship.

Oberlin College announces the following: Assistant Professor Wade Ellis and Assistant Professor E. P. Vance, chairman of the Department of Mathematics, have been promoted to associate professorships.

At Pratt Institute: Mr. F. S. Beckman and Mr. R. J. Kohlmeyer have been promoted to assistant professorships.

Reed College and Swarthmore College announce the following exchange of professors: Professor R. A. Rosenbaum of Reed College will be a member of the faculty of Swarthmore College during the year 1950–1951 and Assistant Pro-

fessor P. W. Carruth of Swarthmore College will join the faculty of Reed College for the year.

Rutgers University announces the following promotions: Associate Professors H. S. Grant, E. R. Ott, M. S. Robertson to professorships; Assistant Professor C. R. Phelps to an associate professorship; Dr. R. M. Cohn to an assistant professorship.

University of Arizona announces the following appointments: Dr. D. G. Duncan, University of Michigan, and Dr. L. E. Payne, Iowa State College, to assistant professorships; Mr. B. R. Cato, Jr. of Duke University and Mr. V. A. Kramer, University of Chicago, to instructorships.

At the University of Delaware: Professor G. C. Webber has been promoted to the position of Chairman of the Department of Mathematics; Assistant Professor R. F. Jackson has been promoted to an associate professorship.

Mr. M. R. Bates of Union College has accepted a position as mathematician at the Bell Aircraft Company, Buffalo, New York.

Dr. R. P. Boas, Jr., executive editor of *Mathematical Reviews*, has been appointed to a professorship at Northwestern University.

Assistant Professor H. K. Brown of Northeastern University has been promoted to the position of Associate Professor of Mechanical Engineering and has been appointed Director of Engineering Graduate Study.

Professor Lamberto Cesari is on leave from the University of Bologna and has been appointed to a visiting professorship at Purdue University.

Associate Professor C. H. Denbow of the United States Naval Postgraduate School has been appointed to a professorship at Ohio University.

Professor Nat Edmonson, Jr., of Johns Hopkins University has a position now with the Fairchild Engine and Aircraft Corporation, Oak Ridge, Tennessee.

Mr. G. W. Evans, II, of New York University has accepted a permanent position with the Argonne National Laboratory, Chicago, Illinois, as an associate mathematician.

Professor William Feller of Cornell University has been appointed Eugene Higgins Professor of Mathematics at Princeton University.

Dr. Sze-tsen Hu has been appointed to an assistant professorship at Tulane University; during the academic year 1950–1951 he will be on leave of absence and will be a member of the Institute for Advanced Study.

Associate Professor F. Burton Jones of the University of Texas has been appointed to a professorship at the University of North Carolina.

Professor E. S. Kennedy of the American University of Beirut, Lebanon, is on leave and has accepted a Rockefeller Fellowship at the Institute for Advanced Study.

Professor W. I. Layton of State Teachers College, Frostburg, Maryland, has been appointed Head of the Department of Mathematics of Stephen F. Austin State Teachers College, Nacogdoches, Texas.

Professor J. C. C. McKinsey of Oklahoma Agricultural and Mechanical College has accepted a position as mathematician at the Rand Corporation, Santa Monica, California.

Professor L. L. Merrill of Clarkson College of Technology has been promoted to the position of Chairman of the Department of Mathematics; he is also serving as a consultant for the Stromberg Carlson Company, Rochester, New York.

Miss Elsie C. Muller has been appointed to an instructorship at LaSalle-Peru Township High School and Junior College, LaSalle, Illinois.

Dr. J. H. Mulligan, Jr., of Allen D. DuMont Laboratories, Inc. has been appointed to an assistant professorship of electrical engineering at New York University.

Professor W. R. Murray of Franklin and Marshall College has been appointed to an associate professorship at Wagner College.

Associate Professor Ivan Niven of the University of Oregon has been promoted to a professorship.

Assistant Professor G. G. Roberts of Berea College has been promoted to an associate professorship.

Professor J. B. Rosenbach, assistant head of the Department of Mathematics, Carnegie Institute of Technology, has been promoted to the position of Head of the Department of Mathematics.

Professor A. C. Schaeffer of Purdue University has been appointed to a professorship at the University of Wisconsin.

Mr. D. J. Smith of the University of Notre Dame has accepted a position with the Royal-Liverpool Group of Insurance Companies, New York, New York.

Mrs. Dorothy M. Stone of Manchester, England, has been appointed Mary Whiton Calkins Visiting Professor of Mathematics at Wellesley College for the year 1950–1951.

Mr. H. J. Vandort is now Assistant Rector, Grace Episcopal Church, Grand Rapids, Michigan.

Professor Abraham Wald, chairman of the Department of Mathematical Statistics, Columbia University, gave a series of lectures on the theory of statistical decision functions at the Naval Ordnance Test Station, April 3–7, 1950.

Dr. J. V. Wehausen has been appointed Executive Editor of *Mathematical Reviews*.

Professor R. L. Wilder of the University of Michigan was visiting lecturer at the University of Oregon for the week May 15–20, 1950.

Professor Constantin Carathéodory of Munich, Germany, died February 2, 1950.

Mr. B. H. Lowe of the University of Akron died on April 24, 1950.

Professor Raleigh Schorling, University of Michigan, died on April 22, 1950.

Professor J. A. G. Shirk of Kansas State Teachers College died April 15, 1950. He was a charter member of the Association.

Mr. G. M. V. Tryon of Fenton, Michigan, died May 1, 1950.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1950 by a mail vote of the membership of the Association in the Sections indicated:

Illinois	A. E. GAULT, Bradley University
Iowa	E. N. OBERG, University of Iowa
Louisiana-Mississippi	T. A. BICKERSTAFF, University of Mississippi
Maryland-District of Columbia-Virginia	G. R. CLEMENTS, U. S. Naval Academy
Michigan	J. S. FRAME, Michigan State College
Minnesota	R. H. CAMERON, University of Minnesota
Philadelphia	G. E. RAYNOR, Lehigh University
Southern California	C. G. JAEGER, Pomona College
Texas	E. H. HANSON, North Texas State College

Great interest is shown by our members in these elections for Sectional Governors. In eight of the nine Sections more than half of those eligible to vote cast their votes in time to be counted. The highest percentage of votes cast was 70 per cent in the Louisiana-Mississippi Section.

It should also be noted that each vote is important in these elections. In the case of two Sections, the elected Governor received one more vote than the losing candidate. In the case of a third Section where there were more than two nominees, the winning candidate received two more votes than two other candidates who were tied for second place.

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 119 persons have been elected to membership by the Board of Governors on applications duly certified:

L. J. ABBEDUTO, Student, Illinois Institute of Technology, Chicago, Ill.

WALTER ABRAMOWITZ, Student, Polytechnic Institute of Brooklyn, N. Y.

R. L. ADEY, Student, Oregon State College, Corvallis, Ore.

KANHAYALAL AGGARWAL, B.A. (Panjab) Proprietor, Messrs. Hariram Aggarwal & Sons, Majith Mandi, Amritsar, India

A. N. AHEART, A.M. (Harvard) Instructor, West Virginia State College, Institute, W. Va.

L. W. AKERS, M.A. (Kansas) Asso. Professor, Central Missouri College, Warrensburg, Mo.

W. H. BARTELS, B.S. in E.E. (Michigan S.) Assistant Electrical Engineer, Board of Water and Light Commission, Lansing, Mich.

ALEXANDER BASIL, Student, Hofstra College, Hempstead, N. Y.

D. C. BENSON, Student, Pomona College, Claremont, Calif.

JONAS BERARU, B.A. (U.C.L.A.) Research Assistant, North American Aviation, Inc., Downey, Calif.

GERALD BERMAN, M.A. (Toronto) Teaching Fellow, University of Toronto, Ont.

A. F. BERNDT, Student, Cooper Union, New York, N. Y.

C. C. BRAUNSCHWEIGER, Student, Alfred University, N. Y.

H. O. BREMER, B.A.(Hofstra) 46—21 156 Street, Flushing, N. Y.

E. B. BRIDGFORTH, A.B.(Duke) Grad. Student, University of Chicago, Ill.

M. D. R. BROWN, Student, University of Kansas, Lawrence, Kans.

MARGUERITE A. BROWN, Student, Carleton College, Northfield, Minn.

C. N. CAMPOPIANO, B.S.(Rutgers) Grad. Student, Rutgers University, New Brunswick, N. J.

DOROTHY I. CARPENTER, M.A.(Michigan) Instructor, Denison University, Granville, Ohio.

R. E. CARROLL, M.S.(Illinois) Instructor, Emory University, Ga.

B. R. CATO, JR., A.M.(Duke) 2032 Greenway Avenue, Charlotte, N. C.

I. J. CHERRY, B.S.(Oregon S.C.) Grad. Student, Oregon State College, Corvallis, Ore.

H. J. CHESTON, JR., M.A.(Maryland) Instructor, Montgomery Junior College, Bethesda, Md.

P. F. CLEMENS, M.A.(New York S.C.T., Albany) Instructor, Rensselaer Polytechnic Institute, Troy, N. Y.

M. J. CLEVELAND, M.S.(Florida) Instructor, University of Florida, Gainesville, Fla.

H. M. COCHRAN, Col., U. S. A., 101 Miramonte Avenue, Palo Alto, Calif.

D. A. DODGE, Student, University of Kansas, Lawrence, Kans.

R. F. DURIG, Student, Kent State University, Ohio.

JAMES EASTCOTT, Student, McMaster University, Hamilton, Ont.

C. C. FAITH, Student, University of Kentucky, Lexington, Ky.

H. M. FELDMAN, Ph.D.(Washington U.) Lecturer, Washington University; Teacher, Board of Education, St. Louis, Mo.

A. M. FEYERHERM, M.S.(Iowa) Instructor, Iowa State College, Ames, Iowa.

F. H. FISHER, A.B.(Washington & Jefferson) Teaching Fellow, West Virginia University, Morgantown, W. Va.

W. L. FRANK, Student, Illinois Institute of Technology, Chicago, Ill.

R. F. GABRIEL, M.A.(Columbia) Instructor, St. Francis College, Brooklyn, N. Y.

W. V. GAMZON, M.A.(U.C.L.A.) Teacher, Washington Jr. High School, Long Beach, Calif.

ROBERT GARFUNKEL, Student, Rutgers University, New Brunswick, N. J.

ISIDORE GOLDMAN, B. of M.E.(Cooper Union) Mechanical Engineer, New York Naval Shipyard, Brooklyn, N. Y.

W. T. GRAYBEAL, A.M.(North Carolina) Asst. Professor, Emory and Henry College, Emory, Va.

J. E. HAFSTROM, M.A.(Minnesota) Instructor, University of Minnesota, Duluth, Minn.

W. C. HAMILTON, Student, Oklahoma A. and M. College, Stillwater, Okla.

J. B. HANSEN, Student, Harvard University, Cambridge, Mass.

L. H. HARRIS, Student, Polytechnic Institute of Brooklyn, N. Y.

L. L. HELMS, Student, Bradley University, Peoria, Ill.

N. K. HERSH, M.A.(Kansas) Instructor, Tulane University, New Orleans, La.

R. J. HEYMAN, Student, Illinois Institute of Technology, Chicago, Ill.

P. A. HINRICHS, Student, Hofstra College, Hempstead, N. Y.

R. V. HOGG, M.S.(U. of Iowa) Instructor, State University of Iowa, Iowa City, Iowa.

N. C. HOOVER, M.A.(Kansas) Instructor, Washburn Municipal University, Topeka, Kans.

DAVID HORWITZ, Student, Illinois Institute of Technology, Chicago, Ill.

L. N. HOWARD, Student, Swarthmore College, Pa.

EDWARD KEEGAN, B.S.(Kansas S.T.C.) Grad. Student, Kansas State Teachers College, Pittsburg, Kans.

R. A. KENNEDY, Student, Illinois Institute of Technology, Chicago, Ill.

DON KIRKHAM, Ph.D.(Columbia) Professor, Iowa State College, Ames, Iowa.

J. D. E. KONHAUSER, B.S.(Penna. State) Instructor, Pennsylvania State College, State College, Pa.

JACOB KOREVAAR, Ph.D.(Leiden) Visiting Lecturer, Purdue University, Lafayette, Ind.

S. B. KRAMER, Student, Polytechnic Institute of Brooklyn, N. Y.

H. C. KRANZER, Student, New York University, N. Y.

K. E. LAKE, Student, Washburn University, Topeka, Kans.

A. S. LEE, JR., Student, Southern Methodist University, Dallas, Texas.

FERNAND LEMAY, Student, Laval University, Quebec, P. Q.

KENNETH LEWIS, B.P.(Toronto) Asst. Professor, University of Florida, Gainesvile, Fla.

J. W. LINDSAY, Instructor, Texas Technological College, Lubbock, Texas.

DOROTHY K. LOONEY, Student, Regis College, Weston, Mass.

I. J. LOWE, Student, Cooper Union, New York, N. Y.

A. W. MALL, A.B.(Clark) Instructor, St. Lawrence University, Canton, N. Y.

A. P. MATTUCK, Student, Swarthmore College, Pa.

C. H. MCCALL, JR., Student, George Washington University, Washington, D. C.

JESSIE MCLEAN, M.S.(Washington U.) Teacher, Harris Teachers College, St. Louis, Mo.

K. W. MCVOY, JR., Student, Carleton College, Northfield, Minn.

BERT MENDELSON, B.A.(Columbia) Grad. Assistant, University of Nebraska, Lincoln, Nebr.

R. J. MIHALEK, Student, Illinois Institute of Technology, Chicago, Ill.

BENJAMIN MITTMAN, Student, Illinois Institute of Technology, Chicago, Ill.

J. A. MORALES, M.A.(Florida) Instructor, University of Florida, Gainesville, Fla.

J. H. MULLIGAN, JR., Ph.D.(Columbia) Asst. Professor, New York University, N. Y.

T. A. NEWTON, B.S.(Colorado A. & M.) Grad. Student, University of Georgia, Athens, Ga.

H. A. OSBORN, A.B.(Princeton) Teaching Assistant, Stanford University, Calif.

A. J. OWENS, M.S.(Iowa) Grad. Student, University of Florida, Gainesville, Fla.

W. D. PAXTON, JR., Student, Pomona College, Claremont, Calif.

A. F. PAYTON, Student, Rutgers University, New Brunswick, N. J.

J. M. PIKE, B.S.(Kansas S.T.C.) Grad. Student, Kansas State Teachers College, Pittsburg, Kans.

W. P. REID, Ph.D.(Pittsburgh) Asst. Professor, U. S. Air Force Institute of Technology, Wright Field, Ohio.

G. A. REILLY, B.S.(St. Mary's) Clerk, American Hospital and Life Insurance Co., San Antonio, Texas.

T. L. REYNOLDS, M.A.(North Carolina) Grad. Student and Part-time Instructor, University of North Carolina, Chapel Hill, N. C.

AUDREY I. RICHARDS, M.A.(Columbia) Instructor, Utica College, N. Y.

W. J. RISSLER, A.M.(New Jersey S.T.C.) Instructor, State Teachers College, Frostburg, Md.

VIRGINIA E. ROBERTS, M.A.(Texas Tech.) Instructor, Texas Technological College, Lubbock, Texas.

J. M. ROBERTSON, B.S.(Clemson) Part-time Instructor, University of Florida, Gainesville, Fla.

SAUL ROSEN, Ph.D.(Pennsylvania) Asst. Professor, Drexel Institute of Technology, Philadelphia, Pa.

D. T. Ross, Student, Oberlin College, Ohio.

F. E. Ross, A.B.(Kentucky) Part-time Instructor, University of North Carolina, Chapel Hill, N. C.

J. D. RUTLEDGE, Student, Swarthmore College, Pa.

MRS. RUTH SCHOLTEN, M.A.(Drake) Instructor, South Dakota State College, Brookings, S. D.

D. W. SEIBEL, B.S.(Denison) Grad. Assistant, Kent State University, Ohio.

HAROLD SHNIAD, Ph.D.(U.C.L.A.) Instructor, University of Southern California, Los Angeles, Calif.

L. G. SIGLER, Student, Oklahoma A. and M. College, Stillwater, Okla.

SISTER AUGUSTINE MARIA, Ph.D.(Fordham) Principal, The Mary Louis Academy High School, Jamaica, N. Y.

SISTER M. IRMA, M.A.(Michigan) Instructor, Siena Heights College, Adrian, Mich.

W. R. SMYTHE, JR., Student, Rollins College, Winter Park, Fla.

E. S. SPIEGELTHAL, Student, Washington Square College, New York University, N. Y.

J. A. STANDERFER, B.A.(N. Texas S.C.) Grad.

Assistant, North Texas State College, Denton, Texas.

C. E. STEWART, Student, Illinois Institute of Technology, Chicago, Ill.

R. C. STEWART, M.A. (Yale) Assistant, Yale University, New Haven, Conn.

E. O. STROBEL, A.B. (Pennsylvania) Statistician, United Nations, Lake Success, N. Y.

E. A. TRABKA, A.B. (Rochester) Grad. Instructor, University of Rochester, N. Y.

GEORGE TROTZUK, Automobile Engineer, Chrysler Corporation, Detroit, Mich.

O. L. WADKINS, JR., B.E.E. (Florida) Electronic Scientist, White Sands Proving Grounds, Las Cruces, New Mexico.

R. E. WALTERS, B.A. (Akron) Grad. Student, Kent State University, Ohio.

L. E. WARD, JR., A.B. (California) Grad. Fellow, Tulane University, New Orleans, La.

BURTON WENDROFF, Student, New York University, N. Y.

P. A. WHITE, Ph.D. (Virginia) Asso. Professor, University of Southern California, Los Angeles, Calif.

J. A. WILSON, M.S. (Case) Asst. Professor, Baldwin-Wallace College, Berea, Ohio.

J. L. Wilson, Student, Wabash College, Crawfordsville, Ind.

L. M. WINER, Student, Swarthmore College, Pa.

W. H. WINNIS, Student, Iona College, New Rochelle, N. Y.

HERBERT WOLF, M.A. (North Carolina) Grad. Student, University of North Carolina, Chapel Hill, N. C.

A. W. WORTHAM, M.S. (Oklahoma A. & M.) Instructor, Oklahoma A. and M. College, Stillwater, Okla.

F. M. YANARI, Student, George Pepperdine College, Los Angeles, Calif.

J. W. YOUNG, M.A. (Florida) Instructor, University of Florida, Gainesville, Fla.

JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The twelfth annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of California, Berkeley, on Saturday, January 28, 1950. Professor H. M. Bacon, Chairman of the Section, presided.

The attendance was seventy-six including the following forty-seven members of the Association: H. A. Arnold, H. M. Bacon, G. A. Baker, T. J. Bass, Jr., Alice K. Bell, B. A. Bernstein, M. T. Bird, Elizabeth W. Brownell, A. C. Burdette, K. L. Cooke, M. A. Dernham, Brother Dominic, Roy Dubisch, Hazel E. Eggett, E. B. Eilertsen, G. C. Evans, E. A. Fay, S. A. Francis, C. M. Fulton, W. H. Gregory, W. R. Hanson, C. A. Hayes, Jr., Marjorie Ley Hoffman, Mary Thayer Huggins, D. W. Hullinghorst, Free Jamison, J. L. Kelley, Helena G. Kusick, D. H. Lehmer, Sophia McDonald, E. D. Miller, F. R. Morris, W. H. Myers, Andrewa Noble, C. D. Olds, George Polya, Edris Rahn, E. S. Robbins, E. B. Roessler, Kathyrn B. Rolfe, J. B. Rosser, Falka G. Sturges, Mary V. Sunseri, Irving Sussman, Edwin Tabor, L. A. Walker, A. R. Williams.

At the business meeting the following officers were elected for the coming year: Chairman, S. A. Francis, San Mateo Junior College; Vice-Chairman, D. H. Lehmer, University of California, Berkeley; Secretary-Treasurer, E. B. Roessler, University of California, Davis; Representative on the *California Journal of Secondary Education*, Ruth G. Sumner, Oakland High School.

By invitation of the Section, Professor A. W. Tucker of Princeton University gave an address during the morning session.

The program consisted of the following papers:

1. *On the birectangular quadrilateral*, by Professor C. M. Fulton, University

of California, Davis.

A complete set of formulas for a birectangular quadrilateral in hyperbolic plane geometry is obtained by a simple method. Using Cartesian coordinates, these formulas quite naturally lead to the familiar expressions for arc length and area. They also give the "slope," $-\text{ctn}\alpha=\text{sech}\ yD_x y$.

2. *The M.A. degree in mathematics as offered at San Jose State College*, by Professor W. H. Myers, San Jose State College.

A description of the content of this paper has been printed in this MONTHLY in the report of the March 1950 meeting of the Southern California Section of the Association.

3. *Let us teach guessing*, by Professor George Pólya, Stanford University.

It has been said very often, and certainly with good reasons, that teaching mathematics should be used as an opportunity to teach demonstrative reasoning. In the speaker's opinion, teaching mathematics should also be used as an opportunity to teach plausible reasoning. Inductive reasoning is a very important kind of plausible reasoning; (non-mathematical) induction means (roughly) guessing a general law from particular instances, and testing it by examining further particular instances. Many important mathematical results have been discovered by induction, especially in the theory of numbers. In his classes, the speaker tries, with appropriate suggestive questions, to make the students discover inductively the law by themselves. The speaker gave a practical demonstration of how this is done.

4. *Games of strategy*, by Professor A. W. Tucker, Princeton University.

Two persons play the following game a large number of times. Each chooses independently one of the ten numbers 1, 2, $\cdots$, 10, and the two chosen numbers are then compared: the chooser of the higher number pays the chooser of the lower number one penny or vice versa, according as the difference between the numbers is odd or even—there being no payment if the numbers are the same. *Problem:* What long-run "strategy" should each player formulate to protect himself on the average against loss? *Solution:* Each should choose the nine numbers 1, 2, $\cdots$, 9 at random with equal frequencies, and never choose 10. If the game is varied by making the payoff two pennies when the difference is even, it turns out that the optimal strategy consists in choosing 8, 9, 10 (only) at random with relative frequencies 1:2:1. These are simple examples of *two-person zero-sum games* (see von Neumann and Morgenstern, *The Theory of Games and Economic Behavior*, Princeton, 1943). The main theorem for such games is that there always exist optimal strategies for the two players, a strategy for the one player that maximizes his minimum expectation, and a strategy for the second player that minimizes his maximum risk, the max-min equaling the min-max. A proof of this fundamental theorem has recently been found within a direct extension of the algebra of simultaneous linear equations to simultaneous linear inequalities.

Discussions for the general reader of economic and military implications of von Neumann's theory of games have been given by Oskar Morgenstern in the *Scientific American*, May 1949, pp. 22–25, and by John McDonald in *Fortune*, June 1949, pp. 100–110. Other references are: R. A. Fisher, *Mathematical Gazette*, 18 (1934), pp. 294–297; R. Bellman and D. Blackwell, *Proceedings of the National Academy of Science*, 35 (1949), pp. 600–605; and various authors, *Annals of Mathematics Study* No. 24 (1950).

This paper arises from work conducted under Office of Naval Research Contracts at Princeton and Stanford Universities.

5. *Note on elementary function theory*, by Professor J. L. Kelley, University of California, Berkeley.

A proof of a form of the Cauchy theorem is given which, though precise, avoids topological assumptions.

The basic lemma: If f has a continuous complex derivative in an open set G, and P, a func-

tion on the unit square with values in G, has piecewise continuous second partial derivatives, then:

$$\frac{d}{ds}\int_0^1 f[P(s,t)]P_t(s,t)dt = \int_0^1 \{f'[P(s,t)]P_t(s,t)P_s(s,t) + f[P(s,t)]P_{ts}(s,t)\}dt$$
$$= f[P(s,t)]P_s(s,t)\Big|_{t=0}^{t=1}.$$

This lemma implies directly that deformation of a path p in a domain where f is analytic does not change the value of $\int_p f(z)dz$ if either: (a) the end points of p are fixed under the deformation; or (b) the deformation is by means of closed curves.

These two rules, together with the following obvious statement, are sufficient for all ordinary calculations with contour integrals. If p and q are paths, and the end point of p is the beginning point of q, and if $\bar{pq}$ is the path obtained by traversing first p and then q, then

$$\int_{\bar{pq}} f(z)dz = \int_p f(z)dz + \int_q f(z)dz.$$

6. *A classroom note*, by Professor C. D. Olds, San Jose State College.

The author discusses an interesting connection between E. T. Whittaker's method for finding the roots of power series (see, for example, this MONTHLY, vol. 49, 1942, p. 462), and some of the recent modifications of Newton's method (H. S. Wall, this MONTHLY, vol. 55, 1948, p. 90). In particular, the approximate formula given by Wall for finding the n-th root of a positive number is obtained directly from the first two terms of Whittaker's formula.

7. *A slant on the problem of crossed ladders*, by Professor H. A. Arnold, University of California, Davis.

Two ladders of lengths a and b connect the bases of parallel vertical walls, z units apart, with points at heights x and y on the opposite walls. They cross at height unity and at distance l from the "y wall." It is required to find z. It is shown that

$$\frac{l}{z} = \frac{1}{x} = \frac{y-1}{y}; \qquad y^2 - x^2 = b^2 - a^2 = K; \qquad x + y = xy.$$

Set $x+y=4u$, $y-x=v$. Then $u=1+(K^2/256)\ (1/u^3)$, a form suitable for computation by successive substitutions. Finally, $v=y-x=(y^2-x^2)/(y+x)=K/4u$, and x and y may be computed from u and v.

E. B. ROESSLER, *Secretary*

MARCH MEETING OF THE MISSOURI SECTION

The Missouri Section of the Mathematical Association of America met at Washington University in St. Louis on Friday afternoon, March 24, 1950, and Saturday morning, March 25, 1950. Professor G. W. Ewing presided at the Friday session, and Professor C. W. Mathews, Chairman of the Section, presided at the Saturday session.

Sixty persons were in attendance including the following thirty-four members of the association: F. L. Barrow, S. Louise Beasley, L. M. Blumenthal, E. W. Bold, S. S. Cairns, J. L. Collins, Mary L. Cummings, R. E. Ekstrom, G. M. Ewing, E. A. Goodhue, Franklin Haimo, F. F. Helton, I. I. Hirschman, Jr., C. A. Johnson, L. O. Jones, Lois Karr, R. E. Lee, Walter Leighton, Ella Martha, C. W. Mathews, Jr., R. R. Middlemiss, Marie A. Moore, R. M. Rankin, Francis Regan, P. R. Rider, Sister Mary Teresine, R. L. Snider, W. L. Stanley, W. J.

Thron, Choy-Tak Taam, W. R. Utz, Jr., W. A. Vezeau, Margaret F. Willerding, W. D. Williams.

The following officers were elected for the coming year: Chairman, F. F. Helton, Central College; Vice-Chairman, L. O. Jones, William-Jewell College; Secretary-Treasurer, Margaret F. Willerding, Harris Teachers College. It was announced that the 1951 meeting will be held at Central College in the spring of 1951. Following the Friday afternoon meeting, tea and coffee were served in the Lounge of the Women's Building.

The following papers were presented:

1. *The geometry of polyhedra*, by Professor S. S. Cairns, University of Illinois.

Professor Cairns was a guest speaker. His paper was expository, and contained a few simple and, for the most part, old results concerning the structure of polyhedra. Despite their simplicity, these results are not widely known, partly because they are out of the main current of traditional geometry, and partly because they are special cases of more general theorems, proved by relatively sophisticated methods. Some of the interest in them stems from the fact that they are denials of plausible statements. Thus, for example, no polyhedron exists with exactly seven edges, though all other numbers greater than five are possible. No polyhedron exists all of whose faces have more than five edges. If none of the faces of a polyhedron are triangular, then some vertex belongs to exactly three faces. No polyhedron has an odd number of odd-sided faces. These and related results were established by elementary arguments, well within the grasp of a high school student.

2. *Notes on Martin's ergodic function*, by Professor W. R. Utz, University of Missouri.

This paper will be published in a subsequent issue of this MONTHLY.

3. *Symmetry in four dimensions*, by Mr. E. W. Bold, St. Louis University

Basic ideas involving the geometry of four dimensions are explained, after which definitions concerning symmetry with respect to a point, line, plane, and hyperplane are considered. Theorems on symmetry in four dimensions are developed by methods of analysis situs. In particular, theorems on perpendicular and coincident elements, as well as those on parallel elements, are discussed.

4. *Hurwitz polynomials in engineering mathematics* by Professor Herman Betz, University of Missouri, introduced by the Secretary.

The purpose of this paper is to call attention to the great importance of Hurwitz polynomials in questions of stability arising from engineering problems. It is shown how the well known Hurwitz criteria or their equivalent may be obtained for those cases which engineers are mostly concerned with, using however, quite elementary methods which are well within the grasp of engineers.

5. *Distribution law of the product of two variables independently distributed in Pearson's type I laws*, by Mr. John S. Hagen, St. Louis University, introduced by the Secretary.

Two variables x and y are known to be independently distributed according to Pearson's Type I laws. In the joint frequency function of x and y, the substitution $y=Z/x$ is made, and the variable x is integrated out, giving a function of the product xy. The conditions under which the series in $f(Z)$ is finite, the general appearance of the curve, and some of the properties of the functions, are discussed.

6. *On quasi-analytic functions of analytic functions*, by I. I. Hirschman, Jr., Washington University.

In the theory of quasi-analytic functions it is sometimes necessary to obtain bounds on the derivatives of $f[g(x)]$ from bounds on the derivatives of f and g separately. Several theorems of this type are presented. It is shown in particular that if g is analytic, and if we possess information concerning the behavior of $g(x)$ in the complex plane, then very precise estimates may be found.

7. *A method of uniformizing grades*, by Marlow Sholander, Washington University, introduced by the Secretary.

The speaker presented a description of the method which has been used with some success at Washington University in multiple section mathematics courses, to reduce the probability that two students with different instructors receive grades incommensurate with their ability and effort.

MARGARET F. WILLERDING, *Secretary*

APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The April meeting of the Metropolitan New York Section of the Mathematical Association of America was held at the City College of the College of the City of New York, New York, on Saturday, April 1, 1950. The Chairman of the Section, Professor B. P. Gill, introduced President Wright, and presided at the business meeting which preceded the afternoon session; the High School Vice-Chairman, Mr. Alan Wayne, presided at the morning session; the Collegiate Vice-Chairman, Professor L. F. Ollmann, presided at the afternoon session.

One hundred and ten persons attended the meeting, including the following sixty-seven members of the Association: M. W. Al-Dhahir, Brother Bernard Alfred, R. C. Archibald, F. C. Beckman, Samuel Borofsky, C. B. Boyer, Paul Brock, A. B. Brown, J. H. Bushey, J. J. Clark, T. F. Cope, W. H. H. Cowles, P. M. Curran, I. A. Dodes, J. N. Eastham, W. H. Fagerstrom, H. F. Fehr, B. P. Gill, G. C. Helme, A. J. Hoffman, E. Marie Hove, T. R. Humphreys, Solomon Hurwitz, L. C. Hutchinson, Alto C. Juelich, C. J. Kaufman, L. S. Kennison, G. A. Keyes, H. S. Kieval, R. J. Kohlmeyer, H. C. Kranzer, C. H. Lehmann, D. R. Lintvedt, E. R. Lorch, D. M. MacEwen, V. S. Mallory, D. May Hickey Maria, F. H. Miller, A. J. Mortola, D. S. Nathan, M. A. Nordgaard, Eugene Odin, C. S. Ogilvy, L. F. Ollmann, O. L. Phillips, W. L. Pickard, E. L. Post, Walter Prenowitz, J. J. Quinn, Moses Richardson, Selby Robinson, H. D. Ruderman, John Salerno, Abraham Schwartz, Aaron Shapiro, James Singer, M. G. Smith, C. G. Solky, E. P. Starke, Mildred M. Sullivan, Annita Tuller, H. E. Wahlert, Alan Wayne, W. W. Winnis, J. M. Wolfe, H. N. Wright, H. J. Zimmerberg.

The following officers were elected for the coming year: Chairman, Brother Bernard Alfred, Manhattan College; Collegiate Vice-Chairman, James Singer, Brooklyn College; High School Vice-Chairman, I. A. Dodes, Stuyvesant High School; Secretary, H. S. Kieval, Brooklyn College; Treasurer, Aaron Shapiro, Midwood High School. Professor W. H. Fagerstrom, Chairman of the Committee on Prizes and Awards, reported on the progress of the Committee and on the forthcoming High School Contest sponsored by the Section. Complete de-

tails of the contest will appear in this MONTHLY.

Dr. H. N. Wright, President of The City College, welcomed the people at the meeting, and then the following papers were presented:

1. *What is mathematical functionality?* by Dr. Barnett Rich, Richmond Hill High School (introduced by Mr. Alan Wayne).

In this paper, the author condensed material contained in his dissertation, which pertained especially to the problems of what constitutes the domain of the subject matter of functionality, and how the basic concepts of functionality or relational variability may be applied to the teaching of mathematics, and utilized further in the teaching of science. The main points treated in the paper concern: (1) The use of a new symbol, namely $\mathfrak{x}$), for the ratio of any two ordered values of a variable; (2) A comparison of the similarity of the uses of the new symbol $\mathfrak{x}$) to those of Δx; (3) The approach to functionality through the use of three-variable relationships of the simplest operational character, such as $z=xy$, $z=x+y$, and $z+x^y$; (4) The emphasizing of sense change and variation in the teaching of both mathematics and science; (5) Continuity in the instruction of functionality from the elementary school to the college.

2. *Inequalities and convex bodies*, by Professor E. R. Lorch, Columbia University.

Basic to the development of analysis are certain inequalities such as those of Cauchy-Schwartz, Holder, and the triangle inequality. The relation between inequalities and the theory of convex bodies has been known since the time of Minkowski. However, the classic theory merely has existential force, and does not give methods for actual computation. This paper presents precise and analytically transparent methods for the derivation of all inequalities of this type. The program is based on a new definition of convexity in which the notions of differentiability, homogeneity, definite quadratic form, and one-to-one transformation play a dominant role. It is shown that bodies which are convex in the classic sense may be approximated arbitrarily closely by those of our type. This circumstance makes it possible to reconsider every problem on convexity along lines in which these new techniques apply.

3. *What mathematics should a high school teacher know?* by Professor H. F. Fehr, Teachers College, Columbia University.

Teacher examinations in large cities demand a knowledge of advanced mathematics of little use in high school teaching. State certifications require from zero to twenty semester hours of mathematics; college major requirements vary from eighteen to forty-six semester hours. Commission reports recommend study beyond the calculus and in the physical sciences. The changing purpose of high school instruction demands other mathematical knowledge than traditional preparatory mathematics. Beyond high school subjects the teacher must know analytic geometry, advanced algebra, the calculus, foundations of modern mathematics, and applications and the history of what he teaches. There is special need for professionalization of high school mathematics, that is, a study from an advanced view-point, and from many avenues of approach, of the high school content. The high school teacher must be a scholar within his field, not necessarily beyond it.

4. *What makes a good problem?* by Professor E. P. Starke, Rutgers University.

Several possible characteristics of a "good" problem are proposed, discussed briefly, and illustrated by selections from the problem departments of this MONTHLY and other journals. A good problem meets one or more of the following requirements: it may extend old results or open up a new field of interest; it may be nicely adapted to develop facility and ingenuity for its solution; it may force out into the open certain misunderstandings or inadequacies; it may have an

unexpected and surprising conclusion; it may require the invention of new methods of attack; or it may possess a wide range of appeal, perhaps through being dressed up in picturesque language. A real need exists for more good problems both for class practice and for the pages of the problem departments.

5. *Analog computers*, by Mr. A. Goetz, Arma Corporation (introduced by Mr. Eugene Odin).

The speaker stressed the role of the machine computer with regard to its utility in the solution of present day problems. He classified the computers into several types, and chose the automatic electromagnetic analog computer for closer examination. The components of such a device were enumerated. A review of the operations of arithmetic, calculus, and function generation was made, and illustrations of various instrumentations of the operations of addition, subtraction, multiplication, division, differentiation, integration, and function generation were given.

The speaker then developed the block diagram of a computer to solve the simple geometric problem of a plane triangle where two sides and the included angle are known and the length of the other side is required.

The hope was expressed that the dissemination of information regarding the capabilities of the analog computer would result in its application in fields as yet unexplored.

JAMES SINGER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December 30, 1950.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May, 1951.

ILLINOIS, University of Illinois, Urbana, May 11–12, 1951.

INDIANA, May 5, 1951.

IOWA, Wartburg College, Waverly, April 20–21, 1951.

KANSAS

KENTUCKY, Eastern Kentucky State College, Richmond, April 28, 1951.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, December 9, 1950.

METROPOLITAN NEW YORK, Spring, 1951.

MICHIGAN, East Lansing, March 24, 1951.

MINNESOTA, Duluth Branch of University of Minnesota, October 7, 1950.

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NEBRASKA, University of Nebraska, Lincoln, May 5, 1951.

NORTHERN CALIFORNIA, University of San Francisco, January 27, 1951.

OHIO, April 21, 1951.

OKLAHOMA, Oklahoma City, November 13, 1950.

PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.

PHILADELPHIA, Lehigh University, Bethlehem, Pennsylvania, November 25, 1950.

ROCKY MOUNTAIN, Colorado State College of Education, Greeley, April, 1951.

SOUTHEASTERN, Vanderbilt University and Peabody College, Nashville, Tennessee, March 16–17, 1951.

SOUTHERN CALIFORNIA, Whittier College, Whittier, March 10, 1951.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1951.

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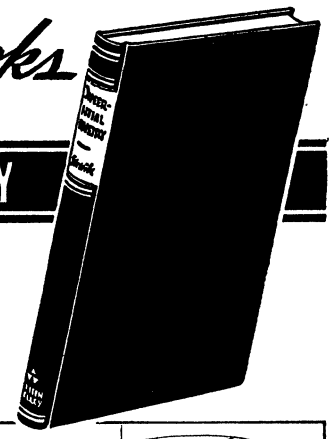
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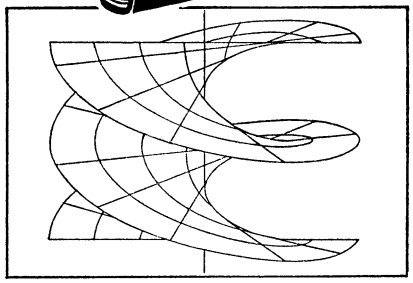
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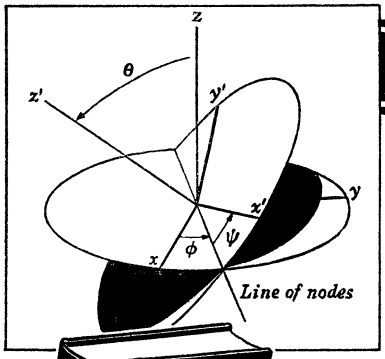
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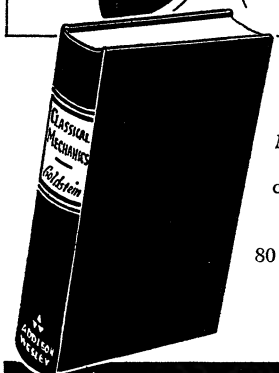
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The index is not a complete index of every item published in the Monthly; rather it includes only certain types of articles as described below:

Included	*Not Included*
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This index has been prepared by the Department of Mathematics of Haverford College, consisting of C. B. Allendoerfer, C. O. Oakley, D. L. Thomsen, and A. H. Wilson. Assistance in many details was provided by C. P. Bruderle of Villanova College.

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A TYPE OF VARIATION ON NEWTON'S METHOD

H. J. HAMILTON, Pomona College

1. Introduction. Newton's Method of successive approximations to a root z^* of $\phi(z)=0$ consists in choosing a first approximation z_0 and then successively applying the formula

$$z_n = z_{n-1} - \phi(z_{n-1})/\phi'(z_{n-1}) \tag{1}$$

for $n=1, 2, \cdots$. Now if $\phi'(z^*)\neq 0$ and $\phi''(z^*)$ exists, and if z_0 is sufficiently near z^*, we have from (1), [1], and the fact that $\phi(z^*)=0$,

$$\begin{aligned}(z_n - z^*) &= (z_{n-1} - z^*) - \frac{(z_{n-1} - z^*)\phi'(z^*) + O[(z_{n-1} - z^*)^2]}{\phi'(z^*) + O[(z_{n-1} - z^*)]} \\ &= (z_{n-1} - z^*)\left\{1 - \frac{1 + O[(z_{n-1} - z^*)]}{1 + O[(z_{n-1} - z^*)]}\right\} = (z_{n-1} - z^*)O[(z_{n-1} - z^*)] \\ &= O[(z_{n-1} - z^*)^2].\end{aligned}$$

Frame [2] and Wall [3], independently, developed the formula

$$z_n = z_{n-1} - \frac{2\phi(z_{n-1})\phi'(z_{n-1})}{2[\phi'(z_{n-1})]^2 - \phi(z_{n-1})\phi''(z_{n-1})} \tag{2}$$

to provide still more rapid convergence of the z_n. We can show by analysis like that used above that, if $\phi'(z^*)\neq 0$, $\phi'''(z^*)$ exists, and z_0 is sufficiently near z^*, then $(z_n - z^*) = O[(z_{n-1} - z^*)^3]$, a fact pointed out by Wall for polynomial $\phi(z)$ [3, p. 94].

The order relations indicated in the preceding two paragraphs suggest the following problem: To find a function $f(z)$ associated with $\phi(z)$ such that, if z_0 is sufficiently near z^* and $z_n = f(z_{n-1})$ for $n=1, 2, \cdots$, then $z_n \to z^*$ with $(z_n - z^*) = O[(z_{n-1} - z^*)^k]$ for $k=2, 3, \cdots$ in general. It is our purpose to discuss this problem. Except for geometric interpretations, our analysis may be assumed to take place in either the real or the complex field.

2. Solution by inversion of series. The efficacy of (1) is usually suggested in the real field by showing that z_n is the z-intercept of the line tangent to the curve $y=\phi(z)$ at the point (z_{n-1}, y_{n-1}), where $y_{n-1}=\phi(z_{n-1})$; and Frame and Wall define the z_r of (2) by approximating with (1) the appropriate z-intercept of the vertically opening parabola which osculates the curve at this point. The z-intercept of the horizontally opening parabola is more easily computed, and this parabola we generalize to osculating curves with the equations

$$z = \psi(y_{n-1}) + (y - y_{n-1})\psi'(y_{n-1}) + \cdots + (y - y_{n-1})^{k-1}\psi^{(k-1)}(y_{n-1})/(k-1)!,$$

where $z=\psi(y)$ is inverse to $y=\phi(z)$. We assume that $\phi'(z^*)\neq 0$ (or that $\phi^{(s)}(z^*)\neq 0$ for some s, in which case we may replace $\phi(z)$ throughout by $g(z)\equiv\phi(z)/\phi'(z)$,

for which $g(z^*)=0$ and $g'(z^*)\neq 0$ [4, p. 119]). Putting $y=0$ we obtain

$$(3)\quad z_n = \psi(y_{n-1}) - y_{n-1}\psi'(y_{n-1}) + \cdots + (-1)^{k-1}y_{n-1}^{k-1}\psi^{(k-1)}(y_{n-1})/(k-1)!,$$

or

$$(4)\quad \begin{aligned} z_n = z_{n-1} &- \phi(z_{n-1})/\phi'(z_{n-1}) - [\phi(z_{n-1})]^2\phi''(z_{n-1})/\{2[\phi'(z_{n-1})]^3\} + \cdots \\ &+ (-1)^{k-1}[\phi(z_{n-1})]^{k-1}\psi^{(k-1)}[\phi(z_{n-1})]/(k-1)!. \end{aligned}$$

For example, taking $k=3$, we have

$$(5)\quad z_n = z_{n-1} - \phi(z_{n-1})\{2[\phi'(z_{n-1})]^2 + \phi(z_{n-1})\phi''(z_{n-1})\}/\{2[\phi'(z_{n-1})]^3\}.$$

Now (4) solves our problem if $\phi^{(k)}(z^*)$ exists. For then $\psi^{(k)}(0)$ exists, and we have (as in the derivation of (8a) in §5 below)

$$(6)\quad \begin{aligned} z^* = \psi(y_{n-1}) &- y_{n-1}\psi'(y_{n-1}) + \cdots \\ &+ (-1)^{k-1}y_{n-1}^{k-1}\psi^{(k-1)}(y_{n-1})/(k-1)! + O(y_{n-1}^k) \end{aligned}$$

in which, since $(z_{n-1}-z^*)/y_{n-1}\to\psi'(0)$, we may replace $O(y_{n-1}^k)$ by $O[(z_{n-1}-z^*)^k]$; and comparison of (6) and (3) shows that $(z_n-z^*)=O[(z_{n-1}-z^*)^k]$.

We shall not content ourselves with this solution, however, since formulas for the derivatives of the inverse function in terms of the direct function and its derivatives [5] are somewhat more involved than the expressions which we shall obtain for $f(z)$ in the following section. Indeed if tractable formulas for the derivatives of the inverse function were available† we could write a useful, immediate expression for z^* itself in the case of analytic $\phi(z)$, namely, the infinite series of which (4) with $n=1$ includes the first k terms.‡

3. Solution by the method of [4]. Perhaps the most direct attack upon our problem is to pursue the method of analysis of [4], by means of which we shall obtain, for regular $\phi(z)$ and $f(z)$, the general $f(z)$ for which $(z_n-z^*)=O[(z_{n-1}-z^*)^k]$ for $k=2, 3, \cdots$. Supposing then that $\phi(z)$ and $f(z)$ are regular at z^*, we assume (or bring about as before) the condition $\phi'(z^*)\neq 0$, and write

$$f(z) = a_0 + a_1(z-z^*) + a_2(z-z^*)^2 + \cdots.$$

Since

$$z_n \equiv f(z_{n-1}) \equiv a_0 + a_1(z_{n-1}-z^*) + a_2(z_{n-1}-z^*)^2 + \cdots$$

must tend to z^*, we see that $a_0=z^*$. Hence

$$f(z) - z^* = a_1(z-z^*) + a_2(z-z^*)^2 + \cdots,$$

† Securing formulas for these derivatives is the problem of inversion of series. The classical results are due to Van Orstrand [5], who lists expressions for $\psi^{(\nu)}$ for $\nu=1, 2, \cdots, 13$. The referee notes that a paper by Reynolds [6] suggests an easily understood process for calculating these expressions.

‡ I am indebted to the referee for calling my attention to a paper by Corey [7] which notes this fact.

so that

$$z_n - z^* = a_1(z_{n-1} - z^*) + a_2(z_{n-1} - z^*)^2 + \cdots .$$

Since we require that $(z_n - z^*) = O[(z_{n-1} - z^*)^k]$ it follows that $a_\nu = 0$ for $\nu < k$. So

$$f(z) = z^* + a_k(z - z^*)^k + \cdots \equiv z - (z - z^*) + a_k(z - z^*)^k + \cdots . \tag{7}$$

Conversely, if $f(z)$ is of the form (7) with $k \geqq 2$ it is clear that $z_n \to z^*$ (provided that z_0 is sufficiently near z^*; this is assumed throughout) with $(z_n - z^*) = O[(z_{n-1} - z^*)^k]$.

It remains to approximate the powers of $(z - z^*)$ in terms of $\phi(z)$ and its derivatives. To this end we first expand $\phi(z)$ about the point z and evaluate at z^*:

$$\phi(z^*) = \phi(z) + (z^* - z)\phi'(z) + (z^* - z)^2\phi''(z)/2! + \cdots .$$

Taking account of the vanishing of $\phi(z^*)$ and rewriting, we obtain

$$0 = \phi(z) - (z - z^*)\phi'(z) + (z - z^*)^2\phi''(z)/2! - \cdots . \tag{8}$$

Multiplying (8) by the successive powers of $(z - z^*)$ and adjoining the resulting equations to a rearrangement of (8), we obtain the system (in which the argument z is omitted for brevity)

$$\begin{aligned}
&-(z - z^*)\phi' + (z - z^*)^2\phi''/2! - (z - z^*)^3\phi'''/3! + \cdots = -\phi;\\
&(z - z^*)\phi - (z - z^*)^2\phi' + (z - z^*)^3\phi''/2! - \cdots = 0;\\
&(z - z^*)^2\phi - (z - z^*)^3\phi' + \cdots = 0;\\
&(z - z^*)^3\phi - \cdots = 0;\\
&\cdots .
\end{aligned} \tag{9}$$

We may now find an expression which shall differ from $(z - z^*)^s$ to the order of $(z - z^*)^{t+1}$ (s, t arbitrary, with $t \geqq s$) by merely transposing all terms beyond the tth in each of the equations (9) and eliminating all powers of $(z - z^*)$ except the sth from the left-hand sides of the first t of the resulting equations. So doing, we obtain

$$(z - z^*)^s = (-1)^{s+1}\Delta_{t,s}/\Delta_t + O[(z - z^*)^{t+1}],$$

where Δ_t is the symmetric determinant,

$$\begin{vmatrix}
\phi' & \phi''/2! & \cdots & \phi^{(t-2)}/(t-2)! & \phi^{(t-1)}/(t-1)! & \phi^{(t)}/t! \\
\phi & \phi' & \cdots & \phi^{(t-3)}/(t-3)! & \phi^{(t-2)}/(t-2)! & \phi^{(t-1)}/(t-1)! \\
0 & \phi & \cdots & \phi^{(t-4)}/(t-4)! & \phi^{(t-3)}/(t-3)! & \phi^{(t-2)}/(t-2)! \\
\cdots & \cdots & & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \phi & \phi' & \phi''/2! \\
0 & 0 & \cdots & 0 & \phi & \phi'
\end{vmatrix}$$

which, since $\phi(z^*)=0$ and $\phi'(z^*)\neq 0$, does not vanish at z^*, and where $\Delta_{t,s}$ is the same determinant with the sth column replaced by a column whose first element is ϕ and whose others are all zero. We note that

$$\Delta_{t,1}=\phi\Delta_{t-1},\qquad (\Delta_0\equiv 1),\qquad \Delta_{t,t}=\phi^t,$$
$$\Delta_t=\Delta_{t-1}\phi'-\Delta_{t-2}\phi\phi''/2!+\Delta_{t-3}\phi^2\phi'''/3!-\cdots+(-1)^k\Delta_0\phi^{k-1}\phi^{(k)}/k!, \tag{10}$$

the last relation being obtained by expanding Δ_t by the first column and using induction.

Returning to (7) we now see that $f(z)$ as there represented is of the form

$$\begin{aligned} f(z) &= z-\Delta_{k-1,1}/\Delta_{k-1}-O[(z-z^*)^k]+a_k(z-z^*)^k+\cdots \\ &= z-\Delta_{k-1,1}/\Delta_{k-1}+b_k(z-z^*)^k+b_{k+1}(z-z^*)^{k+1}+\cdots \\ &= z-\Delta_{k-1,1}/\Delta_{k-1}+(z-z^*)^k[b_k+b_{k+1}(z-z^*)+\cdots]. \end{aligned}$$

Hence, because z^* is a simple root of $\phi(z)$, we have (h is to be read a function which is regular at z^*)

THEOREM 1. *A necessary and sufficient condition that* $z_n\to z^*$ *with* $(z_n-z^*)=O[(z_{n-1}-z^*)^k]$ *for* $k\geqq 2$ *is that*

$$f(z)=z-\Delta_{k-1,1}/\Delta_{k-1}+h\phi^k\equiv z-\phi\Delta_{k-2}/\Delta_{k-1}+h\phi^k.$$

Since from (7) we have

$$(z_n-z^*)/(z_{n-1}-z^*)=a_k(z_{n-1}-z^*)^{k-1}+a_{k+1}(z_{n-1}-z^*)^k+\cdots,$$

the z_n will ultimately oscillate about z^* in the real domain if and only if k (assumed $\geqq 2$) is odd. We therefore have

THEOREM 2. *A necessary and sufficient condition that* $z_n\to z^*$ *with* $(z_n-z^*)=O[(z_{n-1}-z^*)^2]$ *and in the real domain ultimately oscillate about* z^* *is that*

$$\begin{aligned} f(z) &= z-\Delta_{2l+1,1}/\Delta_{2l+1}-a\Delta_{2l+1,2l+1}/\Delta_{2l+1}+h\phi^{2l+2} \\ &\equiv z-\phi(\Delta_{2l}+a\phi^{2l})/\Delta_{2l+1}+h\phi^{2l+2}, \end{aligned}$$

where $a>0$ *and* l *is some positive integer. Necessarily,* $(z_n-z^*)=O[(z_{n-1}-z^*)^{2l+2}]$. (Compare [4, p. 121].)

4. Special cases. The earlier formulas. Taking $h=0$ and $k=2$ in the formula of Theorem 1 and noting that $\Delta_1=\phi'$, we obtain $f(z)=z-\phi/\phi'$, the iterative function of Newton's Method. Taking $h=0$ and $k=3$ in the same formula and using (10) to find that $\Delta_2=\phi'^2-\phi\phi''/2$, we obtain $f(z)=z-2\phi\phi'/(2\phi'^2-\phi\phi'')$, which is the iterative function of Frame and Wall. Taking $k=3$ in the same formula but leaving h for the moment unspecified we obtain

$$f(z) = z - \frac{2\phi\phi'}{2\phi'^2 - \phi\phi''} + h\phi^3 = z - \frac{\phi}{\phi'} \Big/ \left(1 - \frac{\phi''\phi}{2\phi'^2}\right) + h\phi^3$$

$$= z - \frac{\phi}{\phi'}\left(1 + \frac{\phi\phi''}{2\phi'^2}\right) - \phi^3\left(\frac{\phi''^2}{4\phi'^5} + \cdots\right) + h\phi^3;$$

if we now take

$$h = \frac{\phi''^2}{4\phi'^5} + \cdots$$

we finally obtain $f(z) = z - \phi(2\phi'^2 + \phi\phi'')/(2\phi'^3)$, which is the iterative function of (5).

Taking $h=0$, $l=1$, and $a=1$ in the formula of Theorem 2 and using (10) to find that $\Delta_3 = \phi'^3 + \phi^2\phi'''/6 - \phi\phi'\phi''$, we obtain $f(z) = z - 3\phi(2\phi^2 + 2\phi'^2 - \phi\phi'')/(6\phi'^3 + \phi^2\phi''' - 6\phi\phi'\phi'')$, a particular iterative function for which $(z_n - z^*) = O[(z_{n-1} - z^*)^4]$ and, in the real domain, the z_n ultimately oscillate about z^*.

In the real domain the graphical interpretation (see [4, p. 114]; compare [3, Figure 2]) of the situation in which the z_n oscillate about z^* is simply that the curve $y = f(z)$ descends in a deleted neighborhood of the point (z^*, z^*), and the rapidity of convergence of the z_n in Theorem 1 corresponds to the high order of contact ($\geqq (k-1)$th) of that curve with its horizontal tangent at (z^*, z^*).

5. Conclusion. *Remarks.* If instead of the regularity of $\phi(z)$ at z^* we have merely the existence of $\phi^{(k)}(z^*)$ (or of $\phi^{(2l+2)}(z^*)$), and if we impose no regularity requirements on $f(z)$, then the conditions of Theorems 1 and 2 are sufficient for the purposes indicated. This follows from a slight modification of the preceding analysis, starting with the relation (which we shall at once derive)

$$0 = \phi(z) - (z - z^*)\phi'(z) + \cdots + (-1)^{k-1}(z - z^*)^{k-1}\phi^{(k-1)}(z)/(k-1)! + O[(z - z^*)^k] \tag{8a}$$

instead of (8).

To validate (8a) we note first that $\phi^{(k-1)}(z)$ exists in a neighborhood of z^*, define R by

$$0 = \sum_{\mu=0}^{k-1} (z^* - z)^\mu \phi^{(\mu)}(z)/\mu! + R, \tag{11}$$

expand the $\phi^{(\mu)}(z)$ about z^*, rewrite (11), and collect terms. Thus

$$R = -\sum_{\mu=0}^{k-1} (z^* - z)^\mu \sum_{\nu=0}^{k-1-\mu} (z - z^*)^\nu \phi^{(\mu+\nu)}(z^*)/(\mu!\nu!) + O[(z - z^*)^k]$$

$$= -\sum_{\lambda=0}^{k-1} \left\{ \sum_{\nu+\mu=\lambda} [\lambda!/(\mu!\nu!)](z^* - z)^\mu (z - z^*)^\nu \right\} \phi^{(\lambda)}(z^*)/\lambda! + O[(z - z^*)^k]$$

$$= -\sum_{\lambda=0}^{k-1} [(z^* - z) + (z - z^*)]^\lambda \phi^{(\lambda)}(z^*)/\lambda! + O[(z - z^*)^k]$$

$$= -\phi(z^*) + O[(z - z^*)^k] = O[(z - z^*)^k].$$

Hence (11) identifies itself with (8a) and the proof is complete.

Among the remaining questions of possible interest are the following:

(**A**) We have seen that z_1 as given by (4) tends to z^* as k becomes infinite if $\phi(z)$ is regular at z^* and z_0 is sufficiently near z^*. With what other choices of h than those which give us the partial sums (4) of the expansion of the inverse function for regular $\phi(z)$ can the same conclusion be asserted?

(**B**) Are there choices of h other than zero which might make $f(z)$ in some sense simpler? (Compare the derivation of (5) in §4 above.)

(**C**) Are there approximations to the powers of $(z-z^*)$ which are more readily calculable than those of §3?

Although nothing can be proved concerning an infinite process by finite computation, it might be interesting to apply one of our formulas to a numerical case. Let us then approximate $\sqrt{2}$ by taking $\phi=z^2-2$. Here, with $h=0$, $f(z)=[z\Delta_k-(z^2-2)\Delta_{k-1}]/\Delta_k$ and, since $\phi'=2z$, $\phi''=2$, and $\phi'''=\phi^{\text{iv}}=\cdots=0$, we have from (10) $\Delta_k=2z\Delta_{k-1}-(z^2-2)\Delta_{k-2}$. Starting with $\Delta_0=1$ and $\Delta_1\equiv\phi'=2z$, we obtain, successively, $\Delta_2=3z^2+2$, $\Delta_3=4z^3+8z$, $\Delta_4=5z^4+20z^2+4$, $\Delta_5=6z^5+40z^3+24z$, $\cdots$. Taking $k=5$ we have, after simplifications,

$$f(z)=\frac{(z^2+2)(z^4+28z^2+4)}{2z(z^2+6)(3z^2+2)};$$

with $z_0=1$, this yields $z_1=1.41429-$ and with $z_0=1.5$ it yields $z_1=1.414213564+$. Actually, $\sqrt{2}=1.414213562+$.

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EDUCATION VS. LEGISLATION

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It all started in the last part of the nineteenth century; however, the change was so gradual that almost no one fully appreciated the fact that mathematics was losing its place of importance as a secondary school subject and as a general requirement for the bachelor's degree until after the end of World War I. At least, mathematicians were not too well aware of the development. When, in the early twenties, the trend began to make itself felt, mathematicians proved that they were human—they followed the first impulse of a trapped soul—they began to fight. The battle raged in literature and on college campuses; the mathematicians condemned bitterly the professional educators, calling upon the shades of Euclid and Pythagoras to witness their noble stand.* The professional educator was prepared for this onslaught, having his ramparts protected by a battery of tests (the validity of which were unquestionable!), a philosophic concept or two (which were undebatable!), and last, but perhaps the strongest weapon, the ears of the powers that be. This was truly a formidable foe against which the mathematicians had arrayed themselves. The outcome was inevitable; the proponents of mathematics steadily lost ground. The fray continued for at least one decade before there was any evidence of a change of approach on the part of the mathematicians, and this change was by no means a general one. Mathematicians are not to be censored too severely for the tactics used during the twenties; some good was no doubt done—at least the question of the role of mathematics in the curriculum was kept open and not allowed to die in the hands of the victors, as was true of one field of study, the passing of which is mourned by many.

In the thirties some few mathematicians began to work with the educators in curriculum planning, in the evaluation of secondary mathematics in the light of modern concepts of education, in the development of better teaching practices, and in the production of more desirable texts. The effects of this change of approach in many instances were fine; however, favorable accomplishments were not general because so few mathematicians, in comparison with the magnitude of the task, participated. There was still a feeling of animosity between the two groups involved. Had the entire forces of the mathematical band been united in this effort of working with the professional educator rather than trying to force him to accept certain ideas, much more and lasting good would have been accomplished. At least a spirit of friendliness and comradeship in a mutual problem would have been developed, which would have been conducive to to further progress. Since all of this did not happen, a stalemate still exists generally.

When the deplorable conditions pertaining to the mathematical knowledge of the youth of our land were made public by the testing agencies of the armed forces, mathematicians said, "I told you so," and settled back expecting edu-

* The author, a bit young at the beginning, soon became of age and joined the troops

cators to make the necessary adjustments and produce the desired results. Since this has not occurred, the battle has started all over again. Let us recall that which took place from 1920 to 1940, and change our approach to the problem. Unless this is done, conditions will remain unchanged. The balance of this paper will deal with suggestions for a new approach. In particular, an approach is advocated that is built along educational lines rather than through the tactics of force and legislative action.

Before going into a discussion of those things which it is believed will tend to bring about the desired results, it is in order to make a few comments in general on the educational process. Since most of us are in the profession of educating people, we recognize two basic truths connected with the process. They are: (1) the slowness of attaining any noteworthy accomplishment and (2) the permanency and potency of those things which are learned.

One of the greatest failings of mankind is that of impatience. We are in a hurry; we want things done over night and corrections made immediately. Hence, during the twenties and thirties we tried and succeeded to some extent to have the number of units in mathematics required for graduation from high school increased. In those states where we were successful, it did not last long, for many reasons. There are several other examples of this type of thing which will not be mentioned here. Had we not been so perturbed and in such a hurry, a program of an educational nature would have produced by now those things which were desired. When thinking in terms of educating people to our way of believing, we must consider a period of ten, twenty, fifty and even one hundred years—at least, not an interval of one, two or three years.

Now are presented some ideas and suggestions concerning the proposed approach. For the most part, these ideas find expression in the present program in Louisiana, or are involved in the plans for the immediate future. We in Louisiana do not have all the answers even though we believe that we are making progress; therefore, we shall be most happy to receive additional suggestions from any who read this paper.

There are four mathematical organizations operating within Louisiana. They are: (1) The Louisiana-Mississippi Section of the Mathematical Association of America, (2) The Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics, (3) The Mathematics Section of the Louisiana College Conference, and (4) The Mathematics Section of the Louisiana Educational Association. Each of these groups meets annually. At least a part of every program deals with current problems in the field of mathematics, that is, instruction at all levels, certification of teachers, teacher training, teacher procurement, and the like.

Back about 1940, it was decided to form a standing committee composed of three members from each of these four organizations. At this time the militant spirit still ran high. Thus the committee operated as a pressure group. Its chief concern was over the fact that the college freshman was not prepared to do even average work in college mathematics. The answer of course, at that time,

was to get the number of units required for graduation from high school increased from the then existing one unit. As a pressure group, the committee was quite successful. After several conferences with the state superintendent and his staff, both in his office and at an invitational dinner or so, the requirements were raised to three units. No, it did not last long. Pressure was relaxed, mostly due to the war, and as usual with those things which are gained through pressure, they are lost when a period of relaxation takes place. It was just as well, for the college freshman had not improved. Now it is realized that merely making all high school students take three years of mathematics is not the answer. In fact, it is believed that a large per cent of high school students should not take the traditional courses at all. It would be fine for them to take some mathematics to add to their culture and to prepare them for various problems they will meet in their respective spheres of activity, but not the usual courses that were a part of the high school curriculum until more recent years.

The committee is no longer a pressure group. It is now older and wiser, and has become a consultant organization on the one hand and a public relations or educational agency on the other. The time is past when the committee goes to the state superintendent or members of his staff in a body making demands "in the name of common sense." Members of the committee now drop by the offices of these officials occasionally for a chat to offer their services and discuss mutual problems. Be it understood that all of this is sincere—the officials know it and feel it, for they are calling upon the mathematicians of the state for aid and advice. The committee, however, is not all "chit-chat," for it is continually collecting data and other information of importance to the problem at hand.

A few years ago the state department of education felt that the certification regulations needed to be revised and so proceeded to do it. Two outstanding events took place and two outstanding regulations were brought into the code in the process. The events were: (1) A member of the committee was made a member of the certification committee and (2) the committee itself was invited to appear before the certification group in order to make its recommendations. The modifications in regulations were: (1) The mathematical requirement necessary for certification in order to teach high school mathematics was increased from six to eighteen semester hours, and (2) six semester hours was made a part of the general educational requirement for a teaching certificate at any level. This six-semester hour requirement will be elaborated later.

President Conant of Harvard said a few years ago:*

> I am almost tempted to generalize that the more educated the person, the less his knowledge of secondary-school education. Certainly the lack of knowledge among the professors of arts and sciences in our colleges and universities is proverbial. And with lack of information goes lack of understanding and lack of sympathy. As a result, on more than one campus we have almost a state of civil war between those who profess a knowledge of education and those who profess a knowledge of subjects which constitute a modern educational curriculum.

* Conant, J. B., A Truce Among Educators, *Teachers College Record*, December, 1944, pp. 157–163.

> This academic war has been in a sense inevitable, as I propose to show by a brief résumé of history, but to my mind an armistice has been for some years overdue. And it is for such an armistice that I should like to put in a good word this afternoon (and I might remark parenthetically that it takes two to make an armistice quite as much as to make a quarrel). My belief in the need for the cessation of hostilities comes not only from my general tendency to favor pacific methods of handling academic controversy, but also because I am really worried about the present lay reaction to educational matters. I am distressed by both the vehemence and the ignorance with which views about education are expressed publicly and privately by many prominent people. Now we can hardly expect the public to have a very clear understanding about educational problems when education is a house warring against itself. Hence my plea this afternoon for a "cease firing" order.

This statement has been taken seriously in Louisiana. The feeling is that the mathematician still knows best what subject matter the prospective teacher needs to know; nevertheless, the thoughts of the educator are being respected. The result is that the two groups are working together on all the campuses of the state in order to bring about the best possible conditions, and the conditions are greatly improved. The ".. . . state of civil war . . . " of which President Conant spoke is ceasing to exist.

A new project of the committee, which is entering its second year, is a testing program. Each department of mathematics of the state is giving an achievement test to all new students. These tests are sent to a central location where they are processed and various statistical measures are taken. The results are being used in various ways to foster an improvement in secondary instruction and for remedial purposes. As they are used, the educational approach is kept in mind rather than the "we told you so" attitude.

The next venture, which is expected to take shape in 1950, is a mathematical institute of one week to ten days, and a series of week-end conferences. The institute, to be held at some desirable location, will be for all who desire to attend. The conferences are to be held on each of the campuses of the state and are primarily for those who are within driving range. Two colleges have already held such conferences with excellent results. The idea of this work is to give the elementary and secondary teachers some in-service training, inform them of present conditions, and excite them to greater activity.

Another topic which is presently being given consideration concerns graduate study. Traditionally, every student who enters a graduate school to study mathematics is trained as though he is to become a research man. This is desirable for the comparatively few who are going into research. However, every department gets a good many students who like mathematics and desire to teach it—even in high school, but who have just about "reached their level" especially when they get into courses for graduates only. These students usually will become discouraged and quit, or they will be asked to leave, or with a great amount of work from a patient professor they will struggle through to a master's degree. The members of this group, while not the best material for research mathematicians, would make excellent high school teachers—much better than the English and history majors who are forced to teach mathematics

in many schools. It is possible that we are missing a great opportunity to aid our field of study by not setting up a curriculum for this type of student and encourage him towards working in the secondary field. Many such curricula have been suggested during the past ten years or so and are to be found in the literature. One of the most recent is by the writer.*

The last topic to be discussed is a further consideration of the six semester-hours of mathematics in the general education program which is required for certification. Many of us have felt that this requirement was our golden opportunity. To be more specific, the students taking such a six-hour course were to be prospective teachers of the elementary schools and of all high school subjects, except mathematics and those sciences which require several courses of mathematics in their preparation. Thinking further, this group would contain prospective principals and superintendents. This was an opportunity, for the mathematician was going to have a personal contact with that group of teachers (teachers to be) who had been causing much of the trouble. These were the ones who disliked mathematics, could not see any use in it, thought it too difficult and so told the students. Even the prospective student of science and engineering was affected. What better chance could the mathematician have for correcting some of these evils?

When the six-semester hour rule became effective, it was thought by the professional educator that the traditional courses of algebra and trigonometry could be improved upon for this group of students. He thought further that the program should contain some arithmetic for the sake of the prospective elementary teacher. The mathematician was ready for this occasion because he had been conditioned by the current ideas which are spreading abroad as exemplified by such articles as those of Allendoerfer, Newsom and Ore.**

The course that is presently given aims at three ideas. In the first place an effort is made to instruct the student in the more useful phases of elementary mathematics. This begins with a good review of arithmetic, and continues with topics of algebra, geometry, trigonometry and analytics. Secondly, the student is made to realize the part that mathematics has played in the progress of civilization. This is done through the history of mathematics. In the third place, the student is instructed in the importance of mathematics today, and is taught the varying amounts of mathematics which are required for the different professions and occupations. In addition to this program, every effort is made to get the student to have a liking for the subject. In the experience of the writer, several students have finished the course only to change their program and major

* Karnes, H. T., Preparation of Teachers of Secondary Mathematics, *The Mathematics Teacher*, Vol. XXXVIII, p. 3.

** Allendoerfer, C. B., Mathematics for Liberal Arts Students, this MONTHLY, Vol. 54, 1947, p. 573.

Newsom, C. V., A Course in College Mathematics for a Program of General Education, *The Mathematics Teacher*, Vol. XLII, January, 1949.

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in mathematics.

There are some who would criticize strongly such a course as this one. However, there is a rather common belief that the students know as much mathematics, if not more, than those who take three semester hours each of algebra and trigonometry. In addition, they know far more about mathematics and have broadened their cultural background a great deal.

In closing, let us think for a moment what this indoctrination will mean. Miss Smith, teacher of literature, will tell Johnny to learn all the mathematics he can, for one cannot be a creative engineer without it. Miss Jones, teacher of public school music, will tell Tom, who wishes to become an economist, that certain courses of mathematics are most essential if he is to succeed in his chosen field. And so it will continue, ever broadening in scope. Should we pull a pressure act for three units of mathematics? It will not be necessary. Of course, all of this will not take place in two or three years, but ten, fifteen or twenty years will see a great difference in the favor of mathematics.

One last word. This course is beginning to be used by students of the liberal arts for their required six hours. One can find among this group prospective school board members, and most certainly prospective parents, who have not helped the situation in the past by saying in the presence of their children, "I just never could get mathematics, and it certainly does no one any good."

GENERALIZED TERNARY CONTINUED FRACTIONS

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1. Statement of the problem. In [1], J. M. Barbour explained that a Diophantine problem arises in connection with musical scales, namely, to find small integers A, B, and C that stand nearly in the ratio log (5/4), log (3/2), and log 2. Such integers A, B, C are called tuning ratios. Barbour pointed out that this is the same as requiring

$$\frac{\log (5/4)}{\log 2} - \frac{A}{C} \tag{1}$$

$$\frac{\log (3/2)}{\log 2} - \frac{B}{C} \tag{2}$$

to be simultaneously small compared to $1/C$. Accordingly, he attacked the problem by the use of ternary continued fractions, but found it necessary to generalize the ternary continued fraction algorithm quite a bit in order to get a large number of tuning ratios.

Actually, problems of this sort can be solved by use of the generalized Euclidean algorithm. Full instructions for solving just exactly this sort of problem are

set forth in [2], and if these instructions are followed in the present case, practically every tuning ratio given in Barbour's composite table will be found, together with many that are not in Barbour's table.

The purpose of the present note is to give another algorithm for finding tuning ratios. This algorithm has many points in common with the ternary continued fraction algorithm, so much so that it can be considered as a sort of generalization of the ternary continued fraction algorithm. Moreover our algorithm is in a form where extension to the problem of the simultaneous approximation of N irrationals is immediate.

One feature of this algorithm is that the rate of convergence seems to be under the control of the computer. The question of the excellence of the approximations which result from the algorithm is not studied, but the numerical evidence in this particular case suggests that the approximations will generally be very good.

2. The algorithm. Our algorithm is based on the algorithm disclosed in [3]. In [3], let us relax our restrictions on a_n and allow the a_n's to be any set of real numbers of which at least two are incommensurable. Then Theorem 2 holds even after removal of the condition

$$|W_1| + |W_2| + |W_3| + |W_4| > 1,$$

since this condition was used only to insure that at least two of the W's were not zero. This conclusion on the W's now follows from the incommensurability condition, since, if all but one W were zero, we could solve

$$a_1\beta_{i1} + a_2\beta_{i2} + a_3\beta_{i3} + a_4\beta_{i4} = W_i \tag{3}$$

for the a's, and find that the a's are all integer multiples of the non-zero W, thus contradicting our condition that at least two a's must be incommensurable.

When generalized in this fashion, Theorem 2 furnishes the basic step in an algorithm whereby we can find sets of integers β_{ij} with $|\beta_{ij}| = \pm 1$ and successively smaller values of $|W_1| + |W_2| + |W_3| + |W_4|$, where W_i is defined by (3). (Note that $|\beta_{ij}|$ denotes a determinant, whereas $|W_i|$ denotes an absolute value.) This basic step involves two non-zero W's, and replaces the numerically larger by a new smaller W. In case there are more than two non-zero W's, we have our choice of which pair to use in the step. This choice allows us to control the rate of convergence. If we wish rapid convergence and are not concerned with the rate of increase of the sizes of the β_{ij}, we can always choose that pair of W's which produces the least new W. If we wish a slow rate of increase of the sizes of the β_{ij}, we can always choose that pair of W's which provides the new W of "least" β_{ij}. In general, given some desirable attribute of the W's and β_{ij}, we can always choose that pair of W's for which the new W maximizes the attribute in question.

In order to handle Barbour's problem, we seek β_{ij} with $|\beta_{ij}| = \pm 1$ and $|W_i|$ small, where

$$\beta_{i1} \log \tfrac{5}{4} + \beta_{i2} \log \tfrac{3}{2} + \beta_{i3} \log 2 = W_i. \tag{4}$$

Further, we will wish "slow" convergence, in that we are willing for the W_i to go to zero slowly if thereby we keep L_i from increasing rapidly, where

$$L_i = (\beta_{i1})^2 + (\beta_{i2})^2 + (\beta_{i3})^2. \tag{5}$$

We will explain later how to use the β_{ij} to find tuning ratios, but for now let us look at the algorithm for deriving the β_{ij}. In Table I (at the end of the paper), we show the initial steps. We choose the first three W's quite simply. Then our algorithm furnishes the subsequent W's. A record of the definitions of these subsequent W's is appended to the bottom of Table I. The values in the column headed 100 W_i are only approximate, and are computed as follows. For the first three entries, we have simply recorded the integral portion of 100 W_i. Subsequent entries are computed from those above by the definitions at the bottom of the table. Needless to say, accumulation of rounding off errors renders our values of 100 W_i fairly inaccurate as i increases. However, the value of 100 W_i is merely a guide to the operation of the algorithm, and little accuracy is needed.

The important thing is to see the motivation for our definition of the W_i's after the first three. For instance, let us consider the definition of W_5. We have replaced W_3 by W_4, which is $W_3 - W_2$. So we have W_1, W_2, and W_4 to work with. Using pairs of these, we could replace W_2 by any of $W_2 - W_1$, $2W_1 - W_2$, $3W_1 - W_2$, $W_2 - W_4$, or $2W_4 - W_2$, and we could replace W_4 by $W_4 - W_1$ or $2W_1 - W_4$. Of these seven combinations, $W_2 - W_1$ makes L_i the least. So we take W_5 to be $W_2 - W_1$. We then have W_1, W_4, and W_5 to work with. Again, we have the choices of replacing W_4 by $W_4 - W_5$ or $2W_5 - W_4$ or $3W_5 - W_4$ or $W_4 - W_1$ or $2W_1 - W_4$ or replacing W_1 by $W_1 - W_5$ or $2W_5 - W_1$. Of these, $W_4 - W_1$ makes L_i the least, and we choose $W_6 = W_4 - W_1$.

In short, our guiding principle is to choose at each step that W_i derived from two W's which makes L_i a minimum. If two such W_i give the same minimum L_i, we choose that one which gives the smaller value of W_i. This procedure was followed carefully in computing Tables I, II, and III. One could perhaps slow the convergence a bit more by allowing the new W_i to be a linear combination of all three W's. However, it is not clear that this would be worth the considerable amount of extra computation involved in selecting at each step the W_i of minimum L_i.

Because of the accumulating inaccuracies in 100 W_i toward the end of Table I, we start afresh in Table II with W_7, W_9, and W_{10}, these being the W's which are current at the end of Table I. The first three values of 1000 W_i are rounded off from an accurate numerical calculation, but subsequent values are derived from these, so that rounding off errors accumulate again. So after a bit we start in again in Table 3 with three values of $10^4 W_i$ computed afresh.

Let us now see how our β_{ij}'s furnish tuning ratios. For instance consider the first three lines of Table III. They tell us that

$$(6) \qquad -\log \tfrac{5}{4} + 4 \log \tfrac{3}{2} - 2 \log 2 = 0.0124,$$

$$(7) \qquad -2 \log \tfrac{5}{4} - 4 \log \tfrac{3}{2} + 3 \log 2 = 0.0113,$$

$$(8) \qquad 6 \log \tfrac{5}{4} - 5 \log \tfrac{3}{2} + \log 2 = 0.0047.$$

The matrix of the coefficients is

$$B = \begin{Vmatrix} -1 & 4 & -2 \\ -2 & -4 & 3 \\ 6 & -5 & 1 \end{Vmatrix},$$

whose determinant is unity, and whose inverse is

$$B^{-1} = \begin{Vmatrix} 11 & 6 & 4 \\ 20 & 11 & 7 \\ 34 & 19 & 12 \end{Vmatrix}.$$

The constants on the right of (6) and (7) are nearly zero. If they were exactly zero, then log (5/4), log (3/2), and log 2 would have to be in exactly the ratio of the 2-rowed minors of the coefficients of (6) and (7), namely 4, 7, 12. As the constants are nearly zero, log (5/4), log (3/2), and log 2 must stand nearly in the ratio 4, 7, and 12. Hence, 4, 7, and 12 are a set of tuning ratios; in fact they are the familiar set used in the well tempered 12-tone scale.

Since the determinant of B is unity, the elements of B^{-1} are the 2-rowed cofactors of B, and so the columns of B^{-1} are just sets of tuning ratios. Indeed the columns of our B^{-1} are all given in Barbour's composite table of sets of tuning ratios.

The columns of our B^{-1} correspond to the convergents of a ternary continued fraction, so that we have here a generalization of the ternary continued fraction algorithm. The closest equivalent to the coefficients of the ternary continued fraction are our definitions of the successive W's which we have appended at the bottoms of Tables I, II, and III. If the columns of the various B^{-1} are to serve as convergents, this suggests that perhaps we can compute the columns of the various B^{-1} by some sort of recursion procedure. This can be done, but it is a bit complicated, since in general we cannot express a given convergent in terms of the two immediately preceding convergents. The right hand columns of Tables I, II, and III embody our recursion scheme for computing successive convergents. We have listed the convergents in Table IV, at the bottom of which are the recursions used in computing the table, where we let U_i denote the vector of components A_i, B_i, C_i.

To see how we derived our recursions, let us look at the first three lines of Table III. Let W_i denote the vector of components $\beta_{i1}, \beta_{i2}, \beta_{i3}$ as well as the value given by (4). The 15, 13, 12 at the right hand ends of the first three lines of Table III signify that if we take W_{12}, W_{14}, and W_{15} as the three rows of a matrix B, then U_{15}, U_{13}, and U_{12} will be the three columns of B^{-1}. By looking at the exhibited B and B^{-1}, we see that this is indeed the case.

Now we drop W_{14} and add $W_{16}=W_{14}-W_{15}$. What is the corresponding operation on the U's, and what numbers must we now write at the ends of the W_{12}, W_{15}, and W_{16} lines? The corresponding operation on the U's is as follows. We take W_{14} and W_{15}, which are the two W's which we combined to form W_{16}, and look at the ends of their lines. The numbers appearing there are 13 and 12. So we add the two U's with these subscripts, taking $U_{16}=U_{12}+U_{13}$. Next we dropped W_{14} and added W_{16}. So the 13 which was at the end of the W_{14} line is moved down and put at the end of the W_{16} line. Now W_{15} is the other W which went into making up W_{16}, and we replace the 12 as the end of the W_{15} line by a 16, the number of our new U. (The 18 at the end of the W_{15} line gets added later in a similar fashion.) We now have 15, 16, and 13 at the ends of the W_{12}, W_{15}, and W_{16} lines, and the reader may easily verify that if he forms a matrix B with W_{12}, W_{15}, and W_{16} as rows, then B^{-1} will have U_{15}, U_{16}, and U_{13} as columns. Moreover, a careful study of the behavior of the matrix equation $BB^{-1}=1$ will indicate that if we manipulate the rows of B in the manner in which we manipulated the W's to get W_{16}, then we must manipulate the columns of B^{-1} in exactly the manner in which we manipulated the U's, as described above. By the process used, it may be seen that each matrix B is obtained from its predecessor C by subtracting row i from row j and transposing the newly created row to be the bottom row. In the case where $j=3$, no transposition is needed, and we see that $B=PC$, where P is the matrix obtained from the identity matrix I by subtracting row i from row j. Then

$$B^{-1} = (PC)^{-1} = C^{-1}P^{-1}.$$

But P^{-1} is obtained from I by adding row i to row j, and so B^{-1} is obtained from C^{-1} by adding column i to column j. For the cases $j=1$ and $j=2$ an analogous argument can be given.

Since we always form our new W's as differences of two previous W's, we will always form the new U's as sums of two previous U's, and by the procedure indicated above, we can tell at each step which U's go with the current set of three W's. We will go through the next step to illustrate the method once more. The next step is to drop W_{12} and add $W_{17}=W_{12}-W_{16}$. So we define $U_{17}=U_{13}+U_{15}$, since 15 and 13 appear at the ends of the W_{12} and W_{16} lines. The 15 on the end of the W_{12} line is moved down to the end of the W_{17} line when we drop the W_{12} and add the W_{17}. Finally, since W_{16} is the other W involved in this transaction, we put 17, the number of our new U, at the end of the W_{16} line, superseding the 13 which was at the end of that line.

In actual practice, one draws a line through a given W when it is dropped, so that at any time we know which three W's we are dealing with, since they are the only W's not marked out. Moreover, numbers like the 18 at the end of the W_{15} line, which will be added later, have not yet appeared, so that the final numbers then appearing at the ends of our three W lines are the numbers of the corresponding U's. Under these circumstances, our procedure for manipulating the U's is wholly mechanical and can be carried out very quickly. For in-

stance, only three hours were required for the entire computation recorded in Tables I to IV.

A comparison will show that our Table IV compares favorably with Barbour's Table IV, which is a compilation of tuning ratios from many sources.

Quite clearly our algorithm can be used to generate sets of integers standing nearly in the ratios $a_1, a_2, \cdots, a_n$, where the a's are any set of real numbers. If we find $N_0, N_1, \cdots, N_n$ standing nearly in the ratios $1, a_1, a_2, \cdots, a_n$, then we have nearly

$$a_i = \frac{N_i}{N_0},$$

so that our algorithm can be used to approximate several real numbers simultaneously by rationals with the same denominator.

TABLE I

i	β_{i1}	β_{i2}	β_{i3}	100 W_i	
1	1	0	0	22	1, 5, 6
2	0	1	0	40	2, 4
3	0	0	1	69	3
4	0	−1	1	29	3
5	−1	1	0	18	4, 7
6	−1	−1	1	7	3, 8
7	2	−1	0	4	6, 9, 10
8	0	2	−1	11	7
9	−3	0	1	3	8
10	−2	3	−1	7	7

$W_4 = W_3 - W_2$, $W_5 = W_2 - W_1$, $W_6 = W_4 - W_1$, $W_7 = W_1 - W_5$, $W_8 = W_5 - W_6$, $W_9 = W_6 - W_7$, $W_{10} = W_8 - W_7$.

TABLE II

i	β_{i1}	β_{i2}	β_{i3}	1000 W_i	
7	2	−1	0	41	10, 12
9	−3	0	1	24	8, 11, 13
10	−2	3	−1	77	7
11	1	3	−2	53	7
12	−1	4	−2	12	7, 14, 15
13	5	−1	−1	17	12
14	−2	−4	3	12	13
15	6	−5	1	5	12

$W_{11} = W_{10} - W_9$, $W_{12} = W_{11} - W_7$, $W_{13} = W_7 - W_9$, $W_{14} = W_9 - W_{12}$, $W_{15} = W_{13} - W_{12}$.

TABLE III

i	β_{i1}	β_{i2}	β_{i3}	$10^4 W_i$	15
12	− 1	4	− 2	124	
14	− 2	− 4	3	113	13
15	6	− 5	1	47	12, 16, 18
16	− 8	1	2	66	13, 17
17	7	3	− 4	58	15
18	1	8	− 5	11	15, 19, 20, 22
19	− 9	− 7	7	55	17
20	5	−13	6	36	18, 21
21	−14	6	1	19	17
22	−15	− 2	6	8	17, 23, 24, 26, 28
23	16	10	−11	3	22, 25, 27
24	20	−11	0	28	21
25	4	−21	11	25	21
26	19	−19	5	17	21
27	3	−29	16	14	21
28	18	−27	10	6	21

$W_{16}=W_{14}-W_{15}$, $W_{17}=W_{12}-W_{16}$, $W_{18}=W_{17}-W_{15}$, $W_{19}=W_{16}-W_{18}$, $W_{20}=W_{15}-W_{18}$, $W_{21}=W_{19}-W_{20}$, $W_{22}=W_{21}-W_{18}$, $W_{23}=W_{18}-W_{22}$, $W_{24}=W_{20}-W_{22}$, $W_{25}=W_{24}-W_{23}$, $W_{26}=W_{25}-W_{22}$, $W_{27}=W_{26}-W_{23}$, $W_{28}=W_{27}-W_{22}$.

TABLE IV

i	A_i	B_i	C_i	i	A_i	B_i	C_i
1	1	0	0	15	11	20	34
2	0	1	0	16	10	18	31
3	0	0	1	17	17	31	53
4	0	1	1	18	21	38	65
5	1	1	1	19	28	51	87
6	1	1	2	20	49	89	152
7	1	2	3	21	38	69	118
8	1	2	4	22	66	120	205
9	2	3	6	23	83	151	258
10	3	5	9	24	121	220	376
11	2	4	7	25	104	189	323
12	4	7	12	26	159	289	494
13	6	11	19	27	142	258	441
14	7	13	22	28	197	358	612

$U_4=U_2+U_3$, $U_5=U_4+U_1$, $U_6=U_3+U_5$, $U_7=U_4+U_6$, $U_8=U_3+U_7$, $U_9=U_6+U_8$, $U_{10}=U_7+U_9$, $U_{11}=U_7+U_8$, $U_{12}=U_7+U_{10}$, $U_{13}=U_{11}+U_{12}$, $U_{14}=U_7+U_{13}$, $U_{15}=U_{12}+U_{14}$, $U_{16}=U_{12}+U_{13}$, $U_{17}=U_{13}+U_{15}$, $U_{18}=U_{15}+U_{16}$, $U_{19}=U_{15}+U_{17}$, $U_{20}=U_{18}+U_{19}$, $U_{21}=U_{17}+U_{18}$, $U_{22}=U_{17}+U_{20}$, $U_{23}=U_{17}+U_{22}$, $U_{24}=U_{21}+U_{23}$, $U_{25}=U_{21}+U_{22}$, $U_{26}=U_{21}+U_{24}$, $U_{27}=U_{21}+U_{25}$, $U_{28}=U_{21}+U_{26}$.

References

1. J. M. Barbour, Music and ternary continued fractions, this MONTHLY, vol. 55, 1948, pp. 545–555.

2. Barkley Rosser, A generalization of the Euclidean algorithm to several dimensions, Duke Math. Jour., vol. 9, 1942, pp. 59–95.

3. Barkley Rosser, A note on the linear diophantine equation, this MONTHLY, vol. 48, 1941, pp. 662–666.

Editorial Note on the Preceding Paper. After the preceding paper had been accepted, Viggo Brun of the University of Oslo called attention to the fact that he published a paper in 1919 that gives the same numbers as those found by Barbour; the specific reference to Brun's paper is as follows: En generalisation av kjedebrøken (avec un résumé en français) Videnskapsselskapets skrifter, Oslo 1919 og 1920. The method of Brun appears to be a special case of that presented by Rosser.

A DOUBLE SERIES SUMMED GEOMETRICALLY

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L. R. Ford* has used a certain diagram of circles to prove geometrically some results in the theory of numbers. We shall derive another diagram from his and use it to prove the following:

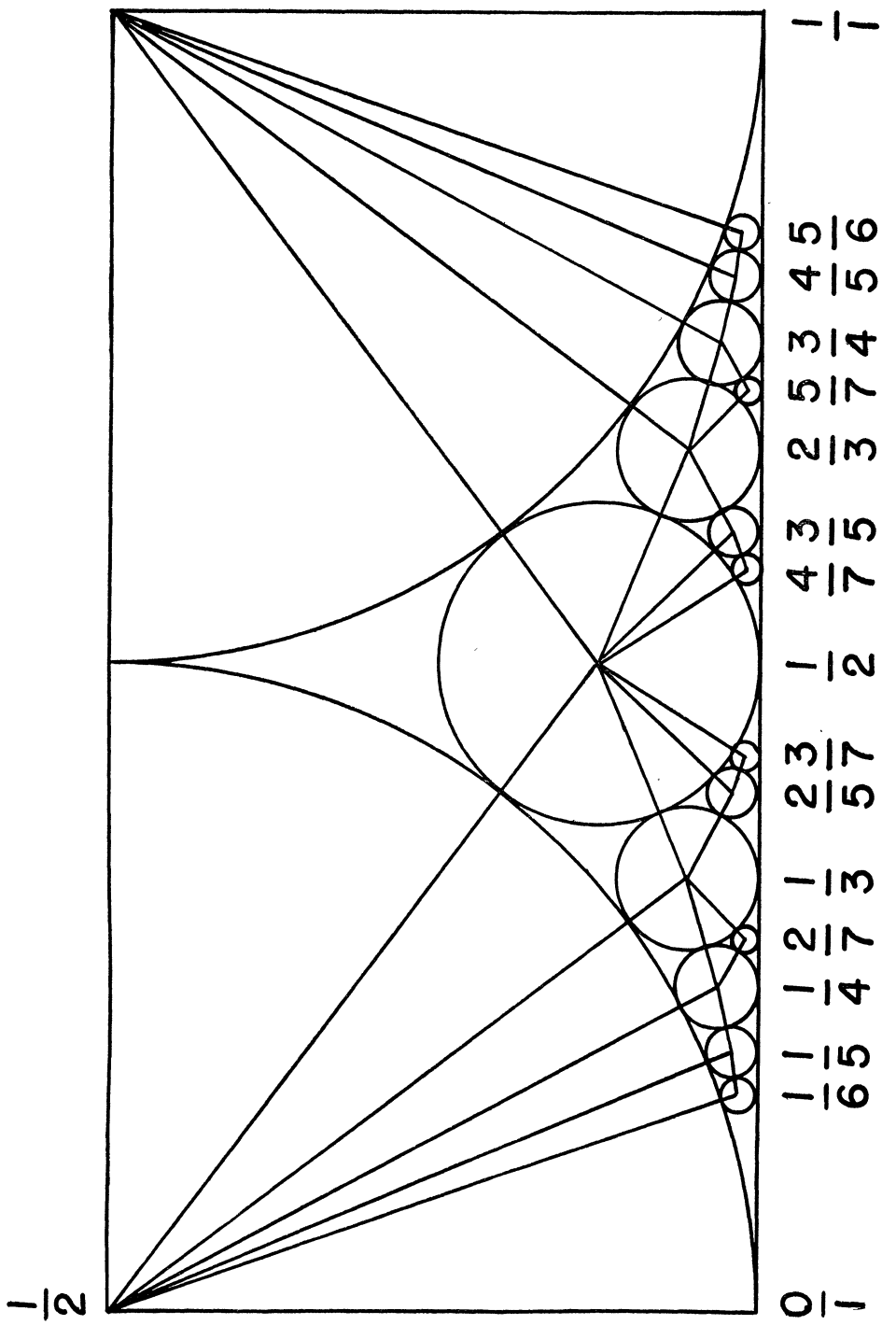
THEOREM. *If $f(p)$ is a monotone function of the positive integers and* $\lim_{p\to\infty} f(p) = \lambda$, *then*

$$(1) \qquad \sum_{\substack{p,q=1 \\ (p,q)=1}}^{\infty} \left[\frac{f(q)}{p(p+q)} + \frac{f(p)}{q(p+q)} - \frac{f(p+q)}{pq} \right] = 2(f(1) - \lambda).$$

Ford's diagram consists of circles with centers at $(u/p,\ 1/2p^2)$ and radius $1/2p^2$, where u/p ranges over all rational numbers expressed in lowest terms, *i.e.*, u and p are integral, $p>0$ and $(u, p)=1$. These circles are non-overlapping and are tangent to the x-axis. The circles corresponding to u/p and v/q are tangent if and only if $uq-vp=\pm 1$ and the circle tangent to both of these and having its center below the line of centers corresponds to $(u+v)/(p+q)$.

Restrict u/p to lie on the closed interval $0 \leqq u/p \leqq 1$. Join the centers of tangent circles with line segments and as a result the rectangular area between the lines $y=0$ and $y=1/2$ and between the lines $x=0$ and $x=1$ is completely filled with non-overlapping triangles.

* L. R. Ford, Fractions, this MONTHLY, vol. 45, 1938, pp. 586–601. See also G. T. Williams and D. H. Browne, A family of integers and a theorem on circles, this MONTHLY, vol. 54, 1947, pp. 534–536. For a bibliography on the diagram of circles see J. F. Koksma, Diophantische Approximationen, 1936, pp. 42–43.


1/1
5/6
4/5
3/4
5/7
2/3
4/7
1/2
3/5
2/7
1/3
2/5
3/7
1/4
4/5
5/6
1/5
6/7
1/6
0/1
-1/2

If instead of the centers $(a/b, 1/2b^2)$ we use as vertices $(a/b, f(b))$ the new triangles are likewise non-overlapping and entirely fill up the area between $f(1)$ and $\lambda = \lim_{b\to\infty} f(b)$. All that has occurred is a distortion of the scale on the y-axis including possibly a reflection.

We now compute the area of a single triangle. If u/p, v/q, and $(u+v)/(p+q)$ are the abscissas of the vertices of a triangle, the area of the triangle is, except for sign,

$$\frac{1}{2}\begin{vmatrix} u/p & f(p) & 1 \\ v/q & f(q) & 1 \\ (u+v)/(p+q) & f(p+q) & 1 \end{vmatrix} = \pm\frac{1}{2}A(p, q; f),$$

where $A(p, q; f)$ is the expression within brackets in (1). If $f(p)$ is decreasing then $A \geqq 0$, for, if $f(p)$ and $f(q)$ are decreased to $f(p+q)$ in A, the expression becomes zero. Similarly if $f(p)$ is increasing, $A \leqq 0$. Because the terms A in the series (1) all have the same sign, all of the areas are being added positively or all negatively. Also the value of the series is independent of the order of summation.

We prove finally that the summation (1) accounts for precisely all the triangles.

By interchanging u/p and v/q the sign of $uq-vp$ changes. Thus we need to show only that for every pair p, q with $(p, q) = 1$ there is one and only one solution of

$$uq - vp = 1 \tag{2}$$

with

$$0 \leqq u \leqq p, \qquad 0 \leqq v \leqq q, \qquad 0 < p, \qquad 0 < q. \tag{3}$$

The existence of a solution is well-known. To show the uniqueness let, u', v' be another solution. Then $u'q-v'p = 1 = uq-vp$, so that $(u-u')q = (v-v')p$. If $u = u'$ then $v = v'$ and the solutions are the same. But if $u \neq u'$, then since $(p, q) = 1$, p divides $u-u'$ and by virtue of (3) this is possible only when $u = 0$ and $u' = p$ (or $u = p$ and $u' = 0$). But if $u = 0$, by (2), $-vp = 1$; this is impossible since $vp \geqq 0$. The proof of the theorem is now complete.

If in (1) we did not require $(p, q) = 1$, then the result is

$$\sum_{p,q=1}^{\infty} A(p, q; f) = 2\sum_{r=1}^{\infty} [f(r) - \lambda]/r^2. \tag{4}$$

This follows from the fact that summing over all positive integral pairs p, q is equivalent to summing over all pr, qr where $(p, q) = 1$ and $r = 1, 2, \cdots$. Now for fixed r,

$$\sum_{(p,q)=1}\left[\frac{f(qr)}{pr(pr+qr)}+\frac{f(pr)}{qr(pr+qr)}-\frac{f(pr+qr)}{prqr}\right]$$

$$=\frac{1}{r^2}\sum_{(p,q)=1}A(p,q;g)=2[g(1)-\mu]/r^2,$$

where $g(p)=f(pr)$ and $\mu=\lim_{p\to\infty}g(p)$. Hence $g(1)=f(r)$ and $\lambda=\mu$, and (4) follows immediately. Both the theorem and (4) may be proved by manipulation of the series directly.

Every function $h(p)$ defined for positive integral p is a sum of two monotone functions h_i and h_d where

$$h_i(p)=\tfrac{1}{2}h(1)+\sum_{s=2}^{p}\max(0,h(s)-h(s-1)),$$

$$h_d(p)=\tfrac{1}{2}h(1)+\sum_{s=2}^{p}\min(0,h(s)-h(s-1)).$$

If either h_i or h_d is bounded then in the theorem and in equation (4), we may replace f by h, thus obtaining a more general result.

In the proof of (4) it was possible to ignore the order of summation because the terms all have the same sign. One may not, however, separately sum each of the terms of $A(p,q;f)$; *e.g.*, if $f(p)=1$ the series

$$\sum_{(p,q)=1}f(p)/q(p+q) \quad \text{is divergent whereas,} \quad \sum A(p,q;f)=0.$$

A few special cases of the theorem may be of interest:

$$\sum_{(p,q)=1}\frac{1}{pq(p+q)}=2, \qquad \text{by taking } f(p)=1/p,$$

$$\sum_{(p,q)=1}\frac{p^2+pq+q^2}{p^2q^2(p+q)^2}=2, \qquad \text{by taking } f(p)=1/p^2,$$

and using (4), we get corresponding results

$$\sum_{p,q}\frac{1}{pq(p+q)}=2\zeta(3), \qquad \sum_{p,q}\frac{p^2+pq+q^2}{p^2q^2(p+q)^2}=2\zeta(4)=\frac{\pi^4}{45}$$

where $\zeta(n)=1/1^n+1/2^n+\cdots$.

By taking $f(p)=1/p^m$ and by means of other considerations, it is possible to evaluate the double harmonic series

$$\sum_{p,q=1}^{\infty}p^{-r}q^{-s}(p+q)^{-t}$$

in terms of the harmonic series $\zeta(n)$ in many cases; *e.g.*, if $r+s+t$ is odd. However $\zeta(n)$ is known only when n is even.

MATHEMATICAL NOTES

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DISCONTINUOUS FUNCTIONS WITH THE DARBOUX PROPERTY

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A real-valued $f(x)$ has the Darboux property if: whenever $f(x_1)<u<f(x_2)$ then $f(x)=u$ for at least one x between x_1 and x_2. Such $f(x)$ need not be continuous[1] nor even Riemann-integrable[2] and may, in fact, be discontinuous for all x.[3] We shall construct such an $f(x)$ which actually takes on every real number as value a continuum number of times on every interval; it is non-Lebesgue measurable on every measurable set of positive measure and hence does not coincide on any such set with any continuous function.

Let $a_1, a_2, \cdots, a_\alpha, \cdots (\alpha<\Omega)$ be a Hamel basis, so that every real number has a unique expression $x=\sum_\alpha r_\alpha a_\alpha$, with rational r_α, of which only a finite number differ from zero.[4] Rearrange this basis into a two-fold transfinite sequence $b_1, b_2, \cdots, b_\alpha, \cdots (\alpha<\Omega)$; $c_1, c_2, \cdots, c_\beta, \cdots (\beta<\Omega)$, and for $x=\sum_\alpha r_\alpha b_\alpha + \sum_\beta s_\beta c_\beta$ set $f(x)=\sum_\alpha r_\alpha a_\alpha$.

For any interval (a, b) and any $u=\sum_\alpha r_\alpha a_\alpha$ we will have $f(x)=u$ if $x=\sum_\alpha r_\alpha b_\alpha+\sum_\beta s_\beta c_\beta$ for all choices of the s_β; in particular, if only one $s_\beta\neq 0$, say s_{β_0}, and it is chosen so that $a-\sum_\alpha r_\alpha b_\alpha<s_{\beta_0}c_{\beta_0}<b-\sum_\alpha r_\alpha b_\alpha$. Hence $f(x)=u$ for continuum many x in (a, b). Since this $f(x)$ is a discontinuous solution of the functional equation $g(x+y)=g(x)+g(y)$, it is not Lebesgue measurable on any set of positive measure.[5]

Using another construction we can define an $f(x)$ with the stronger property that it takes on every real number as value a continuum number of times on every perfect set, including those of measure zero. Let Ω_1 be the smallest ordinal number with continuum many predecessors. The real numbers can be arranged as a transfinite sequence $u_1, u_2, \cdots, u_\alpha, \cdots (\alpha<\Omega_1)$. There are continuum many perfect sets and they can be arranged as a transfinite sequence $P_1, P_2, \cdots, P_\alpha, \cdots (\alpha<\Omega_1)$, so that every perfect set appears continuum many times. Now for each α define p_α, by induction on α, to be the first u not previously selected which is contained in P_α; such a u will exist since, for each α, less than continuum many u's will have been selected previously, and the perfect set P_α contains continuum many numbers. This process will select continuum many p's from each perfect set P since each P appears continuum many

[1] G. Darboux, Annales Scientifiques de l'Ecole Normale Supérieure, 2nd series, vol. 4 (1875), pp. 57–112.

[2] V. Volterra, Giornale de Battaglini, 1881.

[3] H. Lebesgue, Leçons sur l'intégration, Gauthier-Villars, Paris, 1904, page 90. Lebesgue's function is zero except on a set of measure zero.

[4] G. Hamel, Mathematische Annalen, vol. 60 (1905), pp. 459–462.

[5] A. Ostrowski, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 38, 1929.

times as P_α and of course the p's selected from different perfect sets will be mutually exclusive.[6] Now for each perfect set let its selected p's be arranged as a double transfinite sequence $x_{\alpha\beta}(\alpha, \beta<\Omega_1)$, and set $f(x_{\alpha\beta})=u_\alpha$. This $f(x)$ clearly takes on every real number as value a continuum number of times on every perfect set. If $f(x)$ were Lebesgue measurable on any set of positive measure there would be a finite K with $f(x)<K$ on a set of positive measure and hence on a perfect subset (of positive measure). This contradiction shows that $f(x)$ cannot be measurable on any set of positive measure.

DECOMPOSITION OF AN INFINITE-DIMENSIONAL LINEAR SYSTEM INTO UBIQUITOUS CONVEX SETS

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1. **Introduction.** Suppose that L is a (real) linear system (that is, a module over the real number field). A maximal linearly independent subset of L is called a *Hamel basis* for L. It is well known [2] that L must have a Hamel basis and that all its Hamel bases have the same cardinal number $d(L)$, which is called the *dimension* of L.

If x and y are distinct points of L, $[x, y]$ will denote the line segment including its endpoints, $(x, y]\equiv[x, y]-\{x\}$, and so on. For a convex subset X of L, lin X will denote the union of X and the set of all points $y\in L$ such that $(y, x]\subset X$ for some x; X will be called *ubiquitous* if lin $X=L$ (that is, if each point of L is linearly accessible from X). If L is n-dimensional, it is isomorphic with Euclidean n-space and (with X convex) lin X is merely the closure of X, so L has no ubiquitous convex proper subsets. For infinite-dimensional systems, on the other hand, the situation is quite different.

2. **Theorem.** We shall establish the following result.

THEOREM. *Suppose that L is an infinite-dimensional linear system and N is a cardinal number such that $2\leqq N\leqq d(L)$. Then L can be expressed as the union of N pairwise disjoint ubiquitous convex sets.*

Proof. Let H be a Hamel basis for L. Since $d(L)\geqq\aleph_0$, H can be expressed as the union of $N-1$ pairwise disjoint infinite sets; say $H=\bigcup_{1\leqq i<M} H_i$, where M is an ordinal number having $N-1$ predecessors 1, 2, $\cdots$. Each set H_i can be assigned an anti-reflexive linear ordering $<_i$ with respect to which it has no last element.

Now each point $x\in L$ can be expressed in the form $\sum_{h\in H} a(h, x)h$, where all but finitely many of the coefficients $a(h, x)$ are zero. For each i and x, define $h_i(x)$ as follows: if $a(h, x)$ is non-zero for some $h\in H_i$, then $h_i(x)$ is the last such h (in terms of the ordering $<_i$); otherwise (that is, if $h\in H_i$ implies $a(h, x)=0$), $h_i(x)$ is an arbitrary but fixed point of H_i. For each i and x, let $b_i(x)=a(h_i(x), x)$.

[6] This construction was used by W. Sierpinski and N. Lusin, Comptes Rendus (Paris), vol. 165 (1917), pp. 422–424, to subdivide an interval into continuum many parts each of exterior Lebesgue measure equal to the length of the interval.

Let $C_0=\{x\mid b_i(x)\leqq 0$ for all i's$\}$ and for $1\leqq n<M$ define $C_n=\{x\mid b_i(x)\leqq 0$ whenever $i<n$, but $b_n(x)>0\}$. It can be verified that the sets $C_i(0\leqq i<M)$ are pairwise disjoint, cover L, and are all convex. (In fact, each C_n is a convex cone with vertex at the origin, although only C_0 contains the origin.) It remains to show that each set C_n is ubiquitous.

To this end, consider an arbitrary point $x\in L$, an arbitrary index n, and let $K'=\{i\mid b_i(x)\neq 0\}$. Let $K=K'\cup\{n\}$ if $n\neq 0$ and $K=K'$ if $n=0$. For each $i\in K$ pick $y_i\in H_i$ such that $h_i(x)<{}_iy_i$. Let $\epsilon_n=1$, $\epsilon_i=-1$ for $i\neq n$, and define $y=\sum_{i\in K}\epsilon_i y_i$. (Note that K is a finite set.) It can be verified that $x+ty\in C_n$ for each $t>0$, so $(x,\ c+y]\subset C_n$, $x\in\text{lin } C_n$, and the proof is complete.

Notice that if any topology be assigned to L in which every line is a continuous image (under the natural mapping) of the real line, then each ubiquitous convex set is dense. For certain linear spaces and with "ubiquitous" replaced by "dense," the above theorem was proved in [1]. However, the sets obtained in [1] were not ubiquitous and the method of [1] required the existence of a dense *linear* subset, which is not needed here. Hence the present theorem is applicable (to obtain dense convex subsets) to many spaces not covered in [1].

As is pointed out in [1], for a large class of spaces L, $d(L)$ is equal to the cardinal number of L.

The author first became acquainted with ubiquitous convex sets through the comments of a referee of one of his papers.

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ASYMPTOTIC BEHAVIOR OF MEAN VALUE FUNCTIONS

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1. Introduction. The mean value function $M(t)$ of order t of n positive real numbers $x_\nu(\nu=1,\ \cdots,\ n)$, weighted by n positive real numbers $a_\nu(\nu=1,\ \cdots,\ n)$ with $\sum a_\nu=1$, is defined for all real arguments t except $t=-\infty,\ 0,\ +\infty$ by the formula

$$M(t)=\left(\sum a_\nu x_\nu^t\right)^{t^{-1}}.$$

We mention briefly some properties of this well-known function; for a detailed discussion see [1, 2]. At its three singular points, $M(t)$ can be defined so as to become continuous. It is clearly analytic elsewhere. Furthermore, in justification of the term mean value function, $M(t)$ satisfies the inequality

$$\min x_\nu\leqq M(t)\leqq\max x_\nu.$$

Thus $M(t)=x$ for all t if $x_\nu=x$ for all ν. If, however, not all the x_ν are equal, then $M(t)$ steadily increases from the value $\min x_\nu$ at $t=-\infty$ to the value $\max x_\nu$ at $t=+\infty$, with $M'(t)>0$ for $-\infty<t<+\infty$ [2].

In this note we investigate the exact behavior of $M(t)$ at its three singular points.

2. Behavior at zero. The mean value function $M(t)$ may be expressed in the form

$$M(t) = S(t)^{t^{-1}},$$

where

$$S(t) = \sum a_\nu x_\nu^t.$$

Clearly $S(t)$ is analytic at $t=0$, and

$$S(0) = \sum a_\nu = 1;$$

so $\log S(t)$ is also analytic at $t=0$ and $\log S(0)=0$. Therefore

$$\log M(t) = t^{-1} \log S(t)$$

and, consequently, $M(t)$ is analytic at $t=0$ with removable singularities there, these singularities being removed by defining

$$\log M(0) = (\log S)'(0) = S'(0) = \sum a_\nu \log x_\nu.$$

3. Asymptotic equivalence. Let ϕ be a formal power series in an indeterminate σ with real coefficients:

$$\phi = \sum_{h=0}^{\infty} c_h \sigma^h.$$

We say that a function f defined in a neighborhood of $t_0 (-\infty \leqq t_0 \leqq +\infty)$ is weakly asymptotic to ϕ as $t \to t_0$, and write $f \sim \phi$ as $t \to t_0$, if

$$f(t) = \sum_{h=0}^{k} c_h s^h + o(s^k) \qquad (t \to t_0,\ s \to 0),$$

for $k=0, 1, 2, \cdots$, where

$$s = \begin{cases} t - t_0 & \text{if} \quad t_0 \neq \omega, \\ t^{-1} & \text{if} \quad t_0 = \omega, \end{cases}$$

and ω stands for $-\infty$ or $+\infty$. Furthermore, we say that f is strongly asymptotic to ϕ as $t \to t_0$, and write $f \approx \phi$ as $t \to t_0$, if, for $k=0, 1, 2, \cdots$, the kth derivative $f^{(k)}$ of f is weakly asymptotic as $t \to t_0$ to the kth formal derivative $\phi^{(k)}$ of ϕ.

The relations of asymptotic equivalence as defined above subsist between a function and a formal power series. These notions may be extended as follows. We call two functions f and g weakly or strongly asymptotically equivalent as $t \to t_0$, and write $f \sim g$ or $f \approx g$ as $t \to t_0$, provided that f and g are both weakly or are both strongly asymptotic in the original sense as $t \to t_0$ to the same formal power series.

It can be shown that both weak and strong asymptotic equivalence are preserved under addition, multiplication, function composition, and integration, and that strong asymptotic equivalence is preserved under differentiation.

In applications we shall extensively employ these preservation properties of strong asymptotic equivalence. But we cannot operate in a void: one strong asymptotic expansion formula is required, namely: $\theta^t \approx 0$ as $t \to \infty$ if $0 < \theta < 1$, this equivalence being a consequence of the limit relations $t^k\theta^t \to 0$ as $t \to \infty$ for $k = 0, 1, 2, \cdots$.

4. Behavior at infinity. We are here concerned with the strong asymptotic behavior of $M(t)$ as $t \to \omega$. Since $M(t) = x$ for all t if $x_\nu = x$ for all ν, we may assume that $n \geqq 2$ and that $x_1 < \cdots < x_n$. For notational convenience we define a_ω and x_ω as follows:

$$a_{-\infty} = a_1 \qquad a_{+\infty} = a_n$$
$$x_{-\infty} = x_1 \qquad x_{+\infty} = x_n,$$

and let ω be attached as a subscript to inequality signs with the meaning that the inequality is to be reversed if $\omega = -\infty$, but is to be preserved if $\omega = +\infty$. It is understood without explicit mention that $t \to \omega$ in limits and asymptotic expansions.

Consider the following functions of t:

$$M_\omega(t) = x_\omega a_\omega^{t^{-1}},$$
$$I_\omega(t) = S_\omega(t)^{t^{-1}},$$
$$S_\omega(t) = \sum \left(\frac{a_\nu}{a_\omega}\right)\left(\frac{x_\nu}{x_\omega}\right)^t,$$

and note that

$$M(t) = M_\omega(t) I_\omega(t).$$

As we have mentioned, $\theta^t \approx 0$ for $0 < \theta <_\omega 1$, so

$$S_\omega(t) = 1 + \text{terms of type } \theta^t \approx 1,$$
$$I_\omega(t) = \exp(t^{-1} \log S_\omega(t)) \approx 1.$$

Consequently the desired strong asymptotic expansion of the mean value function $M(t)$ as $t \to \omega$ is

$$M(t) \approx M_\omega(t),$$

the function $M_\omega(t)$ being analytic at $t = \omega$ with $M_\omega(t) \to x_\omega$. It follows that $M(t) \to x_\omega$. Furthermore, $M(t)$ in its approach to x_ω is squeezed between $M_\omega(t)$ and x_ω; for $S_\omega(t) > 1$, hence $I_\omega(t) >_\omega 1$ provided $t >_\omega 0$, so

$$M(t) >_\omega M_\omega(t) \qquad (t >_\omega 0).$$

The logarithm of the expansion for $M(t)$ is

$$\log M(t) \approx \log M_\omega(t) = \log x_\omega + t^{-1} \log a_\omega.$$

Differentiating this $k \geqq 1$ times with respect to t, we get

$$(\log M)^{(k)}(t) \approx t^{-k-1} k!(-1)^k \log a_\omega,$$

whence

$$M^{(k)}(t)/M(t) \approx t^{-k-1} k!(-1)^k \log a_\omega + \cdots .$$

Thus the kth derivatives of both $M(t)$ and $\log M(t)$ are >0 near $t = -\infty$, and near $t = +\infty$ are >0 if k is odd and <0 if k is even. In particular, taking $k=2$, we see that $M(t)$ and $\log M(t)$ *as functions of* t *are convex near* $t = -\infty$ *and concave near* $t = +\infty$. Three of the four parts of this result were recently obtained in quite a different fashion by Shniad [3]; his methods did not yield the concavity of $M(t)$ near $t = +\infty$.

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ON A PAIR OF FUNCTIONAL EQUATIONS

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1. Introduction. It is well known that all non-trivial, real, continuous solutions, defined for all $x>0$, of Cauchy's functional equation

$$F(xy) = F(x)F(y) \tag{1}$$

are of the form $F(x) = x^c$, where c is any real number.[1] In a recent issue of this MONTHLY there was considered the functional equation

$$f(xy) = [f(x)]^{(y^\beta)}[f(y)]^{(x^\beta)}, \tag{2}$$

which reduces to the equation (1) if $\beta = 0$.[2] Solutions of equation (2) were found to be of the form $x^{(cx^\beta)}$. In deriving these solutions the differentiability of the solution was assumed. This is an unnecessary assumption, for it follows from Cauchy's result for equation (1) that all real, continuous solutions of (2) which are defined for all $x>0$ are of the form $x^{(cx^\beta)}$, where c is any real constant. To derive this result one needs only to raise both sides of the equation (2) to the power $(xy)^{-\beta}$ and thus obtain

$$f(xy)^{1/(xy)^\beta} = f(x)^{1/x^\beta} f(y)^{1/y^\beta}. \tag{3}$$

[1] A non-trivial solution is one which does not vanish identically. A solution in this paper shall mean a non-trivial solution.

[2] Vol. 56 (1949), p. 414.

This equation is of the form of equation (1) if one takes

$$F(x) = f(x)^{1/x^\beta}.$$

Since all real continuous solutions of (1) are given by x^c, it follows that all real continuous solutions of (2), which are defined for all $x>0$, are of the form $f(x)=x^{(cx^\beta)}$.

It is the object of this note to solve a pair of simultaneous functional equations of which equations (1) and (2) are very special cases.

2. A pair of generalized Cauchy functional equations. We consider the set of functional equations

$$(A) \qquad f(xy) = g(x)^{p(y)}h(y)^{q(x)},$$

$$(B) \qquad r(xy) = p(x)q(y)$$

and look for all real, continuous functions defined for all $x>0$ and satisfying these equations. We must determine all sets of six real, continuous, not identically vanishing functions for which (A) and (B) are satisfied.

We consider first equation (B). In this equation let us set $y=1$. Then $r(x)=p(x)q(1)$. If $q(1)=0$, then $r(x)\equiv 0$. Hence $q(1)\neq 0$, and $p(x)=r(x)/q(1)$. Next let $x=1$; then $r(y)=p(1)q(y)$. Again if $p(1)=0$, then $r(y)\equiv 0$. Hence $q(y)=r(y)/p(1)$. Thus (B) can be rewritten in the form

$$r(xy) = \frac{r(x)r(y)}{p(1)q(1)}.$$

Dividing both sides of this equation by $p(1)q(1)$, we get

$$\frac{r(xy)}{p(1)q(1)} = \frac{r(x)}{p(1)q(1)}\cdot\frac{r(y)}{p(1)q(1)}.$$

In this equation we set $r(x)/[p(1)q(1)]$ equal to $s(x)$. Then $s(x)$ satisfies equation (1); hence $s(x)=x^\beta$ where β is any real number. Hence $p(x)=p(1)x^\beta$ $q(x)=q(1)x^\beta$, $r(x)=p(1)q(1)x^\beta$, and thus all real, continuous solutions of $(B$, are given by $p(x)=c_1x^\beta$, $q(x)=c_2x^\beta$, $r(x)=c_1c_2x^\beta$, where the c_i $(i=1, 2)$ are any real constants different from zero.

We now turn to equation (A). It can now be written as

$$f(xy) = g(x)^{(c_1y^\beta)}h(y)^{(c_2x^\beta)}.$$

In order to have only real solutions, $f(x)$, $g(x)$, $h(x)$ must be assumed to be positive. Let us write $g(x)^{c_1}=u(x)$, $h(y)^{c_2}=v(y)$. Then the last displayed functional equation takes the form

$$f(xy) = u(x)^{(y^\beta)}v(y)^{(x^\beta)}.$$

Raising both sides of this equation to the power $(xy)^{-\beta}$, we get

$$(A') \qquad f(xy)^{1/(xy)^\beta} = u(x)^{1/(x)^\beta}v(y)^{1/(y)^\beta}.$$

If in this equation we write for the left side $F(xy)$, and let $G(x)$ and $H(y)$ be the first and second factors of the right hand side respectively, then (A') becomes

$$(A'') \qquad F(xy) = G(x)H(y).$$

This, however, is equation (B) which was solved above. Hence all continuous solutions of (A'') are given by $G(x) = k_1x^\alpha$, $H(x) = k_2x^\alpha$, $F(x) = k_1k_2x^\alpha$, where α is any real constant and $k_i > 0$ $(i = 1, 2)$. Substituting these values in the equations which define $f(x)$, $g(x)$, and $h(x)$, we get the result stated in the following theorem.

THEOREM. *All real, continuous solutions, defined for all $x > 0$, of the set of functional equations (A) and (B) are of the form*

$$p(x) = c_1x^\beta, \qquad q(x) = c_2x^\beta, \qquad r(x) = c_1c_2x^\beta,$$
$$g(x) = (k_1x^\alpha)^{x^\beta/c_1}, \qquad h(x) = (k_2x^\alpha)^{x^\beta/c_2},$$
$$f(x) = (k_1k_2x^\alpha)^{x^\beta},$$

where α, β are any real constants while c_i and k_i are any real numbers such that $c_i \neq 0$, $k_i > 0$ $(i = 1, 2)$.

3. Special cases. Cauchy's equation (1) is equivalent to the equations (A) and (B) with the additional simultaneous equations $f(x) = g(x) = h(x)$, and $p(x) = q(x) = r(x) = 1$. The solution of Cauchy's equation is described by a special case of this theorem; namely that for which $\beta = 0$, $c_1 = c_2 = k_1 = k_2 = 1$, α arbitrary. The equation (2) is equivalent to the equations (A) and (B) with the additional simultaneous equations $f(x) = g(x) = h(x)$, and $p(x) = q(x) = r(x) = x^\beta$. The solution of equation (2) is described by the special case of our theorem for which $c_1 = c_2 = k_1 = k_2 = 1$, α and β arbitrary.

COROLLARY. *All real continuous solutions, defined for all $x > 0$, of the set of functional equations,*

$$(C) \qquad F(xy) = p(y)G(x) + g(x)H(y),$$

$$(D) \qquad r(xy) = p(x)g(y),$$

are of the form

$$p(x) = c_1x^\beta, \qquad q(x) = c_2x^\beta, \qquad r(x) = c_1c_2x^\beta,$$
$$G(x) = \frac{x^\beta}{c_1}(\alpha \log x + \log k_1), \qquad H(x) = \frac{x^\beta}{c_2}(\alpha \log x + \log k_2),$$
$$F(x) = x^\beta(\alpha \log x + \log k_1k_2),$$

where α, β are any real numbers while c_i and k_i are any real numbers such that $c_i \neq 0$, $k_i > 0$, $(i = 1, 2)$.

To derive this result from the Theorem we write (C) as

$$e^{F(xy)} = e^{G(x)p(y)}e^{H(y)q(x)},$$

which is equation (A) with $f(x) = e^{F(xy)}$, $g(x) = e^{G(x)}$, and $h(x) = e^{H(x)}$. Taking the logarithms of the solutions for $f(x)$, $g(x)$, and $h(x)$ given in the Theorem we have the result of the Corollary.

By taking logarithms or exponential functions of the functions occurring in the given functional equations one can derive other results similar to those given here.

CLASSROOM NOTES

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NOTE ON AN EXTENSION OF THE METHOD OF UNDETERMINED COEFFICIENTS IN SOLVING A LINEAR DIFFERENTIAL EQUATION

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The method of undetermined coefficients for solving a linear differential equation with constant coefficients, as presented in many texts* has the requirement that the method is applicable only if the differential equation

$$F(D)y = (r_0D^s + r_1D^{s-1} + \cdots + r_{s-1}D + r_s)y = f(x) \tag{1}$$

has a right member $f(x)$ such that $f(x)$ and a finite number of its successive derivatives form a set of linear dependent functions. The restriction requires that $f(x)$ satisfy a linear homogeneous differential equation with constant coefficients

$$(b_0D^t + b_1D^{t-1} + \cdots + b_{t-1}D + b_t)f(x) = 0.$$

By the theory for the solution of equations of this type $f(x)$ is a linear combination with constant coefficients of terms of the form

$$x^pe^{qx}, \qquad x^pe^{\alpha x}\cos\beta x, \qquad x^pe^{\alpha x}\sin\beta x \qquad (p \geqq 0) \tag{2}$$

where p, q, α and β are real. Now (2) are essentially functions of the form,

$$Cx^pe^{(\alpha+i\beta)x} \qquad (p \geqq 0).$$

* Reddick, H. W., Differential Equations, Section 36, 1943. Cohen, Abraham, Differential Equations, Section 50, 1906.

It is the purpose of this paper to replace this restriction by the more general requirement that $f(x)$ be of the form,

$$Cx^p e^{(\alpha+i\beta)x^m} \qquad (p \geqq 0) \ (m \text{ a positive integer}). \tag{3}$$

We present a method for solving equations of this type, when the primitive is expressible in finite form.

Denote the general solution† by

$$y = A_0 f(x) + A_1 f'(x) + A_2 f''(x) + \cdots \qquad (A\text{'s are undetermined constants})$$

or

$$y = e^{(\alpha+i\beta)x^m}(a_0 + a_1 x + a_2 x^2 \cdots) \tag{4}$$

with undetermined constant coefficients, $a_0, a_1, a_2, \cdots$. Substituting (4) in (1) gives an equation,

$$H_0 + H_1 x + H_2 x^2 + \cdots = 0 \tag{5}$$

where H_i involves $a_0, a_1, a_2 \cdots$ and the coefficients of $f(x)$ of form (3). Since the sum of the coefficients of each power of x (i.e. H_i) must equal zero, we have the set of linear equations:

$$H_i = 0 \qquad (i = 0, 1, 2, \cdots) \tag{6}$$

to solve for $a_0, a_1, a_2, \cdots$.

Now if the highest power of x in $f(x)$ of form (3) is p then the equations,

$$H_i = 0 \qquad (i = p+1, p+2, \cdots) \tag{7}$$

will be homogeneous. The linear recurrence relation, derived from substituting (4) in (1) by which equations (7) are derived has at most

$$\gamma \equiv \sum_{i=1}^{m} (s + 2 - i) \qquad \text{terms if } m \leqq s+1 \tag{8}$$

or

$$\gamma \equiv \sum_{i=1}^{s} (s+2-i) \qquad \text{terms if } m > s+1.$$

In relation (4), expand $e^{(\alpha+i\beta)x^m}$ in a Maclaurin's series and multiply this expansion by $(a_0+a_1x+a_2x^2 \cdots)$, then if this series is substituted in (1), it can be observed that H_0 has at most the terms $a_0, a_1, \cdots, a_s$; H_1 has at most the terms $a_0, a_1 \cdots a_{s+1}$; and H_i has at most the terms $a_0, a_1, \cdots a_{s+i}$. We now solve the set of linear equations:

† Coble, A. B., this MONTHLY, vol. 26, p. 12, 1919. Conkwright, N. B., this MONTHLY, vol. 41, p. 228, 1934.

$$H_i = 0 \qquad (i = 0, 1, \cdots, p + \gamma) \tag{9}$$

for $a_0, a_1, \cdots a_{p+\gamma}$.

Since y as given by (4) is equivalent to a power series, by the theory for a power series solution‡ for a differential equation of order s, there are s arbitrary constants in the solution. Hence we can assign s of the a's values for a particular solution. The solution is easier if we assign $a_{p+1}, a_{p+2}, \cdots a_{p+s}$ zero and first solve the first $p+1$ equations** of (9) ($p+1$ non-homogeneous linear equations in $p+1$ unknowns) and use these results to solve the last γ homogeneous equations of (9).

Now if $a_{p+s+1}, a_{p+s+2} \cdots a_{p+\gamma-1}$ are equal to zero in the solution of (9), all but one of the terms in the recurrence relation are zero. Therefore the remaining term is zero. Hence for equations (7) (these equations do not contain $a_0, a_1, \cdots a_p$) $a_{p+1}, a_{p+2}, \cdots$ are all zero. In this case we have a particular solution in finite form. Otherwise the solution will be infinite.

In particular, when $m=1$, (8) has the value $s+1$, and $\gamma-s$ has the value 1, so that the equations (7) always have zero solutions. In this case it is always possible to find a finite solution of form (4). The coefficient of a_0 in H_0 is $F(\alpha+i\beta)$. The determinant of the unknowns is $[F(\alpha+i\beta)]^{p+1}$. If $F(\alpha+i\beta)=0$, and $\alpha+i\beta$ is a root of multiplicity m, we cut out $a_0, a_1, \cdots a_{m-1}$ and proceed as before.

An illustration may be presented. Solve the differential equation

$$y'' - 12y = 2e^{x^2}(4x^4 - 3).$$

The recurrence relation is

$$R_n \equiv (n + 2)(n + 1)a_{n+2} + (4n - 10)a_n + 4a_{n-2}.$$

Hence

$$\begin{aligned} H_0 &= 2a_2 - 10a_0 + 6 = 0 \\ H_1 &= 6a_3 - 6a_1 = 0 \\ H_2 &= 12a_4 - 2a_2 + 4a_0 = 0 \\ H_3 &= 20a_5 + 2a_3 + 4a_1 = 0 \\ H_4 &= 30a_6 + 6a_4 + 4a_2 - 8 = 0. \end{aligned}$$

Assign $a_5=0$ and $a_6=0$, then there are 5 equations in 5 unknowns. In this case $a_0=1$, $a_1=0$, $a_2=2$, $a_3=0$, $a_4=0$. Also $a_5, a_6 \cdots$ are all zero. The particular solution is $y=e^{x^2}(2x^2+1)$.

‡ Ince, Ordinary Differential Equations, Chapter VII.

** In each problem the rank of the matrix of the system (9) should be examined to determine the character of the solution. Finally the particular solution should be checked in (1).

THE REMAINDER IN COMPUTING BY SERIES

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In computing by series it is important to have a bound for the remainder after a finite number of terms have been computed. The following method is simple enough to be presented to any class in which the integral test for convergence has been taken up, and it is applicable to any series on which that test may be used; for instance, the common series $\sum_{n=1}^{\infty}(1/n^p)$, commonly called the p-series.

Denote a series of positive terms by $\sum_{n=1}^{\infty} u_n$, and let $u_n=f(n)$ be such that $f(n)$ can be changed to $f(x)$ with x a continuous variable. Suppose that the sum of m terms of the series has been computed. We assume that $\int_m^{\infty} f(x)\,dx$ is finite and can be found; and that, for $m \leqq x$, $f'(x)$ is negative and $f''(x)$ is positive.

Draw the curve $y=f(x)$ and, as in the usual proof of the integral test for convergence, represent the terms u_n by rectangles under the curve of base 1 whose right-hand sides are $y=f(n)$. Also, as in the usual proof of the integral test for divergence, draw rectangles of base 1 that extend above the curve whose left-hand sides are $y=f(n)$. Denote the points $[m, f(m)]$, $[m+1, f(m+1)]$, $[m+2, f(m+2)]$, $[m, f(m+1)]$, $[m+1, f(m+2)]$ by A, B, C, D, E, respectively. Draw the chord AB, and the chord CB, and produce the latter until it cuts $x=m$ at F.

Let $R=\sum_{n=m+1}^{\infty} u_n$. We shall find lower and upper bounds for R.

To find the lower bound we have

$$u_{m+1} > \int_m^{m+1} f(x)dx - \triangle ABD, \qquad \triangle ABD = \tfrac{1}{2}(u_m - u_{m+1})$$

and we have similar expressions for the other terms of R.

Hence

$$R > \int_m^{\infty} f(x)dx - \tfrac{1}{2}[(u_m - u_{m+1}) + (u_{m+1} - u_{m+2}) \cdots]$$

or

$$R > \int_m^{\infty} f(x)dx - \tfrac{1}{2}u_m \quad \text{since} \quad \lim_{n\to\infty} u_n = 0.$$

To find the upper bound consider the triangle DBF=triangle BCE. The hypotheses assumed for $f(x)$ ensure that BF is entirely below the curve. Therefore

$$u_{m+1} < \int_m^{m+1} f(x)dx - \triangle DBF$$

or

$$u_{m+1} < \int_m^{m+1} f(x)dx - \triangle BCE, \qquad \triangle BCE = \tfrac{1}{2}(u_{m+1} - u_{m+2}).$$

Hence

$$R < \int_m^{\infty} f(x)dx - \tfrac{1}{2}[(u_{m+1} - u_{m+2}) + (u_{m+2} - u_{m+3}) \cdots]$$

or

$$R < \int_m^{\infty} f(x)dx - \tfrac{1}{2}u_{m+1}.$$

The above applies directly only to series with all positive terms, but an alternating series can often be changed by pairing the terms to one with all positive or all negative terms, and frequently this method can be used. It usually gives a much closer upper bound for R than $|u_{m+1}|$ which is commonly employed, and in addition it provides a lower bound.

THE BINOMIAL COEFFICIENTS

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When the student of elementary algebra first meets the binomial expansion, it may stimulate his interest to point out certain neat properties of the coefficients which are frequently passed over.

For one thing, most books state the law of formation of the elements of Pascal's Triangle but fail to connect it in any way with the expansion itself. The result is that the student feels that through some mysterious coincidence, the Triangle just happens to yield the right coefficients. The connection becomes immediately apparent if a sample multiplication is presented, say $(a+b)^5$ by $(a+b)$:

1	5	10	10	5	1	
1	1					
1	5	10	10	5	1	
	1	5	10	10	5	1
1	6	15	20	15	6	1

(The instructor should probably fill in signs and letters.) Such a presentation has the added advantage of paving the way for the induction proof which the student is about to meet for the first time.

The surprising fact that all the coefficients (except the first and last) of $(a+b)^n$ are divisible by n if and only if n is a prime may be mentioned, and possibly a proof left as a challenge to the student, although the proof derives more naturally perhaps from the theory of commutative fields.

If the expression for the $(k+1)$th term is written as a times the kth term, the resulting equation reduces to

$$n = ka + k - 1. \tag{1}$$

This means that for *any* a and *any* integer k, some binomial expansion exists (and is exhibited) whose $(k+1)$th term is a times its kth. That such a demand can so easily be met is scarcely intuitively evident. The condition for n to be integral is simply that ka be integral. For example, $a=3/2$, $k=6$, $n=14$, yield 2002 and 3003, the 6th and 7th coefficients of $(a+b)^{14}$.

Setting the $(k+2)$nd term $=b$ times the kth results in

$$n = \tfrac{1}{2}(2k - 1 \pm \sqrt{4bk^2 + 4bk + 1}). \tag{2}$$

The requirement that (1) and (2) hold simultaneously for arbitrary a, b and k exceeds by one the available degrees of freedom. In fact, (1) and (2) together are equivalent to

$$k = \frac{a+b}{a^2-b} = \text{an integer.}$$

Certain special combinations lead to usable (integral) k. As a particular example, note that if $b=1$, k is integral when a is of the form $(c+1)/c$, c integral. This tells how to find an *integral* n such that the middle term of the expansion is $(c+1)/c$ times either adjacent term for any integral c.

Other similar properties are revealed by further very easy explorations. which the student may be encouraged to conduct on his own.

MATHEMATICS FOR BUSINESS STUDENTS

W. O. BUSCHMAN, Vanport Extension Center, Portland, Oregon

1. Introduction. For a number of years there has been considerable dissatisfaction with the traditional college mathematics program among those who are not majoring in the fields of mathematics and natural science. There have been many reports concerning the place of mathematics in General Education, but little has been written about the place of mathematics in the business curriculum.

In order to satisfy the mathematical needs of the business students some schools have arranged special courses for them. In other schools business students take the traditional mathematics courses or none at all. The nature of business itself makes this second type of program unsatisfactory, especially for students who are very weak in mathematics.

The purpose of this study is to consider the mathematical needs of the business student and to examine current trends and policies. Only higher education will be discussed here; this will mean only work beyond the traditional four-year high school. Only regular course work will be considered though it is admitted that outside or extracurricular activities are also important.

2. Mathematical requirements. The most noticeable thing about the mathematical requirements for business students, as shown by examination of various college catalogues, is the wide variation. In the schools of business the trend has been away from the traditional freshman courses: college algebra, trigonometry, and analytic geometry.* The integrated courses, when offered, generally do not differ much from the traditional sequence; the difference is more one of arrangement than of content.

Mathematics courses required of business students range all the way from none to calculus, and even to differential equations in one college. Most colleges require a program between these two extremes, including a course in mathematics of finance or mathematics of investment. College algebra is generally a prerequisite for these courses. Many colleges also require some work in elementary statistics.†

Generally where technical minors such as engineering, science, or similar fields are being combined with business the mathematical requirements are more rigorous and consist of the traditional type of courses plus a course in mathematics of investment. There is reason for this, as engineering, physics, *etc.* require some of the skills and concepts from trigonometry, analytic geometry, and calculus, even at the beginning level. A businessman dealing with people who are working in these fields will need to be acquainted with such topics himself.

In some universities the business program is entirely in the upper division or the graduate school; this plan appears to be part of the trend toward "general education" for all students during their first two years at a college or university. Specialization is thus postponed till after the second year. In these schools the lower division mathematical program of the business curricula will be the same as for "general education." For schools that do not have a "general education" program a special course should be provided in the lower division for all business students.

3. Aims of proposed course. The aims of an elementary mathematics course for all business students should be the development of a basic knowledge of and skills in arithmetic, algebra, and such computational devices as needed for their work. There is need of a working knowledge of processes dealing with loans, interest, investment, discounts, annuities, insurance of various types, and of some of the concepts of the development of these topics. Ability to analyze verbal problems and their solutions is of great importance and needs particular emphasis and development in the business curricula.

Most students entering the field of business have had little background in mathematics and nearly all will need to review fundamentals. For this reason the first course should require only a minimum of previous knowledge and skills

* Monroe, Paul, Encyclopedia of Educational Research, Macmillan Co., New York, N. Y., 1950, p. 720.

† Hammond, Clela D., An Analysis of Mathematics Course Offered in Public Junior Colleges of California, Junior College Journal, Dec. 1949, p. 219.

in mathematics. At the same time, however, such a course must not be so elementary as to allow the student to lose interest, but should proceed to new topics as rapidly as possible. Elementary skills which may be lacking should not all be reviewed at the first of such a course as is the usual custom. These skills should be developed at the time they are extended and applied to more advanced topics. In this way the student will understand the need for these skills and the applications.

All students should be required to take a year course of 3 to 5 hours credit in mathematics; this should start with the fundamentals, including arithmetic concepts.‡ In most courses it has become customary to do all review and drill work first. This is very tedious for the better student, and he becomes bored. Review will be more effective if taken as needed, with the class going forward to new topics immediately.

4. General Outline of Proposed Course. 1. Development of algebraic notation and symbolism. This should be done through the use of equations and formulas taken from the field of business. Nearly all problems should be of the verbal type so that the formulas do not become drill exercises.

2. Percentage and its applications. Problems of cash discount, mark-ups, profit, loss, etc. can be used to develop facility with percentage. The use of index numbers in the statistical treatment of data will furnish additional applications or this may be taken later in the course thus providing a review of percentage.

3. Taxation. Income, property, and sales taxes, will furnish further applications of percentage and at the same time establish a familiarity with the processes of taxation which each student will meet sooner or later.

4. Graphical presentation of data. Graphs and formulas should be studied as a part of the development of the function concept. Problems of variation can readily be adapted to this aim.

5. Elementary concepts of statistics. A study of the measures of central tendency will provide further applications of formulas and give practice in the operations of arithmetic. If needed these operations can be reviewed. Considerable emphasis in the work of statistics should be on interpretation of the data. Statistical computations have little use or meaning without the interpretation of results. Most people will have many more occasions to interpret the meaning of statistical data than to compute it.

6. Time series. This will provide further applications of graphing and statistics, through a study of trends, seasonal as well as linear.

7. Further topics from statistics. Measures of central tendency will have developed the processes needed for a study of dispersion. Simple work with the binomial distribution and regression and correlation could be introduced if desired.

8. Progressions and series. These will be needed for further work in simple

‡ Newsom, C. V., A Course in College Mathematics for a Program of General Education, Mathematics Teacher, January 1949, pp. 19–24.

and compound interest. Applications of the progressions should be taken from verbal business problems as far as possible.

9. Computational aids such as logarithms and special tables will be needed before extensive work is done with interest and annuities. Calculators should be used if available.

10. Simple interest. This topic should be studied as an outgrowth of the arithmetic progressions. Emphasis should be on the development of the particular progression involved rather than direct applications of the formulas to develop the required concepts.

11. Compound interest. Amount and present value problems should be developed as an outgrowth of the geometric progression as indicated above for simple interest. The use of the compound interest and discount tables should be included.

12. Effective and equivalent rates of interest. Only through a study of these topics can the student gain an understanding of the high costs of deficit financing. These processes will give methods of comparing investments and loans. Applications should be taken from class situations, purchasing cars, houses, etc.

13. Annuities. For most students a treatment of annuities with simple data will be sufficient. Emphasis should be on the development of formulas and concepts of annuities. Some students will be reaching the limit of their ability to deal with abstractions and will have to deal with problems of annuities only as applications of formulas. Concepts should be developed as far as possible, however.

14. Amortization. This topic might well be introduced along with the one above to show the methods of liquidating debts. Sinking funds should also be included here. Comparisons of the methods of amortization and annuities should be stressed.

15. Bonds. Particular emphasis should be placed on the smaller government bonds with which the student is likely to come in contact. Bonding and bond investment on a larger scale can be used with local data such as municipal bonds to keep the problems within the area in which the student is familiar.

16. Depreciation. This is a topic about which all businesses are concerned, as it is important in taxation as well as for properly estimating profits and losses.

17. Building and Loan associations. The nature of this type of enterprise should be considered and compared with other types of investment.

18. Insurance. The various types of insurance are dealt with by almost everyone today. Life insurance, health insurance, car insurance, and many other types should all be considered as they are sure to be encountered sooner or later.

The general program outlined above could serve best for a year course to include all of the titles listed. It could be adjusted to either a "three-hour" or "five-hour" program; the material could be satisfactorily covered by a three hour course or more extensively in a four or five hour course. Parts of items 16, 17, and most of 15 could be omitted if the shorter course is desired. In the

earlier sections the review can be adjusted to the class, and most review omitted if the class does not need it.

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ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 931. *Proposed by H. D. Larsen, Albion College*

Ten balls numbered from 0 to 9 inclusive are placed in an urn. Five of the balls are then drawn at random (without replacement) and arranged in a row. What is the probability that the number thus formed is divisible by 396?

E 932. *Proposed by Paul Erdös, University of Illinois*

(1) Prove that every positive integer not exceeding $n!$ is the sum of n or fewer distinct divisors of $n!$.

(2) Denote by $E(n) = 2^{a_2}3^{a_3} \cdots p^{a_p}$ the least common multiple of the positive integers not exceeding n. Prove that every positive integer not exceeding $E(n)$ is the sum of $a_2+a_3+\cdots+a_p$ or fewer distinct divisors of $E(n)$.

E 933. *Proposed by Joseph Langr, Prague, Czechoslovakia*

Let A', B', C' be the feet of the altitudes of a triangle ABC whose circumcircle and nine point circle are orthogonal. Show that (1) the areas of triangles ABC and $A'B'C'$ are equal, (2) $(AC')(BA')(CB') = (AB)(BC)(CA)/2$. (3) Construct the triangle ABC given side BC and the position of A' on BC.

E 934. *Proposed by C. O. Oakley, Haverford College*

Let C_1 and C_2 be two arbitrarily given curves whose parametric equations are $x_1 = x_1(t)$, $y_1 = y_1(t)$ and $x_2 = x_2(t)$, $y_2 = y_2(t)$. Show that there exists a third curve C, $x = x(t)$, $y = y(t)$, with the "focal" property that, if P_1, P_2, and P are corresponding points on C_1, C_2, and C respectively, then P_1P and P_2P make equal angles with the tangent to C at P.

E 935. *Proposed by C. D. Olds, San Jose State College*

Give an example of a function which is discontinuous in an everywhere dense set, and which is also differentiable in an everywhere dense set.

SOLUTIONS

An Unsolved Problem

E 534 [1942, 475, 1943, 261]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that 4, 5, 7 are the only values of n for which $n!+1$ is a perfect square.

Solution by H. D. Grossman, New York, N. Y. The only values of $n<10$, for which $n!+1$ is a perfect square, are 4, 5, and 7. Then let $n \geqq 10$ and let all letters in the discussion be positive integers.

If $n!+1 = k^2$, then $n! = (k+1)(k-1)$. Let the integers $k+1$, $k-1$ be r, s in unknown order. Since r, s differ by 2 and their product, $n!$, is even, they must both be even. Also, they can have no common even factor but 2 and no common odd factor at all (besides unity).

Then 2 divides only one of them, say r, while (4) (6) (8) (10) $\cdots$ divides the other. Then (3) (5) (9) also divides s. At most r is divisible by 2 and by those odd primes whose doubles exceed n. These restrictions make r so much smaller than s that r and s cannot possibly differ by only 2.

In the following table the greatest possible chain of factors of r is in the left column. Each of these is exceeded by the corresponding factor of s in the right column.

r		s
2		(3)(4)
7		(5)(6)
$6t-1$	$(t>1)$	$(6t-2)(6t-3)$
$6t+1$	$(t>1)$	$6t(6t-4)$

Whereas the product of factors in the right column is much greater than that in the left;

And whereas, as n increases, every odd prime $6t\pm1$ in the left column must be permanently transferred to the right column as soon as n reaches $2(6t\pm1)$;

And whereas all composite $6t\pm1$ do not even fall initially in the left column;

And whereas no integer is repeated anywhere in the table;

Be it hereby resolved that $n!+1$ can never be a perfect square if $n\geqq10$.

A Dissected Cube

E 897 [1950, 34]. *Proposed by W. R. Ransom, Tufts College*

A cube is divided into equal parts by the plane of a hexagon whose vertices are midpoints of its edges. In what ratio is a cube divided by a parallel plane that divides an edge in the ratio $a:(1-a)$?

Solution by C. M. Sandwick, Easton High School, Easton, Pa. Imagine an observer at a point outside a unit cube in line with the diagonal r which is perpendicular to the cutting plane. Imagine that the plane passes through the cube while approaching the observer. The plane travels a distance $r/3$ from the time that it passes the most remote vertex until it contains the next three vertices simultaneously and during this interval it cuts three edges. The plane travels a second distance $r/3$ until it again contains three vertices simultaneously, during which time it intersects six edges. Finally the plane travels the distance $r/3$ until it leaves the cube as it passes the last vertex, cutting the three remaining edges as it traverses the final interval. In each of the three stages assume that a varies from 0 to 1 as the plane approaches the observer. Let v denote that part of the volume which is on the side of the plane away from the observer. In a unit cube $r=\sqrt{3}$. In each stage, since the projection of an intersected edge on r is $\sqrt{3}/3$, we see that $dr=da/\sqrt{3}$. Let K denote the area of the intersection of the plane and the cube.

In the first stage, as a increases from 0 to 1, v increases from 0 to 1/6, and $K=a^2\sqrt{3}/2$. Integrating $K\,dr$, we obtain $v=a^3/6$. Therefore, in this region,

$$v:(1-v)=a^3:(6-a^3).$$

In the second stage, as a increases from 0 to 1, v increases from 1/6 to 5/6, and it is easy to show that $K=\sqrt{3}(1/2+a-a^2)$. Integrating $K\,dr$ we then find that $v=(1+3a+3a^2-2a^3)/6$. Hence

$$v:(1-v)=(1+3a+3a^2-2a^3):(5-3a-3a^2+2a^3).$$

In the third stage, as a increases from 0 to 1, v increases from 5/6 to 1, and $K=(1-a)^2\sqrt{3}/2$. Hence $v=[6-(1-a)^3]/6$, and

$$v:(1-v)=[6-(1-a)^3]:(1-a)^3.$$

Also solved, for the second stage, by W. G. Brady, D. H. Browne, Roger

Lessard, L. A. Ringenberg, Nathan Schwid, C. W. Trigg, and the proposer.

Trigg further noted that, in the second stage, the surface area of the cube is divided in the ratio $(1+2a):(3-2a)$, which is also the ratio of the edges (and portions of edges) of the cube contained in the two pieces. The perimeter of the hexagonal section of the unit cube is $3\sqrt{2}$.

Construction of an Equilateral Triangle

E 899 [1950, 35]. *Proposed by E. R. Bowersox, Chicago, Illinois*

Construct an equilateral triangle given the distances of a point in its plane from the three vertices.

Solution by C. W. Trigg, Los Angeles City College. CONSTRUCTION. We take the distances $a \geqq b \geqq c$ from the point P. On $PD=a$ construct an equilateral triangle PDA and triangles PDC and PDC' with $PC=c=PC'$ and $DC=b=DC'$. On the other side of AC from D construct the required equilateral triangle ABC. Similarly, construct equilateral triangle $AB'C'$. Clearly there are two ($a<b+c$ and a, b, c not all equal), one ($a=b+c$ or $a=b=c$) or no ($a>b+c$) solutions.

PROOF. Draw PB and PB'. By construction, $PA=DA=a$, $CA=BA=x$, and $\measuredangle DAC+\measuredangle CAP=60^\circ=\measuredangle CAP+\measuredangle PAB$, so $\measuredangle DAC=\measuredangle PAB$. Hence, triangles DAC and PAB are congruent, so $PB=DC=b$. In like manner, $PB'=b$.

Also solved by Norman Anning, J. H. Braun, D. H. Browne, W. E. Buker, W. B. Clarke, L. P. Coggin, William Douglas, R. R. Eddy, B. H. Gundlach, Frank Herlihy, H. K. Humphrey, John Jones, Jr., L. M. Kelly, M. S. Klamkin, Roger Lessard, Prasert Na Nagara, C. C. Oursler, A. P. Rhodes, L. A. Ringenberg, Joseph Rosenbaum, N. C. Scholomiti, Alfred Sylvester, and the proposer.

Editorial Note. Most of the solutions were based upon either a stretch-rotation or intersecting circles of Apollonius, and could easily be generalized to yield the construction of a triangle of assigned species having its vertices at given distances from a given point. This generalized problem is sometimes referred to as *Carnot's problem.* For a solution see, *e.g.*, N. A. Court, *College Geometry*, sec. 56, p. 49. For a discussion and history of Carnot's problem see problem 2761 [1920, 138]. Here it is shown, for the given problem, that

$$x^2 = (a^2 + b^2 + c^2)/2 \pm 2\sqrt{3}\,K,$$

where K is the area of triangle PDC. Also see problems 514 [1918, 21] and 3904 [1941, 75]. For a generalization in a different direction see *Proc. Ed. Math. Soc.*, vol. 27 (1909), pp. 70–75 and vol. 28 (1910), pp. 95–120. Trigg gave several references to problems in *School Science and Mathematics* where the configuration of this problem has received some study. It is intimately related to the identity connecting the six distances of four points in a plane.

Some Cospherical Circles

E 900 [1950, 35]. *Proposed by N. A. Court, University of Oklahoma*

If four spheres may be divided into pairs so that the circle common to one pair is cospherical with the circle common to the second pair, then the circle determined by any two of the given spheres is cospherical with the circle determined by the remaining two spheres.

Solution by L. M. Kelly, Michigan State College. Let the spheres be (A), (B), (C), (D), and suppose the circles of intersection of (A) and (B) and of (C) and (D) are on a sphere (S). The center of (S) is on the line of centers AB, and also on the line of centers CD. Thus A, B, C, D are coplanar. The spheres (A), (B), (S) have a common radical plane, as have (C), (D), (S). These two radical planes intersect in a line perpendicular, at P say, to the plane of centers of the four spheres. The powers of P relative to the four spheres are all equal. Let X_1, X_2, and Y_1, Y_2 be the points where the circles of intersection of (A), (C) and of (B), (D) cut the plane of centers. Then $(PX_1)(PX_2) = (PY_1)(PY_2)$, and the points X_1, X_2, Y_1, Y_2 are concyclic. It follows at once that the two circles are cospherical. A similar argument may be applied to the remaining pair of circles.

Also solved by Roger Lessard and the proposer.

Editorial Note. The analogous theorem in the plane is also true, and reads as follows: If four circles may be divided into two pairs so that the intersections of one pair are concyclic with those of the second pair, then the intersections of any two of the circles are concyclic with those of the remaining two circles.

Bonus Problem

E 901 [1950, 113]. *Proposed by Leo Moser, University of Manitoba*

A company offers its 350 employees a bonus of \$10.00 to each male and \$8.25 to each female. All the females accept, but a certain percentage of the males refuse to accept. Knowing what this percentage is, one can deduce the total amount paid out. What is the total amount paid to the women?

Solution by Bart Park, Michigan College of Mining and Technology. Let m be the number of men and let r be the fraction of men refusing a bonus. Then the amount paid out is given by

$$T = 10m(1 - r) + (350 - m)(8.25),$$

which will be independent of m if and only if $r = 0.175$. We find that $T = 2887.50$ dollars. Both m and $0.175m$ must be integers, with $m < 350$. The following values of m satisfy these requirements: 40, 80, 120, 160, 200, 240, 280, 320. There are therefore eight possible amounts, given by $(350 - m)(8.25)$, which may be paid to the women.

Also solved by W. E. Buker, M. S. Klamkin, C. S. Ogilvy, W. O. Pennell, J. V. Pennington, A. P. Rhodes, L. A. Ringenberg, C. W. Trigg, G. W. Walker, and the proposer.

Editorial Note. As was pointed out by some solvers, the solution could have been made unique by paying a more judicious bonus to each woman, say \$8.15 or \$8.35, instead of \$8.25. There are other values of r for which T has a single value, but here finding T would involve finding m. Walker found all such solutions in addition to those given above.

Rational Points on a Sphere

E 902 [1950, 113]. *Proposed by R. V. Andree's class in Solid Analytics, University of Oklahoma*

A similar argument to that used in E 832 [1949, 407] may be used to "prove" the following theorem: "If a sphere has a center with at least one irrational coordinate, then there are at most three points on the sphere with rational coordinates." Give a counter example to show that this theorem is not true and discuss the point at which the "proof" breaks down.

Solution by L. A. Ringenberg, Eastern Illinois State College. A counter example is any sphere in the family $x^2+y^2+z^2+Az=1$, where the parameter A is any irrational number. Four rational points on this sphere are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$. The proof of E 832 cannot be extended to the case of a sphere; for while it is true that if three points lie on a circle, then they lie on only one circle, it is not true that if four points lie on a sphere, then they lie on only one sphere.

Also solved by Paul Baird, Vern Hoggatt, M. S. Klamkin, Roger Lessard, C. S. Ogilvy, and the proposers.

Editorial Note. Of course we do have the true theorem: "If a sphere has a center with at least one irrational coordinate, then there cannot be four *non-coplanar* points on the sphere with rational coordinates."

On the other hand, it is easy to show that there exist spheres of arbitrary size, having non-rational centers, and passing through infinitely many (co-planar) rational points. To accomplish this we may proceed as follows. It is known that $x^2+y^2=1$ is satisfied by infinitely many pairs of rational values for x and y (see, *e.g.*, H. N. Wright, *First Course in Theory of Numbers*, p. 93), and therefore the circle C: $x^2+y^2=r^2$, where r is rational, passes through infinitely many rational points. Any sphere through C thus passes through infinitely many rational points. For the center of the sphere we may choose any non-rational point on the z-axis.

Even Integers as Sum of Two Abundant Numbers

E 903 [1950, 113]. *Proposed by F. A. E. Pirani, Carnegie Institute of Technology*

Show that every even integer greater than 46 can be expressed as the sum of two abundant numbers. (See E 848 [1949, 478].)

I. *Solution by Leo Moser, University of North Carolina.* An explicit representation of every even integer greater than 46 as the sum of two abundant num-

bers is given by

$$48 + 6k = 30 + (3 + k)6,$$
$$50 + 6k = 20 + (5 + k)6,$$
$$52 + 6k = 40 + (2 + k)6,$$
$$k = 0, 1, 2, \cdots .$$

II. *Solution by John Selfridge, University of Washington.* Any even number $m > 46$ can be written as $6k$, $6k+2$, or $6k+4$, where $k \geqq 8$. We have

$$6k = 6(k - 2) + 12,$$
$$6k + 2 = 6(k - 3) + 20,$$
$$6k + 4 = 6(k - 6) + 40.$$

Since 20, 40, and $6n$ $(n > 1)$ include all abundant numbers less than 46 it follows that 46, 34, 28, 26, and all even numbers less than 24 are the only even numbers not expressible as the sum of two abundant numbers. However, allowing the summands 6, 28, and 0 we may say: Any even number greater than 22 is non-deficient or can be expressed as the sum of two non-deficient numbers.

Also solved by Roger Lessard, C. W. Trigg, and the proposer.

Trigg further showed that every even integer greater than $12M+26$ is expressible as the sum of M abundant numbers. He also observed that 24, 36, 48, 60, 66, 84 are the smallest integers which can be expressed as the sum of two abundant numbers in 1, 2, 3, 4, 5, 6 ways, respectively.

A Cubic Locus of Centers of Curvature

E 904 [1950, 113]. *Proposed by C. S. Ogilvy, Columbia University*

Find the equation of the locus of centers of curvature of $y = x^n$ at the point (1, 1) as n varies continuously from $-\infty$, $+\infty$.

Solution by S. T. Thompson, Tacoma, Washington. Using the standard formulas for center of curvature one obtains, as parametric equations for the locus,

$$x = 1 - (n^2 + 1)/(n - 1), \qquad y = 1 + (n^2 + 1)/n(n - 1).$$

Eliminating the parameter by means of the relation

$$n = -(x - 1)/(y - 1),$$

one obtains the cartesian equation

$$(1 - x)^2(2 - y) + (1 - y)^2(2 - x) = 0.$$

This represents a unicursal acnodal cubic with its isolated double point at U:(1, 1). The point U, however, is not part of the required locus, for it corresponds to $n = \pm i$, and n is restricted to real values only. The cubic is tripartite having symmetry in the line $y = x$ and having $x = 2$, $y = 2$, $x = -y$ as asymptotes.

The branches correspond to the parametric ranges $-\infty < n < 0$, $0 < n < 1$, $1 < n < \infty$. The first branch is in the first quadrant, passing through the point (2, 2) and asymptotic to both $x = 2$ and $y = 2$. The second branch is in the fourth quadrant having $x = 2$ and $x = -y$ as asymptotes and lying within the infinite sector bounded by these asymptotes. The third branch is the reflection of the second branch in the line $x = y$. The curve is not found in the third quadrant nor in the infinite strips

$$-2\sqrt{2} - 1 < x < 2\sqrt{2} - 1 \quad \text{and} \quad -2\sqrt{2} - 1 < y < 2\sqrt{2} - 1.$$

The parameter n has simple geometric significance. Let P be any point on the curve and let UX be parallel to the positive x-axis. Then

$$n = -\cot (XUP).$$

If UX is chosen as a polar axis, the polar equation of the curve can be written as

$$r \sin \theta \cos \theta(\sin \theta + \cos \theta) = 0.$$

Also solved by W. G. Brady, P. L. Chessin, Ragnar Dybvik, Kurt Eisemann, Frank Herlihy, C. J. Kaufman, M. S. Klamkin, Roger Lessard, J. E. Morgan, A. Sisk, C. G. Solky, O. E. Stanaitis, John Torian, and the proposer.

A Property of the Orthocenter of a Triangle

E 905 [1950, 113]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let A_1 and A_2, B_1 and B_2, C_1 and C_2 be the points of intersection with the sides BC, CA, AB of a triangle ABC of the polars of a point P with respect to the escribed circles (I_b) and (I_c), (I_c) and (I_a), (I_a) and (I_b). For what position of P will the points A_1 and A_2, B_1, and B_2, C_1 and C_2 be, simultaneously, symmetric with respect to the midpoints of BC, CA, AB?

Solution by the Proposer. This problems leads to an interesting and probably new property of the orthocenter H of a triangle.

LEMMA. *A necessary and sufficient condition for the polars of two points A_1 and A_2, taken on the side BC of a triangle ABC, with respect to the escribed circles (I_b) and (I_c), to intersect on the altitude AA' through vertex A, is that A_1 and A_2 be symmetric with respect to the midpoint of BC.*

Let A_1 and A_2 be symmetric with respect to the midpoint of BC, and let (I_b) and (I_c) be tangent to BC at D_b and D_c respectively. Let the polar of A_1 with respect to (I_b) cut D_cI_c in M and the polar of A_2 with respect to (I_c) cut D_bI_b in N. Then, from the pairs of similar right triangles $D_bI_bA_1$, D_cD_bM and $D_cI_cA_2$, D_bD_cN, and the fact that $A_1D_b = A_2D_c$, we have

$$ND_b : MD_c = D_bI_b : D_cI_c = AI_b : AI_c.$$

The point of intersection, A'', of the polars D_cN and D_bM thus lies on a parallel to D_bI_b and D_cI_c through the external center of similitude, A, of (I_b) and (I_c). That is, A'' lies on the altitude AA'. Conversely, by reversing the above argument, it may be shown that if A'' is on the altitude AA', then A_1 and A_2 are symmetric with respect to the midpoint of BC. This proves the lemma.

It now follows that the altitude line AA' is the locus of a point P whose polars with respect to (I_b) and (I_c) will pass through a pair of isotomic points A_1, A_2 of BC. Similarly, the altitude lines BB', CC' of triangle ABC are the loci of points P for which the polars with respect to (I_c) and (I_a), (I_a) and (I_b) cut CA and AB in pairs of isotomic points. In order that A_1 and A_2, B_1 and B_2 C_1 and C_2 be, simultaneously, isotomic points on BC, CA, AB, it is necessary and sufficient that P coincide with the orthocenter H of ABC.

Also solved by Roger Lessard.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4410. *Proposed by C. D. Olds, San Jose State College, California*

Show that, for every positive integer n, there is a polynomial $f(x)$ of degree n and a related polynomial $\phi(x)$ which satisfy

$$[f(x)]^2 - 1 = (x^2 - 1)[\phi(x)]^2.$$

4411. *Proposed by L. C. Hsu, National Tsing-Hua University, Peiping, China.*

Show that if α is a real number >0, then for any given $\epsilon>0$ and any integer $k>0$ there can be found a number $\delta=\delta(\epsilon)>0$ and an integer $N\geqq k$ such that, if

$$\Delta x_1 + \Delta x_2 + \cdots + \Delta x_N = \Delta x_1' + \Delta x_2' + \cdots + \Delta x_N' = \alpha;$$
$$S_k = \sum \Delta x_{\nu_1} \cdots \Delta x_{\nu_k}, \qquad S_k' = \sum \Delta x_{\nu_1}' \cdots \Delta x_{\nu_k}',$$

then

$$|S_k - S_k'| < \epsilon$$

whenever $0<\Delta x_i<\delta$, $0<\Delta x_i'<\delta$, where S_k and S_k' are extended over all possible terms with distinct subscripts $\nu_1, \cdots, \nu_k$.

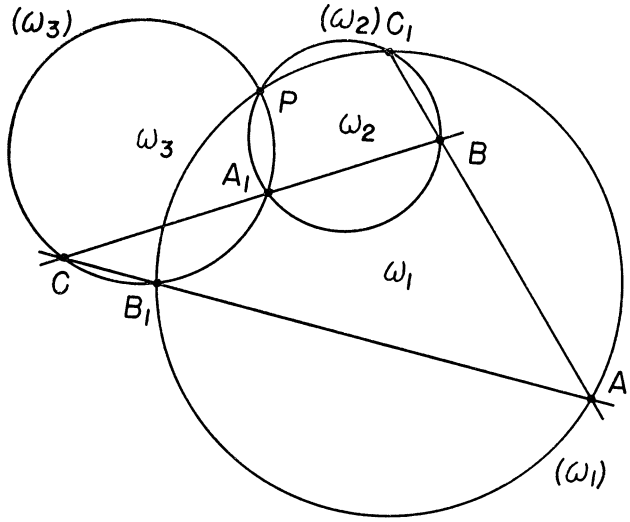
SOLUTIONS

Circumcircles of a Family of Similar Triangles

4301 [1948, 369]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider similar triangles ABC, the sides BC, CA, AB of which pass through the fixed points A_1, B_1, C_1. (1) The locus of the circumcenters O of these triangles is a circle. (2) The circumcircles (O) are orthogonal to a fixed circle. (3) When the points A_1, B_1, C_1 are collinear, the envelope of the circles (O) is a cardioid. What is this envelope otherwise?

Solution by W. E. Byrne, Lexington, Virginia. We prefer the following statement of the problem:* *Given three circles* $AB_1C_1 \equiv (\omega_1)$, $A_1BC_1 \equiv (\omega_2)$, $A_1B_1C \equiv (\omega_3)$, *having the fixed point* P *in common. If* A, B, C *vary subject to the condition that* BC, CA, AB *always pass through the distinct fixed points* A_1, B_1, C_1, *show that the locus of the circumcenters* Γ *of* ABC *is a circle, show that the circumcircles* (Γ) *are orthogonal to a fixed circle if* A_1, B_1, C_1 *are not collinear, and find the envelope of the circumcircles* (Γ). *Discuss also the special case for which* A_1, B_1, C_1 *are collinear.*



It is convenient to use a system of complex coördinates $z = x + iy$, $\bar{z} = x - iy$. Let us choose as a unit reference circle the circumcircle of triangle $\omega_1\omega_2\omega_3$,† of center $\omega(z=0)$, with the point P $(z=a)$ on the positive half of the real axis. If t_1, t_2, t_3 denote the points ω_1, ω_2, ω_3, we have as the equation of (ω_k), $k = 1, 2, 3$,

$$z\bar{z} - \bar{t}_k z - t_k \bar{z} = a^2 - \bar{t}_k a - t_k a.$$

* As originally stated the problem gives part of the loci here obtained, due to the fact that only portions of the circles (ω_k) are involved.

† ω_1, ω_2, ω_3 are not collinear since otherwise the circles (ω_k) would have an axis of symmetry and A_1, B_1, C_1 would coincide in a point symmetric to P.

The coördinates of A_1, B_1, C_1 are

$$z_{A_1} = t_2 + t_3 - at_2t_3, \qquad z_{B_1} = t_3 + t_1 - at_3t_1,$$
$$z_{C_1} = t_1 + t_2 - at_1t_2.$$

If the position of A is fixed by $z_A = t_1 + \mu(a - t_1)$, $|\mu| = 1$, then it follows that

$$z_B = t_2 + \mu(a - t_2), \qquad z_C = t_3 + \mu(a - t_3).$$

The equations of the perpendicular bisectors of AB and BC, are, respectively,

$$z + \mu t_1t_2\bar{z} = a(\mu + t_1t_2), \qquad z + \mu t_2t_3\bar{z} = a(\mu + t_2t_3),$$

and hence the circumcenter Γ of triangle ABC has the affix $z = a\mu$, and we conclude that the locus of Γ is the entire circle $z\bar{z} = a^2$ of center ω and radius a.

The equation of the circumcircle (Γ) is

$$z\bar{z} - \mu a\bar{z} - \bar{\mu}az + a^2 = 2 - \mu - \bar{\mu}.$$

The condition that (Γ) be orthogonal to the circle having equation $z\bar{z} - b\bar{z} - \bar{b}z + c = 0$ is $ba\bar{\mu} + \bar{b}a\mu = a^2 + \mu + \bar{\mu} - 2 + c$, or

$$\mu(1 - a\bar{b}) + \bar{\mu}(1 - ab) + a^2 - 2 + c = 0.$$

If we choose $b = \bar{b} = 1/a$, $c = 2 - a^2$, we have a fixed circle

$$z\bar{z} - (z + \bar{z})/a + 2 - a^2 = 0,$$

of center $z = 1/a$ and radius $|1 - a^2|/a$, orthogonal to the circles (Γ).

A_1, B_1, C_1 are collinear if and only if

$$\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{t_1t_2t_3}(a^2 - 1) = 0.$$

In this case the circle orthogonal to the circles (Γ) reduces to a null circle (P) and P is on the circumcircle of $\omega_1\omega_2\omega_3$.

In both cases the envelope of (Γ) is given by the discriminant of

$$(1 - a\bar{z})\mu^2 + (z\bar{z} + a^2 - 2)\mu + 1 - az = 0,$$

where use is made of the fact that $\mu\bar{\mu} = 1$, that is by the equation

$$(z\bar{z} + a^2 - 2)^2 - 4(1 - a\bar{z})(1 - az) = 0.$$

If the point P is chosen as the pole in a system of polar coördinates, with ωP serving as the initial line, the envelope appears as the graph of $r = 2(1 - a\cos\theta)$, which is a limaçon. It is a cardioid for $a = 1$.

Also solved by Robert Bouvaist and the Proposer.

Editorial Note. The Proposer's solution uses pure geometry. The first result follows very easily through the use of the special triangle $A_0B_0C_0$ determined by the perpendiculars drawn through A_1, B_1, C_1 to PA_1, PB_1, PC_1. Let Γ_0 be the

circumcenter of $A_0B_0C_0$. Since P is the center of similitude of the triangles $A_0B_0C_0$ and ABC, triangles $P\Gamma_0A_0$ and $P\Gamma A$ are similar, and so $P\Gamma_0\Gamma$ and PA_0A are similar. But the angle at A of triangle PA_0A is a right angle, and hence Γ describes the circumference of diameter $P\Gamma_0$.

Sequence of Integers No One of Which Divides Another

4332 [1949, 112]. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Let $a_1<a_2<\cdots$ be an infinite sequence of integers. Prove that from the sequence a_i+a_j, $i=1, 2, \cdots, j=1, 2, \cdots$, one can always select an infinite subsequence such that no element divides another.

Solution by the Proposer. We distinguish two cases.

Case I. In every infinite subsequence, $b_1, b_2, \cdots$, of the a's there exists at least one sum b_i+b_j and a corresponding infinite subsequence $b_{i_1}, b_{i_2}, \cdots$ satisfying

$$2b_{i_k} \not\equiv 0 \pmod{(b_i + b_j)}.$$

It follows that we can select infinitely many of them all in the same residue class. In other words, there exists an $a_{i_1}+a_{j_1}$ and infinitely many a's (for convenience we denote them by $a_1^{(2)}, a_2^{(2)}, \cdots$) such that

$$(1) \qquad 2a_k^{(2)} \not\equiv 0 \pmod{(a_i + a_j)}, \qquad a_1^{(2)} \equiv a_2^{(2)} \equiv \cdots \pmod{(a_i + a_j)}.$$

In the same way we see that there again exists $a_{i_2}^{(2)}+a_{j_2}^{(2)}$ such that there is an infinite subsequence of the $a_i^{(2)}$'s (say $a_1^{(3)}, \cdots$) satisfying

$$2a_k^{(3)} \not\equiv 0 \pmod{(a_{i_2}^{(2)} + a_{j_2}^{(2)})}, \qquad a_1^{(3)} \equiv a_2^{(3)} \equiv \cdots \pmod{(a_{i_2}^{(2)} + a_{j_2}^{(2)})}.$$

From (1) it follows clearly that

$$a_{i_2}^{(2)} + a_{j_2}^{(2)} \equiv 2a_{i_2}^{(2)} \not\equiv 0 \pmod{(a_i + a_j)}.$$

Similarly we obtain

$$a_{i_3}^{(3)} + a_{j_3}^{(3)}, \; a_{i_4}^{(4)} + a_{j_4}^{(4)}, \cdots,$$

and clearly no one divides any other.

Case II. There exists a subsequence, $b_1, b_2, \cdots$, of the a's such that to every b_i+b_j there are only a finite number of b_k's for which $2b_k \not\equiv 0 \pmod{(b_i+b_j)}$. Hence for all n sufficiently large

$$2b_n \equiv 0 \pmod{(b_i + b_j)}.$$

Consider then the sequence $b_1+b_{n_1}, b_1+b_{n_2}, \cdots$, where $n_1<n_2<\cdots$ tend to infinity fast enough so that

$$2b_{n_k} \equiv 0 \pmod{(b_1 + b_{n_i})}, \qquad i < k.$$

Further, since $0<2b_1<b_1+b_{n_i}$,

$$2b_1 \not\equiv 0 \pmod{(b_1 + b_{n_i})}.$$

Thus

$$2(b_1 + b_{n_k}) \equiv 2b_1 \pmod{(b_1 + b_{n_i})}$$

and clearly no one divides another.

The Bennett Point of a Quadrangle

4334 [1949, 112]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that the feet of the six perpendiculars from the Bennett point on the sides of the complete quadrangle lie on a conic.

Solution by the Proposer. This will follow if the locus of the feet of perpendiculars from the Bennett point* on the asymptotes of conics through four points is proved to be a conic.

To find the degree of the locus when the point is general, consider the points it has in common with the line at infinity. These will occur when an asymptote passes through a circular point or is the line at infinity itself. There are two conics one through each of the circular points and there are two touching the line at infinity. The curve is therefore a quartic.

In the particular case when the point is the intersection of the tangents at the circular points to the respective conics through these points, that is the Bennett point of the quadrangle, these tangents are evidently part of the locus. The remainder is therefore a conic.

It may be remarked that this conic has asymptotes perpendicular to those of the conic which is the locus of centers of the conics through the four points—since it passes through the harmonic conjugates with respect to the circular points of the points of contact of the two parabolas with the line at infinity—and passes through the Bennett point and the center of the unique rectangular hyperbola through the four points.

Fibonacci Numbers

4341 [1949, 268] *Proposed by D. H. Browne, Buffalo, New York*

The sequence $\{a\} = 3, 7, 47, 2207, 4870847, \cdots$, used for the determination of primality of the Mersenne numbers, is usually defined by

$$a_{n+1} = a_n^2 - 2.$$

Show that it may also be defined by

$$a_k = f_{2^{k+1}}/f_{2^k},$$

where the f's are the Fibonacci numbers, 1, 1, 2, 3, 5, 8, $\cdots$.

* The Bennett point is the point from which the circumcircles of the four points three at a time look the same size. See Morley and Morley, *Inversive Geometry*, p. 254, ex. 7, also p. 250.

Solution by John Todd, East Molesey, Surrey, England. It is well known that $f_n=(r^n-s^n)/\sqrt{5}$, where r and s $(-r^{-1})$ are the roots of $x^2-x-1=0$. Hence, writing $a=2^n$, we have

$$\frac{f_{4a}}{f_{2a}}-\left(\frac{f_{2a}}{f_a}\right)^2+2=\frac{r^{4a}-s^{4a}}{r^{2a}-s^{2a}}-\left(\frac{r^{2a}-s^{2a}}{r^a-s^a}\right)^2+2$$

$$=r^{2a}+s^{2a}-(r^a+s^a)^2+2=-2(rs)^a+2=0.$$

Therefore $f_{2^{k+1}}/f_{2^k}$ satisfies the same recurrence relation as a_k and since $a_1=3=f_4/f_2$ the desired result is proved.

Also solved by A. C. Aitken, Murray Barbour, W. R. Cowell, J. S. Frame, R. Goormaghtigh, Hansraj Gupta, B. F. Hadnot, R. N. Haskell, Vern Hoggatt, Free Jamison, Irving Kaplansky, M. S. Klamkin, N. D. Lane, Roger Lessard, D. C. B. Marsh, Leo Moser, C. D. Olds, S. T. Parker, C. F. Pinzka, L. B. Rall, Joseph Rosenbaum, Alex Rosenberg, Azriel Rosenfeld, N. C. Scholomiti, Robert Steinberg, J. D. Swift, and the Proposer.

Editorial Note. Several solvers point out that a_k may be given also as

$$a_k = f_{2^k+1} + f_{2^k-1}, \tag{1}$$

$$a_k = g_{2^k} \tag{2}$$

where $\{g\}$ is defined by $g_{n+2}=g_{n+1}+g_n$; $g_1=1$, $g_2=3$.

Cowell, Goormaghtigh and Rosenberg consider the more general quadratic $x^2-bx+c=0$, whose roots are r and s. If $u_n=(r^n-s^n)/(r-s)$, we have

$$u_{2^{n+1}}/u_{2^n} = r^{2^n} + s^{2^n},$$

whence, if the left member is represented by a_n, it is easily verified that

$$a_{n+1} = a_n^2 - 2c^{2^k}.$$

Further generalization and applications may be found in Kaplansky, *Lucas's Test for Mersenne Numbers*, This MONTHLY [1945, 188].

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

You Can't Win. By E. E. Blanche. Washington. Public Affairs Press, 1949. 155 pages. $2.00.

This book is concerned with various aspects of gambling in this country. It consists largely of descriptive material and does not attempt very profound analyses of games of chance. The author seems thoroughly convinced of the "pernicious nature" of gambling and that all professional gamblers are essentially dishonest. In some cases the title assertion is supported by accounts of the classical mathematical analyses, but no very complete study is made of any particular game.

In so far as the reviewer noted, odds and probabilities quoted are correct. The reviewer is somewhat dubious concerning an argument which seems to maintain that the second card in a Bingo game is a much poorer investment than the first one. He also feels that the emphasis on confidence men, carnival games and dishonest practices serves to decrease the potential interest of the book to the mathematician. The reader who wants some information concerning the rules of various games and the underlying mathematics and who is somewhat predisposed to the author's general attitude should find this an interesting work.

E. M. BEESLEY

Probability Theory for Statistical Methods. By F. N. DAVID. Cambridge University Press, 1949. 10+230 pages. $3.50.

A leading British statistician has written this book to serve as an elementary textbook on probability theory for students who wish to study statistics. Actually, however, this book discusses a number of topics in the statistical theory of estimation. Two introductory chapters are followed by four chapters on the binomial distribution. Then follow chapters entitled as follows: VII Probabilities a posteriori, Confidence limits, VIII Simple genetical applications, IX Multinomial theorem and simple combinatorial analysis, X Random variables. Elementary laws and theorems, XI Moments of sampling distributions, XII Random variables. Inequalities. Laws of large numbers. Lexis theory, XIII Simple estimation. Markoff theorem on least squares, XIV Further applications of the Markoff theorem. Chapters XV, XVI, XVII deal with characteristic functions.

The author of the book writes with an enthusiasm for her subject which is bound to communicate itself to the student. There are many lively and meaningful examples, drawn from statistical applications, which should convince the

student that the field of application of probability theory goes far beyond problems about gambling or about very dull urns which somehow, in an unexplained manner, always manage to contain balls of several colors. The mathematical argument is kept within the reach of a student with a knowledge of what in the United States is called advanced calculus, or perhaps even less. There is a great lack of good books on probability with these qualities.

The chief fault of the book is that no distinction is made between probability theory and the theory of statistical estimation. The arrangement of the chapters and certain remarks of the author heighten the confusion which must arise in the mind of the student. For example, on page 71 the author precedes what is meant to be (and actually is) the proof of an innocuous result known as Bayes' theorem with the dire warning that "the modern point of view is that, strictly speaking, the theorem is wholly fallacious except under very restricted conditions." (She is obviously referring to the use of Bayes' theorem for statistical estimation.) On page 72 it is said that, "unless $P\{E_t\}$ $\cdots$ is known the formula cannot be valid." Obviously the validity of a formula is not a function of our knowledge.

There are a number of mathematical inadequacies, of which we cite a few. The definition of probability on page 13 is the one hallowed by use in many textbooks, but unsatisfactory nevertheless. On page 22 the author states that "the favorite problem of the probability of drawing balls from an urn is incalculable unless it is assumed that all balls have an equal probability of being drawn. . . ." Why? The definition of a continuous chance variable on page 112 is really that of an absolutely continuous chance variable. The student should not be misled into thinking that every continuous monotonic function is the integral of its derivative. In the proof of Liapounoff's theorem on page 217 there is no need to bound the variances above or to bound the absolute third moment below. No doubt these inadequacies will be corrected in subsequent editions of the book.

J. WOLFOWITZ

Reissner Anniversary Volume: Contributions to Applied Mechanics. Edited by the Staff of the Department of Aeronautical Engineering and Applied Mechanics of the Polytechnic Institute of Brooklyn, J. W. Edwards, Ann Arbor, Michigan, 1949. VIII+493 pp. $6.50.

This volume has been dedicated to Professor Hans J. Reissner of the Polytechnic Institute of Brooklyn on the occasion of his seventy-fifth birthday (Jan. 18, 1949). In addition to a short biography of Professor Reissner and a list of his scientific publications from 1899 to 1948 (92 entries) the volume contains 32 papers contributed by friends and former students of Professor Reissner. Obviously, the space available for this revue does not permit the reviewer to do justice to the individual contributions: a list of the authors and the fields of their contributions will have to suffice. There are 5 papers on aerodynamics (by Bergman, Harrington and Libby, Lew, Riabouchinsky, Tollmien), 5 papers on dynamics (by Goland, Lieber and Hamilton, Oldenburger, Williams, Yuan

and Morduchow), 7 papers on elasticity (by Donnell, Friedrichs, Gran Olsson, E. Reissner, Schleusner, Schnadel, Stoker), 5 papers on aircraft structures (by Hoff, Salerno, Liebowitz, Boley, Nardo), 2 papers on electricity (by Foster, Rüdenberg), 3 papers on mathematical methods (by Geiringer, Grammel, Weinstein), 3 papers on plasticity (by v. Mises, Nadai, Odqvist), and 2 papers on propulsion (by v. Kármán, Torda).

W. PRAGER

Theory and Application of $\int_0^z e^{-x^2}dx$ *and* $\int_0^z e^{-p^2y^2}dy\int_0^y e^{-x^2}dx$. (Part I. Methods of Computation.) By J. B. Rosser. New York, Mapleton House, 1948. 4+192 pages $8.00.

Part II of the monograph *Theory and Application of* $\int_0^z e^{-x^2}dx$ *and* $\int_0^z e^{-p^2y^2}dy\int_0^y e^{-x^2}dx$ has not appeared in full as it contains information which is restricted. All secrecy on the contents of Part I has been lifted and it has been reproduced by arrangement with the Office of Technical Services Department of Commerce. A large share of Part I which is entitled *Methods of Computation* is devoted to means of computation of the single integral $\int_0^z e^{-x^2}dx$. Many known methods of evaluating this integral are given in the first few pages. These, in general, were developed for computation when z^2 was a real or pure imaginary number. The known methods are extended and new ones introduced for handling more general values of z. Ways of estimating the accuracy of the computation based on the various types is also given. Power series, asymptotic series, and continued fractions are the main devices used for making the evaluations. As by-products, methods for evaluating a great many other types of integrals are given; of these we mention only the Fresnel integrals.

Pages 102 to 179 are devoted to various means of evaluating the double integral mentioned in the title. Here, in general, the problem is reduced to evaluation of series and certain single integrals.

The last pages of the book are devoted to two tables. Let the functions $Rr(x)$ and $Ri(x)$ be defined by the equation

$$Rr(x) + iRi(x) = \frac{\sqrt{2}}{\pi}\int_0^\infty \frac{e^{-\pi x^2 y^2}}{1 - iy^2}\,dy, \qquad i = \sqrt{-1}.$$

Table 1 gives the values of $Rr(x)$, $Ri(x)$, $Rr^2(x)+Ri^2(x)$, and $\int_0^x Rr(y)dy$ to 12 decimal places for the range $x = +.06, -.04, \cdots, 3.75$. Table 2 gives the values of

$$e^{-x^2}\int_0^x e^{y^2}dy, \qquad e^{x^2}\int_x^\infty e^{-y^2}dy, \qquad \int_0^x e^{-y^2}dy\int_0^y e^{t^2}dt, \qquad \int_0^x e^{y^2}dy\int_y^\infty e^{-t^2}dt$$

to 10 decimal places for the range $x = -.20, -.15, \cdots, 6$.

Considering the large number of formulas in the book there are relatively few misprints. Unfortunately, the format is not pleasing and there is little moti-

vation of the exposition. However, the book should prove of great value to those interested in computation of definite integrals.

F. G. DRESSEL

Modern-School Solid Geometry. By R. R. Smith and J. R. Clark. Yonkers-on-Hudson, N. Y. World Book Company, 1949. 8+256 pages. $1.76.

In the preface the authors state that the primary purposes to be achieved in the study of solid geometry are four-fold: (1) discovery of geometrical truths of three-dimensional space and their establishment by logical methods; (2) development of computational skill; (3) practice of perceptual analysis leading to the ability to see relationships in space; (4) extension of the understanding of the nature of logic.

This reviewer is in sympathy with this and believes that an able student, aided by a competent instructor, should certainly achieve the first three of the above purposes. It is questionable whether or not he will make much advance in the fourth.

The formal subject matter is contained in some twenty postulates, sixty-one theorems, and one hundred and twenty corollaries. There are nearly a thousand exercises most of which are numerical. For reference, there is a list of plane geometry theorems and constructions. In addition there are four place tables of trigonometric functions and common logarithms.

The format of the book is very pleasing. Diagrams and illustrations are excellent. It should be a very teachable book.

J. N. RICE

Kurvenintegrale and Begründung der Funktionentheorie. By Lothar Heffter. Berlin, Springer, 1948. 4+48 pp. DM 5.40.

This pamphlet is a unified treatment of seven papers of the author appearing between 1902 and 1941. The objective is to prove the Cauchy integral theorem: $\int_c f(z)dz=0$, under the weakest possible hypotheses. The classical proof assumes the *existence* and *continuity* of $f'(z)$ in the domain considered. As is well known, Goursat in 1900 weakened the hypotheses, assuming only the *existence* of $f'(z)$. The author avoids assuming the existence of $f'(z)$, replacing it by a difference equation.

The discussion is reduced to a consideration of real line integrals $\oint f(x, y)dx + g(x, y)dy$ which are in turn reduced to integrals around rectangles with sides parallel to the axes. The vanishing of the line integral around such a rectangle is then equivalent to an elementary difference equation.

The interest of the book lies with the following items:

1. The care and completeness with which the book is written. Chapters I through IV contain the requisite material on functions of one and two variables, making the work self-contained. The book is recommended to the student desiring to deepen his understanding of the Cauchy Theorem and at the same time improve his reading ability in German.

2. The author's proof of the Cauchy-Goursat theorem and his remarks concerning the excessive strength of the assumption that $f'(z)$ exists.

3. Chapter VII which consists of a bibliography (incomplete but containing the important items) and remarks on the contents of the papers listed.

The main contribution of the author could perhaps best be described as a new and easily understood proof of the Cauchy-Goursat Theorem. The fundamental difference equation is not, however, a local condition on $f(z)=u+iv$. On the other hand the Looman-Menchoff Theorem proves the Cauchy Theorem under essentially weaker hypotheses of a local nature (u and v merely possessing partial derivatives satisfying the Cauchy-Riemann equations at "most" points of the domain). The only mention of the Looman-Menchoff Theorem is in Chapter VII where the author remarks that its proof is quite complicated whereas his method leads to a simple proof.

M. E. SHANKS

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1949–50

Mathematics Club, Haverford College

At the four meetings of the Haverford College *Mathematics Club* the following papers were presented:

The fourth dimension, by Irving Hollingshead

The cubic equation, by Charles Hughes

Hyperbolic trigonometry, by John Woll

Constructions: ruler and compass, by John Lamperti.

At each meeting prizes were awarded to members of the club for the best solutions to problems which had been posted previous to the meeting. The presentation of the solutions by the winners constituted a special feature of the club's activities.

Ricci Mathematics Academy, Boston College

The following lectures were presented to the members of the *Ricci Mathematics Academy* during the year:

A plane geometry which cannot be embedded in space, by F. A. Guerrera

The meaning of a mathematics academy, by Rev. A. J. Eiardi, S.J.

Elements of mathematical chance and probabilities, by P. T. Banks

Functions, functionals and operators, by Dr. H. G. Haefeli
Random samples, by Prof. Adelnour Thomas
Nomography, by Dr. F. E. White.

The second annual mid-term social and dance, sponsored by the Academy was held at the Princess Ballroom of the Hotel Somerset in Boston.

The officers of the Academy are: President, John Downes; Vice-President, Thomas Lynch; Secretary, William Durante; Treasurer, Raymond Walton; Program Committee, Julian Demeo, Francis Collins, Joseph Crosby, and Mark Devane.

Mathematics Club, Swarthmore College

The Swarthmore *Mathematics Club* had an active year during 1949–50. The following topics were presented by students, faculty members, and visiting professors:

Curves of unit diameter, by Louis Howard
Plane curves in projective differential geometry, by Prof. Alice Schafer
A locus problem, by Louis Winer
Certain problems in the foundations of geometry, by Prof. J. R. Kline of the University of Pennsylvania
Lattice points and number theory, by Prof. Hans Rademacher of the University of Pennsylvania
The calculus of finite differences, by Robin Briehl
Counting the infinite, by Prof. Philip Carruth
The story of numbers, by Mary Yntema
Complex and hypercomplex numbers, by Prof. Richard Schafer of the University of Pennsylvania
Some theorems on algebraic curves, by Prof. Arnold Dresden
Paper folding, by Robert Eisinger
An equation for Swarthmore, by Prof. C. O. Oakley of Haverford College
Symbolic logic, by Joseph Rutledge.

In addition to its regular meetings, the club sponsored two contests, each consisting of three problems of graded difficulty. A maximum mathematical preparation for each problem was specified as part of the eligibility requirements.

Officers for the year 1949–50 were: President, Robert Eisinger (fall) and Arthur Mattuck (spring); Treasurer, Mary Yntema.

The new officers elected are: President, Joan Berkowitz, Treasurer, Louis Winer.

Mathematics Club, Brown University

The *Mathematics Club* of Brown University, organized in 1915, heard the following papers during the year:

The probabilities in bridge, by Frank Stewart
The Königsberg bridge problem, by Margaret Hashimora
Oscillograph patterns, by George Anderson

Electronic computing machines, by E. F. Moore
Galois theory, by Philip Curtis
Pi, by Barbara Bruce
Mathematical tricks, by Robert Peabody
Continued fractions, by Joseph Crudele
Constructions by means of two parallel lines, by Gerald Freileich.

A winter social and a spring picnic were held. An informal period for discussion during which refreshments were served was held after each regular meeting.

The staff of the Mathematics Club were: Chairman, Alexander Marshall; Treasurer, Joseph Crudele; Secretary, Dorothy Smith; Committee members, Barbara Bruce, Philip Curtis, James Gray, Simone Matteodo, Robert Peabody, Chester Thomas, Barbara Webb: Faculty Representative, Prof. H. Federer.

Mathematics Club, Immaculate Heart College

The following papers were presented by student members of the club:
The humanistic side of mathematics, by Mary Parker
Approximate computation, by Marie Kreuper
History of calculus, by Rita Ricciardi
The life of Newton, by Helen Clark
The life of Leibnitz, by Glenna Christie
Playing with numbers, by Joan Pfisterer
Review of G. H. Hardy's A Mathematician's Apology, by Rita Anderson
Magic squares, by Mary Heffern
The slide rule, a motion picture.

At each meeting a period was devoted to mathematical recreations after which refreshments were served.

Officers for 1949–50 were: President, Mary Claire Heffern; Secretary-Treasurer, Glenna Christie.

Pi Mu Epsilon, University of Washington

The following papers were presented at regular meetings of the *Washington Beta* chapter of *Pi Mu Epsilon* during 1949–50:

Proof of the Weierstrass approximation theorem, by Dr. Edwin Hewitt
Proof that e is a transcendental number (a modification of Edourd Maillet's proof), by Halton McQueen
The fifth postulate and some consequences of its omission, by Calvin Thimsen
Spherical conics, by Dr. Mo-Yeh
Statistical aspects of the Geiger counter, by Richard Titman
Some properties of continued fractions, by C. Y. Lee
Evaluating an infinite product, by Richard Woollett
Application of calculus in high school physics, by Orlando Furno.

The regular officers for the 1950–51 will be elected in the Fall. However, Richard Woollett has been elected temporary Director and Orlando Furno has been elected temporary Secretary-Treasurer.

Kappa Mu Epsilon, The College of Wooster

During the academic year 1949–50, the following papers were presented at meetings of the *Ohio Beta* chapter of *Kappa Mu Epsilon*:

Mathematics and philosophy, by Prof. Wilford Bower

The calculus of variations, by Prof. M. P. Fobes

Trisection of the angle, by Charles Sauder

The cooperative engineering plan of the University of Cincinnati, by Prof. Emeritus J. H. Kindle of the University of Cincinnati.

The program also included a picnic at the home of Prof. C. O. Williamson, two initiations and social meetings.

Officers for 1950–51 are: President, Richard Holroyd; Vice-President, Morley Russell; Secretary, Milton Snyder; Treasurer, James Kister; Sponsor, Prof. M. P. Fobes.

Mathematics Club, Albany State College for Teachers

The program of the *Mathematics Club* at the New York State College for Teachers at Albany was composed of talks by students, a panel discussion led by students, a trip to the plant of General Electric at Schenectady, and a report of a professional meeting by a faculty member.

The following talks were given:

The Moebius strip, by Wayne Palmer

What other mathematics clubs have done, by Virginia Roth

Celestial navigation, by Jack Mooney

The unit circle, by Virginia McDonald

The binomial theorem, by Ifigenia Aliferis.

The panel discussion on high school mathematics clubs was conducted by Shirley Casler and Henrietta Daub, a discussion that was prompted by a desire to learn how to organize and develop a mathematics club in high school. The discussion, which involved an interesting piece of research, provoked considerable interest.

The trip to the General Electric plant was made to see the differential analyzer in operation and to learn something about how it worked. Dr. Caroline Lester gave the report of the Christmas meeting in New York.

The officers for the year were: President, Ruth Marschner, Secretary, Marjorie Hills; Treasurer, Rosemary Lessard; Faculty Adviser, Nura Turner.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

VACANCIES AT U. S. NAVAL ORDNANCE TEST STATION

The United States Naval Ordnance Test Station, Inyokern, China Lake, California, is recruiting a large number of technical personnel as a result of the present world situation. The Station is establishing a group of senior mathematicians and statisticians for Weapons Evaluation work. It is also seeking theoretical physicists with specialized interests in Solid State Physics, Atomic Physics, Fluid Mechanics, Hydro or Aero Dynamics, and in almost any field of Classical Physics. Vacancies exist also in most fields of Engineering. There are openings for recent graduates with degrees ranging from Bachelor of Arts to Doctor of Philosophy in the above fields.

Full information and application forms can be obtained from: J. B. Hamilton, Head Professional Placement Branch, U.S.N.O.T.S., Inyokern, China Lake, California.

RESEARCH FELLOWSHIPS IN PSYCHOMETRICS

The Educational Testing Service is offering for 1951–52 its fourth series of research fellowships in psychometrics leading to the Ph.D. degree at Princeton University. Open to men who are acceptable to the Graduate School of the University, the two fellowships each carry a stipend of $2,375 a year and are normally renewable.

Fellows will be engaged in part-time research in the general area of psychological measurement at the offices of the Educational Testing Service and will, in addition, carry a normal program of studies in the Graduate School. Competence in mathematics and psychology is a pre-requisite for obtaining these fellowships. Applications must be submitted before January 19, 1951. Information and application blanks may be obtained from: Director of Psychometric Fellowship Program, Educational Testing Service, 20 Nassau Street, Princeton, New Jersey.

GRADUATE INSTITUTE FOR APPLIED MATHEMATICS AT INDIANA UNIVERSITY

Indiana University has recently established a Graduate Institute for Applied Mathematics. The work of the Graduate Institute will deal primarily at present with problems in gas dynamics, hydrodynamics and turbulence. Members of the Graduate Institute will operate under exceptionally favorable teaching arrangements to allow ample time for their own researches and for personal contacts with students. It is advisable for students entering the Graduate Institute to have the M.A. degree and at least one year of graduate work in mathematics is required. A number of research assistantships open to students who wish to

work toward the Ph.D. degree in applied mathematics have been established by Indiana University under contract with the Office of Naval Research.

CONFERENCES

The second annual Conference of Teachers of Mathematics was held at Marshall College, Huntington, West Virginia, on March 24–25, 1950. At this conference the teachers of college mathematics of West Virginia organized the Association of College Mathematics Teachers of West Virginia. The following officers were elected: President, Professor A. V. Kozak of Concord State College; Secretary-Treasurer, Professor J. K. Sterrett, Marshall College.

On March 18, 1950 Millersville State Teachers College, Millersville, Pennsylvania, held a High School Mathematics Conference. Another conference is scheduled for March 10, 1951.

PERSONAL ITEMS

An honorary degree of Doctor of Laws was conferred upon Dean E. A. Bailey by Florida Southern College on March 17, 1950 for his twenty-five years of service as Dean of LaGrange College.

Arizona State College, Flagstaff, reports the following: Dr. Arthur Adel was an invited speaker at the Centenary Meeting of the Royal Meteorological Society which was held on March 27–April 4, 1950 at Oxford and London, England; Professor J. H. Butchart was the travelling lecturer for the Southwestern Section of the Association for the year 1949–50.

Arizona State College, Tempe, announces: Associate Professor F. C. Gentry has been promoted to a professorship; Assistant Professor R. S. Fouch of the University of Chicago has been appointed to an assistant professorship.

Baldwin-Wallace College announces the following: Associate Professor P. R. Annear, head of the Department of Mathematics and Astronomy, has been promoted to a professorship; Assistant Professor D. L. Robb has been promoted to an associate professorship in the Department of Mathematics and Astronomy.

Fullerton Junior College, California, reports: Mr. C. A. Worsley, chairman of the division of Mathematics, Science and Engineering, has retired; Mr. L. W. Wheatley has succeeded him.

Indiana University makes the following announcements: Professor T. Y. Thomas, Professor Eberhard Hopf and Assistant Professor David Gilbarg have been transferred from the Mathematics Department to the newly established Graduate Institute for Applied Mathematics; Dr. C. A. Truesdell of the Naval Research Laboratory, Washington, D. C., has been appointed to a visiting professorship for the year 1950–51.

At Montana State University: Dr. George Marsaglia, previously graduate student at Ohio State University, has been appointed to an assistant professorship; Mr. G. A. Craft, graduate student at the University of Indiana, and Mr. A. L. Duquette, graduate student at Columbia University, have been appointed to instructorships; Dr. H. B. Mann of Ohio State University was a

visiting lecturer during the 1950 Summer Session.

North Dakota Agricultural College announces the following: Instructor G. A. Walrath has been promoted to an assistant professorship; Mr. David Gosslee has been appointed to an instructorship.

North Georgia College reports the following: Professor J. C. Barnes, head of the Department of Mathematics, has retired; Dr. M. C. Wicht of Louisiana State University has been appointed Professor and Head of the Department of Mathematics; Mr. B. F. Hadnot, graduate student at the University of Georgia, has been appointed to an instructorship; Associate Professor G. N. Wollan is engaged in graduate study at the University of Georgia.

Pomona College reports: Professor C. G. Jaeger has been elected Governor of the Southern California Section of the Association; Mr. C. J. A. Halberg, Jr. has left Pomona College and is doing graduate work at the University of California at Los Angeles.

Trinity College announces the appointments of Mr. R. C. Stewart and Mr. J. N. Williams to instructorships.

The United States Naval Postgraduate School announces the following promotions: Associate Professors W. E. Bleick and C. C. Torrance to professorships; Assistant Professor Walter Jennings to an associate professorship.

At University of Bridgeport: Associate Professor Simon Mowshowitz has been promoted to the position of Professor and Chairman of the Department of Mathematics; Professor E. M. Bigsbee, formerly chairman of the Department of Mathematics, has been appointed Dean of the Junior College of Connecticut.

University of California, Santa Barbara College, reports the following: Associate Professor S. E. Rauch, chairman of the Department of Mathematics, is on sabbatical leave during the year 1950–51 and is studying in France; Assistant Professor L. F. Walton is Acting Chairman of the Department for the year.

University of Washington makes the following announcements: Associate Professor Z. W. Birnbaum has been promoted to a professorship; Assistant Professor Edwin Hewitt has been promoted to an associate professorship; Professor Kôsaku Yosida of the Mathematical Institute of Nagoya University has been appointed Walker-Ames Professor for the autumn quarter; Dr. F. H. Brownell III of Princeton University and Dr. R. B. Leipnik of the Institute for Advanced Study have been appointed to assistant professorships; Dr. R. P. Peterson of the University of California at Los Angeles has been appointed to an instructorship; Assistant Professor Edward Paulson has returned from a leave of absence spent at Columbia University; Lecturer D. M. Sandelius has returned to Sweden; Dr. Yeh Mo has returned to China.

At University of Western Ontario: Professor H. R. Kingston has resigned from the position of Head of the Department of Mathematics and Astronomy to accept the position of Principal of University College and Dean of Arts and Science; Associate Professor G. R. Magee has been promoted to a professorship and named Head of the Department of Mathematics and Astronomy; Associate Professor R. H. Cole has been promoted to a professorship; Dr. G. P. Henderson has been promoted to the position of Lecturer.

Dr. Shmuel Agmon has been appointed Visiting Lecturer in Mathematics at the Rice Institute.

Dr. Gerald Berman has been appointed to an instructorship at Illinois Institute of Technology.

Assistant Professor F. C. Bolser of Florida State University has accepted an assistant professorship at the University of Tampa.

Mr. C. B. Eaton has been appointed to an instructorship at Pasadena City College.

Associate Professor C. H. Fischer of the University of Michigan has been promoted to a professorship of actuarial mathematics in the College of Literature, Science and the Arts and has also been appointed Professor of Insurance in the School of Business Administration.

Professor J. S. Frame, head of the Department of Mathematics of Michigan State College, is spending the year 1950–51 at the Institute for Advanced Study.

Instructor Hunter Hardman of Marshall College has been promoted to an assistant professorship.

Assistant Professor C. B. Helms of Pennsylvania Military College has been promoted to an associate professorship.

Assistant Professor P. B. Johnson has been appointed Acting Chairman of the Department of Mathematics of Occidental College and promoted to an associate professorship.

Assistant Professor R. E. Lowney has been promoted to an associate professorship at Montana State College.

Mr. R. C. Lundquist has been appointed to an instructorship at State Teachers College, Frostburg, Maryland.

Assistant Professor Walter Lyche of Augustana College, Sioux Falls, South Dakota, is now a graduate student at the University of Minnesota.

Assistant Professor H. C. McKenzie of Western State College of Colorado has been appointed to an instructorship at the University of Colorado.

Mr. R. B. Merrill has accepted a position as statistician in the Quality Control Department of Wisconsin Steel Works, Chicago, Illinois.

Associate Professor N. H. Mewaldt, Northern State Teachers College, Aberdeen, South Dakota, has been elected to membership in the Beadle Club, an honorary educational society in South Dakota.

Dean E. F. Moore of Hannibal-LaGrange College has been appointed Professor of Statistics at Baylor University.

Dr. H. B. Ribeiro of the University of California at Berkeley has been appointed to an associate professorship at the University of Nebraska.

Associate Professor W. H. Spragens, Jr. of Florida State University has been appointed to an associate professorship at the University of Mississippi.

Professor I. L. Stright of State Teachers College, Indiana, Pennsylvania, is on leave of absence for the year 1950–51 and is teaching at the Higher Teachers Training College, Baghdad, Iraq.

Miss Mabel G. Whiting, head of the Department of Mathematics of Santa Ana Junior College, California, has retired.

Professor Henry Blumberg of Ohio State University died on June 28, 1950. He was a charter member of the Association.

Professor W. J. Bruns of Utica College, Syracuse University, died July 18, 1950.

Assistant Professor R. I. Pepper, acting head of the Department of Mathematics of Winthrop College, died July 2, 1950.

Miss Elinor Schneider, instructor at Pasadena City College, died on January 1, 1950.

Professor Emeritus E. W. Sheldon of Acadia University died June 14, 1950. He was a charter member of the Association.

Professor W. M. Smith, head of the Department of Mathematics of Lafayette College and a charter member of the Association, died on July 3, 1950.

Professor G. R. Trott of the University of Mississippi died on July 8, 1950. He was Chairman of the Louisiana-Mississippi Section for the year 1949–50

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following one hundred and one persons have been elected to membership by the Board of Governors on applications duly certified.

W. A. BALLOU, Ph.D. (Pennsylvania) Professor, Morehead State College, Ky.

E. H. BATHO, Student, Fordham University, New York, N. Y.

J. R. BATT, Student, University of Buffalo, N. Y.

S. K. BERBERIAN, M.S. (Michigan S.) Grad. Student, Michigan State College, East Lansing, Mich.

H. G. BERGMANN, Ph.D. (N.Y.U.) Instructor, City College of the City of New York, N. Y.

JEROME BERKOWITZ, B.S. (C.C.N.Y.) Research Assistant, New York University, N. Y.

R. J. BICKEL, M.A. (Northwestern) Asst. Professor, Drexel Institute of Technology, Philadelphia, Pa.

H. F. BLIGH, M.S. (Syracuse) Asst. Professor, Hobart College, Geneva, N. Y.

C. R. BONNELL, B.S. (Central Missouri C.) Instructor, Missouri School of Mines & Metallurgy, Rolla, Mo.

J. C. BRADFORD, Student, North Texas State College, Denton, Texas

C. L. BRADSHAW, B.S. (Tennessee Poly. Inst.) Grad. Assistant, University of Tennessee, Knoxville, Tenn.

MRS. HELEN E. BROWN, A.M. (Michigan) Instructor, University of Tennessee, Knoxville, Tenn.

C. C. BUCK, M.S. (Michigan) Instructor, University of Nebraska, Lincoln, Nebr.

M. D. BURACK, M.S. (DePaul) Instructor, Wilson Junior College, Chicago, Ill.

SARAH M. BURKHART, M.A. (Kansas) Asst. Professor, University of Tulsa, Okla.

B. A. BURRILL, Student, University of Buffalo, N. Y.

G. J. CAMPBELL, Student, University of Detroit, Mich.

R. S. CHRISTIAN, M.S. (Washington U.) Asst. Professor, Georgia Institute of Technology, Atlanta, Ga.

S. I. CIOLKOWSKI, Student, New York University, N. Y.

J. P. CORY, Student, New York University, N. Y.

Q. A. DARMSTADT, Student, Oberlin College, Ohio

D. J. DAVIS, B.S. (Harvard) Mathematician, Ballistic Research Laboratories, Aberdeen Proving Ground, Md.

RUTH M. DAVIS, Student, American University, Washington, D. C.

ANNE F. DOWNEY, Student, Regis College, Weston, Mass.

R. H. EDWARDS, Ph.D. (Illinois Inst. of Tech.) Research Physicist, Hughes Aircraft Co., Culver City, Calif.

J. O. ETHRIDGE, B.A. (Southwestern at Memphis) Grad. Assistant, University of Georgia, Athens, Ga.

T. C. FRANCIS, Accountant, Hanley & Bird, Bradford, Pa.

LOUIS GATT, Student, University of California at Los Angeles, Calif.

R. H. GILLESPIE, Student, Albion College, Mich.

N. A. GOLDSMITH, M.A. (Illinois) Asso. Professor, Illinois Wesleyan University, Bloomington, Ill.

H. H. GOODE, M.A. (Columbia) Supervisor, Aerophysics Group, Aeronautical Research Center, University of Michigan, Ann Arbor, Mich.

E. E. GREEN, B.A. (Buffalo) Grad. Student, University of Buffalo, N. Y.

R. E. GREEN, Student, Kansas State Teachers College, Pittsburg, Kan.

B. W. GREGORY, Student, Oregon State College, Corvallis, Ore.

J. C. HARDEN, JR., M.A. (Tennessee) Instructor, Clemson College, S. C.

J. J. HART, B.S. (Memphis S. C.) Grad. Student, University of Alabama, University, Ala.

R. F. HAYS, A.B. (Transylvania) Grad. Student, University of Kentucky, Lexington, Ky.

P. G. HODGE, JR., Ph.D. (Brown) Asst. Professor, University of California at Los Angeles, Calif.

SZE-TSEN HU, Ph.D. (Manchester, England) Asst. Professor, Tulane University, New Orleans, La.

H. A. JAMES, M.S. (DePaul) Instructor, Wilson Junior College, Chicago, Ill.

W. P. JONES, M.S. (Iowa) Grad. Student, State University of Iowa, Iowa City, Iowa

C. E. KERR, B.A. (LaSalle) Instructor, LaSalle College, Philadelphia, Pa.

C. A. KIRBY, A.B. (California) Chief Computer, Western Geophysical Co., Sacramento, Calif.

E. L. KRETSCHMAR, JR., B.S.Ed. (Florida) Grad. Student, University of Florida, Gainesville, Fla.

MORTON KUPPERMAN, M.A. (George Washingington) Statistician, Office of the Surgeon General, Department of the Army, Washington, D. C.

JOSEPH KUSHNER, Student, New York University, N. Y.

G. D. KYLE, Ph.D. (Minnesota) Professor, A. M. & N. College, Pine Bluff, Ark.

R. D. LARSSON, A.B. (Maine) Instructor, Clarkson College of Technology, Potsdam, N. Y.

L. L. LAYTON, M.S. (Illinois) Asst. Professor, James Millikin University, Decatur, Ill.

MARGARET M. LINNEY, Student, Regis College, Weston, Mass.

A. E. LIVINGSTON, M.A. (Oregon) Grad. Assistant, University of Oregon, Eugene, Ore.

S. S. LOTZ, Lieutenant, U. S. Navy; Student, Tufts College, Medford, Mass.

J. R. MACY, Student, University of Chicago, Ill.

JOY MAHACHEK, Ph.D. (Pittsburgh) Head of Department, State Teachers College, Indiana, Pa.

C. B. MAILE, JR., A.B. (Hofstra) Metallurgist, Sperry Gyroscope Co., Lake Success, N. Y.

ZELJKO MARKOVIC, Sc.D. (Zagreb) Professor, University of Zagreb, Yugoslavia

H. B. MARKS, Student, Southern Methodist University, Dallas, Tex.

H. F. MATTSON, JR., Student, Oberlin College, Ohio

MARY L. MCLAUGHLIN, Student, Regis College, Weston, Mass.

D. ST. C. MELVIN, Student, University of Maryland, College Park, Md.

WILLIAM MENDENHALL, 3RD, M.S. (Bucknell) Grad. Assistant, Pennsylvania State College, Pa.

C. F. MILLER, Student, George Pepperdine College, Los Angeles, Calif.

IRWIN MILLER, Student, Alfred University, N. Y.

W. L. MURDOCK, A.B. (Cornell) Instructor, Cornell University, Ithaca, N. Y.

E. F. MYERS, A.B. (Geneva) Instructor, University of Pittsburgh, Pa.

R. K. NEUMANN, Student, George Washington University, Washington, D. C.

W. A. NIELSEN, M.Ed. (U.C.L.A.) Instructor, Bakersfield Junior College, Calif.

L. P. OLBRICH, Student, St. John's University, Brooklyn, N. Y.

J. H. OPPENHEIM, Ph.B. (Chicago) 409 E. Green Street, Connellsville, Pa.

J. E. OXLEY, Student, Cooper Union, New York, N. Y.

E. J. PALMBACH, B.S. (Lawrence) 1412½ N. Durkee Street, Appleton, Wis.

H. G. PARKER, M.S. in Ed. (Alabama Poly. Inst.) Professor, Southern Union College, Wadley, Ala.

H. A. PATIN, S.M. (Chicago) Instructor, Wilson Junior College, Chicago, Ill.

ELISABETH J. PETSCHEK, A.B. (Vassar) Student, Vassar College, Poughkeepsie, N. Y.

LOUISA G. PLUMMER, M.S. (Minnesota) Asst. Professor, West Liberty State College, W. Va.

MARIE E. PORTER, M.S. (Stanford) Instructor, California State Polytechnic College, San Luis Obispo, Calif.

W. N. PRENTICE, A.M. (Brown) Instructor, Clarkson College of Technology, Malone Branch, N. Y.

C. H. REID, JR., M.Ed. (Colorado) Teacher, Sargent Consolidated School, Monte Vista, Colo.

H. B. RIBEIRO, D.Sc. (Zurich) Instructor, University of California, Berkeley, Calif.

J. P. RUSSELL, Sc.M. (Brown) Asst. Professor, Polytechnic Institute of Brooklyn, N. Y.

W. G. SAUNDERS, B.S. (Randolph-Macon) Asst. Instructor, Virginia Military Institute, Lexington, Va.

SISTER MARY CHARLES, M.A. (Catholic) Instructor, Immaculate Heart College, Los Angeles, Calif.

D. J. SMITH, M.S. (Notre Dame) Actuarial Trainee, Royal-Liverpool Group of Insurance Co.'s, New York, N. Y.

A. L. STAMPER, Student, Eastern Kentucky State College, Richmond, Ky.

H. R. STEVENS, Student, University of Buffalo, N. Y.

E. W. SWOKOWSKI, M.S. (Marquette) Teaching Assistant, University of Wisconsin, Madison, Wis.

A. V. SYLWESTER, Student, Oregon State College, Corvallis, Ore.

JEAN E. TEATS, M.A. (Pittsburgh) Asst. Professor, University of Pittsburgh, Pa.

D. E. THORO, Student, University of Florida, Gainesville, Fla.

H. S. THURSTON, Ph.D. (Wisconsin) Professor, University of Alabama, University, Ala.

N. W. TORGERSON, Student, George Pepperdine College, Los Angeles, Calif.

BRADFORD TYE, M.Sc. (N.Y.U.) Professor, Bethany College, W. Va.

G. F. VAUGHN, Student, Butler University, Indianapolis, Ind.

KATHLEEN V. VINCENT, Student, Regis College, Weston, Mass.

N. D. VLACHOS, A.M. (Pennsylvania) Instructor, Drexel Institute of Technology, Philadelphia, Pa.

C. R. WAMPOLE, Student, St. John's University, Brooklyn, N. Y.

W. G. WEIDEMAN, B.E.E. (Marquette) Grad. Student, Marquette University, Milwaukee, Wis.

W. J. WELLS, M.S. (Michigan) Instructor, State Teachers College, Mankato, Minn.

G. W. WHITE, M.A. (California) Professor, College of the Pacific, Stockton, Calif.

MARY WHITE, Student, Regis College, Weston, Mass.

W. G. YOUNKIN, Student, St. Ambrose College, Davenport, Iowa

SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Association was held at the University of Richmond in Richmond, Virginia, on Saturday, May 6, 1950. Professor L. M. Kells, Chairman of the Section, presided at both the morning and afternoon sessions.

Fifty-seven people registered attendance at the meeting including the following forty-three members of the Association: M. W. Aylor, R. P. Bailey, W. E. Barnes, R. H. Bing, J. W. Blincoe, Eleanor C. Calkins, Randolph Church, G. R. Clements, C. H. Frick, B. C. Getchell, M. V. Gibbons, Michael Goldberg, R. A. Good, E. S. Grable, Clem Grabner, Jr., E. C. Gras, D. W. Hall, J. R. Hammond, Isabel Harris, J. E. Ikenberry, S. B. Jackson, Walter Jennings, Sidney Kaplan, L. M. Kells, V. L. Klee, Jr., D. B. Lowdenslager, Joseph Milkman, J. F. Milos, A. K. Mitchell, Dewey Moore, Mary Pettus, F. M. Pulliam, O. N. Robinson, R. C. Simpson, Jr., T. McN. Simpson, Jr., E. R. Sleight, Robert Edward Smith, John Tyler, C. H. Wheeler, III, P. M. Whitman, G. T. Whyburn, Verba M. Wood, J. W. Wright.

The following officers of the Section were elected for the year 1950–51: Chairman, O. J. Ramler, Catholic University; Secretary, S. B. Jackson, University of Maryland; Executive Committee, E. R. Sleight, University of Richmond, and R. P. Bailey, U. S. Naval Academy. Professor Kells announced that the Bureau of Standards in Washington, D. C. has invited the Section to meet there for one of the meetings in 1951. He also announced that Professor G. R. Clements of the Naval Academy has been elected Regional Governor from this Section for the next three years. The next meeting of the Section will be in December 1950 at a place to be announced later.

The morning session consisted of five contributed papers, as follows:

1. *The constants of integration of a differential equation as differential forms*, by Professor John Tyler, United States Naval Academy.

If the equation $y=f(x, c_0, c_1, \cdots, C_K)$ is differentiated K times, the resulting $K+1$ equations may be solved for the C_i as functions of x, y, and the first K derivatives of y. Thus the c_i are expressed as differential forms in x, y, and its derivatives, namely, $c_i=c_i(x, y, y', \cdots y^{(K)})$, $i=0, 1, \cdots, K$. These differential forms have certain interesting properties, notably that $dC_i=\lambda_i D dx$ where $D=0$ is a differential equation of order $K+1$ satisfied by the function f, and the λ_i are integrating factors. If the C_i are substituted into the equation $F(C_0, C_i, \cdots C_K)=C$, where F is an arbitrary function, there arises an equation of Clairaut type for which the function f is a solution.

2. *Some characterizations of compactness*, by Professor V. L. Klee, Jr., University of Virginia.

Niemytzke and Tychonoff showed by an adaptation of the Alexandroff-Urysohn metrization proof that a metrizable space is compact if and only if it is complete in each of its allowable metrics. Hausdorff proved that each allowable metric on a closed subset of a metric space can be extended to an allowable metric on the whole space. By use of the fact, he gave a simple proof of the above result, and showed that "complete" can be replaced by "bounded." Vaughan improved on these characterizations by further use of Hausdorff's extension theorem. In this note there is established an elementary embedding theorem which is then applied to give rather simple and pictorial proofs

for Vaughan's principal results as well as to establish two new characterizations of compactness which seem not to be directly obtainable from the extension theorem.

3. *Rotors in polyhedra*, by Mr. Michael Goldberg, Bureau of Ordnance, Navy Department.

A rotor in a polyhedron was defined as a closed convex surface which can be rotated continuously through every orientation within a given polyhedron while keeping contact with all the faces of the polyhedron. Rotors in cubes, which are rotors in other rhombic parallelepipeds also, have been studied by Minkowski. By elementary methods, Mr. Goldberg derived a class of non-spherical rotors for the regular tetrahedron and exhibited working models of them. An outline was presented of E. Meissner's derivation of all the possible rotors for the regular polyhedra. For the dodecahedron and the icosahedron, the sphere is the only possible rotor.

4. *On finding* $\sqrt{a+bi}$, by Mr. Sidney Kaplan, U. S. Department of Labor.

The standard methods for computing $\sqrt{a+bi}$, namely De Moivre's theorem and the solution of a system of quadratics, both present difficulties for machine computation, the first because of the number of steps, the second because of the loss of significant figures. A slight modification of the second method, suggested by Professor Garrett Birkhoff, proves to be both accuracy preserving and labor saving.

5. *Undergraduate research*, by Professor E. R. Sleight, University of Richmond.

This paper dealt with the need and value of a program that will challenge the student of unusual ability. Professor Sleight feels that some form of undergraduate research has created a great deal of interest, and is already operating in many institutions. He discussed the introduction and maintenance of this type of work in various institutions with which he has had acquaintance.

6. *Some applications of partitionings*, by Professor R. H. Bing, University of Wisconsin and University of Virginia.

This was an invited address. Suppose X is a compact locally connected metric continuum. An ϵ-partitioning of X is a finite ϵ-collection $G=(g_1, g_2, \cdots, g_n)$ of mutually exclusive open subsets of X such that $\sum g_i$ is dense in X. If $G_1, G_2, \cdots$ is a sequence of partitionings, it is called a decreasing sequence if G_{i+1} refines G_i and the diameters of the elements of G_i approach 0 with $1/i$. The following applications of the fact that X has a decreasing sequence of partitionings were discussed. There is a convex metric for X. Each pair of points of X belongs to an arc whose complementary domains in X have property S. There is a countable basis for X such that the intersections of the elements of this basis are connected and uniformly locally connected. The notion of partitionings may be used to characterize simple closed curves, simple surfaces, and simple solids.

S. B. JACKSON, *Secretary*

APRIL MEETING OF THE UPPER NEW YORK STATE SECTION

The annual meeting of the Upper New York State Section of the Mathematical Association of America was held at Syracuse University, Syracuse, New York, on Saturday, April 22, 1950. The Chairman of the Section, Professor Walter H. Durfee of Hobart and William Smith Colleges, presided at the morning session; the Vice-Chairman, Professor B. C. Patterson of Hamilton College, presided at the afternoon session. Tea was served at the conclusion of the afternoon session.

One hundred twenty-five persons attended the meeting including the follow-

ing seventy-four members of the association: E. B. Allen, H. P. Atkins, D. C. Barton, M. R. Bates, R. W. Beals, Jr., R. L. Beinert, Dorothy L. Bernstein, W. W. Bessell, Jr., C. G. Burger, Jr., F. J. H. Burkett, E. A. Butler, Ethel B. Callahan, J. D. Campbell, I. S. Carroll, W. B. Carver, P. F. Clemens, Nancy Cole, E. J. Downie, Walter H. Durfee, Jean B. Feidner, C. W. Foard, A. H. Fox, J. E. Freund, H. M. Gehman, B. H. Gere, H. E. Goheen, R. D. Gordon, Lillian Gough, J. B. Greeley, N. G. Gunderson, May N. Harwood, M. B. Haslam, Lucille F. Hetzelt, R. G. Hill, H. K. Holt, A. W. Jones, J. R. F. Kent, D. E. Kibbey, R. D. Larsson, J. V. Limpert, R. R. R. Luckey, June M. McArtney, L. L. Merrill, Rudolf Meyer, F. E. Milliman, Harriet F. Montague, Mabel D. Montgomery, D. S. Morse, Abigail M. Mosey, C. W. Munshower, C. V. Newsom, Ruth B. Noller, B. C. Patterson, Ruth M. Peters, Theresa L. Podmele, M. H. Protter, J. F. Randolph, H. W. Reddick, C. E. Rhodes, M. K. Rosskopf, Edith R. Schneckenburger, Joseph Seidlin, Sister Noel Marie, William A. Smith, Ruth W. Stokes, Mary C. Suffa, G. L. Tiller, J. P. van Alstyne, S. I. Vrooman, R. J. Walker, J. F. Wardwell, E. T. Welmers, Ina W. Welmers, R. C. Yates.

A progress report was received from the committee studying the relation between secondary school and college mathematics in New York State, Professor M. F. Rosskopf, Syracuse University, Chairman.

The following officers were elected: Chairman, B. C. Patterson, Hamilton College; Vice-Chairman, C. W. Munshower, Colgate University; Secretary, N. G. Gunderson, University of Rochester. The 1951 meeting will be held at Hamilton College, Clinton, New York, and the 1952 meeting at Hobart and William Smith Colleges, Geneva, New York.

The following papers were presented:

1. *Models and methods*, by Professor R. C. Yates, U. S. Military Academy.

A few items intended to stimulate student interest in both subject matter and technique were presented. These included folding the conics, the current misuse of Euclid's compass, dissection of plane polygons, and simple derivations of normal acceleration, the Euler exponential form, and the probability integral. Several models were exhibited as suggested aids to the teacher.

2. *The problem of eight points*, by Professor W. B. Carver, Cornell University.

This paper has been published in this MONTHLY, May, 1950.

3. *Some educational problems in New York State*, by Dr. C. V. Newsom, Assistant Commissioner for Higher Education, New York State.

This speaker gave a brief discussion of the state educational program. In particular, emphasis was given to the role that mathematics departments might play in the training of elementary and secondary teachers. Comments were also made upon the inferior level of college instruction in many institutions.

4. *On space frames and plane trusses*, by Professor H. E. Goheen, Syracuse University.

The speaker proposed the following definition for use by the student of elementary mechanics: A space frame (or plane truss) will be called statically determinant if and only if the $3j$ (or $2j$)

conditions for equilibrium at the j joints determine uniquely the m bar forces and the 6 (or 3) forces necessary to restrain the structure.

5. *Analytic solutions of non-linear differential equations*, by R. D. Larsson, Clarkson College of Technology.

The purpose of this paper is to present a general method for solving non-linear differential equations with polynomial coefficients. The method of Laplace transforms has been used to achieve an analytical solution. Some equations of this type have been solved by the method of approximations; but the method presented here gives the solution in a form in which the role of the boundary values is apparent for any particular case. The general solution reduces to known results in special cases. For the purpose of illustration a second order differential equation with constant coefficients, non-linear in the first derivate, is used.

6. *Difference and differential equations in probability theory*, by Dr. Murray Rosenblatt, Cornell University, introduced by Professor R. J. Walker.

Models of 1 and 2-dimensional Brownian motion and corresponding 1 and 2-dimensional random walks are defined. The probability density of the particle in Brownian motion is a fundamental solution of a parabolic differential equation while the probability that the particle in the random walk be at a lattice point is a solution of a corresponding difference equation. Absorption problems for Brownian motion and the random walk are discussed and related to boundary value problems of the corresponding differential and difference equations. This is made plausible by considering in some detail the 1-dimensional Brownian and random walk. The probability distribution of the Brownian motion is a limit of the solution of the random walk problem.

7. *The tutorial system at the University of Buffalo*, by Professor Harriet F. Montague.

The tutorial system at the University of Buffalo provides that all students in their last two years of undergraduate work shall be given an opportunity for independent study and investigation in addition to regular course work. The tutorial program of the department of mathematics is divided into three parts. In the junior year the students meet in small groups to study and discuss the history and foundations of mathematics, and its various branches. The first half of the senior year is devoted to independent work on an individual research problem. The last half of the senior year is concerned with a general review and correlation of all courses taken in mathematics. This is climaxed by a comprehensive examination.

C. W. MUNSHOWER, *Secretary*

APRIL MEETING OF THE KANSAS SECTION

The thirty-fifth annual meeting of the Kansas Section of the Mathematical Association of America was held at Kansas State Teachers College, Pittsburg, Kansas, on Saturday, April 22, 1950. Sessions were held in the morning and afternoon. Professor R. G. Smith presided at these sessions. The morning session was a joint meeting with the Kansas Association of Teachers of Mathematics.

The attendance was one hundred sixty-seven including the following fifty members of the Association: R. W. Babcock, C. H. Brown, W. R. Cowell, L. E. Curfman, Paul Eberhardt, W. C. Foreman, Albert Furman, W. H. Garrett, F. C. German, Laura Z. Greene, B. H. Gundlach, J. R. Hanna, N. C. Hoover, S. L. Hull, Emma Hyde, Edward Keegan, Helen F. Kriegsman, L. E. Laird,

C. F. Lewis, K. L. Loewen, Anna Marm, Margaret E. Martinson, Eloise McCord, Sabrina Morlan, Thirza A. Mossman, John D. Neff, Agnes E. Nibarger, S. T. Parker, J. M. Pike, P. S. Pretz, G. B. Price, O. M. Rasmussen, C. B. Read, C. A. Reagan, L. M. Reagan, D. P. Richardson, R. G. Sanger, Sister M. Helen, Sister Jeanette, G. W. Smith, R. G. Smith, E. C. Stopher, W. T. Stratton, Wilmot Toalson, C. B. Tucker, Gilbert Ulmer, E. B. Wedel, A. E. White, Ferna E. Wrestler, P. M. Young.

The program consisted of the following papers:

1. *Mathematics in general education*, by Mr. H. W. Charlesworth, East High School, Denver, Colorado.

This paper was given in the interest of better mathematics teaching in both secondary schools and colleges. Mathematics must make its maximum contribution to all students, recognizing their individual interests and needs. College instruction must give more attention to the art of teaching. Mathematics education for the prospective teacher of mathematics must be sound mathematically and educationally. Teacher training schools have a great responsibility in properly training teachers for this new era in mathematics. Mathematics can make its proper contribution to the general education of the individual and yet retain its value as a mode of thought. The training of our leaders in mathematics and science must begin early and be done more expertly. Greater coordination of high school and college teaching is needed. The Mathematical Association of America and The National Council of Teachers of Mathematics should do more cooperative work on solving their common problems arising out of general education.

2. *Sign lines and tangent carriers, a speedy check for curve tracing*, by Miss Henrietta Courtright, Arkansas City Junior College, introduced by the Secretary.

Since any function or factor occurring to the first power divides the plane into a negative and a positive region, this fact may be used as a rapid check for curve tracing. A square of any factor does not divide the plane into two regions but instead carries the point of tangency with a linear factor on the other side of the equation. Nodes and isolated points may be checked by looking for factors to the second power on each side of the equation, or a cusp may be checked in like manner when a factor to the second power is paired with a factor to the third power.

3. *Linear transformations of point lattices*, by Mr. W. R. Cowell, Kansas State College.

A unimodular transformation defines a mapping of a point lattice onto itself. It is possible to establish conditions for one lattice point to be carried into another under such a transformation, and also conditions for two corresponding pairs of lattice points to determine uniquely a unimodular transformation. This type of transformation preserves any order of collinear lattice points and also leaves invariant the number of lattice points on a closed convex plane curve and in the interior of the curve. Associated with a unimodular transformation of determinant $+1$ is an invariant binary quadratic form. This makes possible the use of the transformation as a recursion relation for the solutions of certain Diophantine equations.

4. *Selected applications in difference equations*, by Prof. J. R. Hanna, Wichita University.

This speaker introduced some of the notation, definitions, and procedures pertaining to difference equations. Consideration was given to the use of difference equations in solving problems relative to geometry, continued fractions, Fibonacci's numbers, and mathematics of investment.

5. *A philosophic synthesis of the sciences*, by Sister M. Helen Sullivan, Mount St. Scholastica.

This paper is an epistemological discussion of the findings of science, and shows the place and function of these findings in the general hierarchy of learning. Speculative sciences are classified according to the three degrees of formal abstraction. The specific role of each science as well as its relationship to other branches of knowledge is clearly brought out. The philosophic synthesis is established when a comparative study is made of all the sciences in the light of ultimate explanations. The synthesis is confirmed when science is regarded as one among several rational approaches to truth rather than as a mere technique.

6. *Bessel functions*, by Mr. L. E. Trapp, Kansas State College, introduced by the Secretary.

This paper is concerned with some of the aspects of the theory and application of Bessel functions. The theory involves the discussion of such topics as a second solution to Bessel's differential equation, zeros of $J_0(x)$, solution of the general equation, differentiation and recursion formulas, integral forms of $J_n(x)$, and orthogonality of Bessel functions. The application involves Bessel functions as applied to the solution of Kepler's problem on planetary motion.

ANNA MARM, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December 30, 1950.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May, 1951.

ILLINOIS, University of Illinois, Urbana, May 11–12, 1951.

INDIANA, May 5, 1951.

IOWA, Wartburg College, Waverly, April 20–21, 1951.

KANSAS

KENTUCKY, Eastern Kentucky State College, Richmond, April 28, 1951.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, December, 1950.

METROPOLITAN NEW YORK, Spring, 1951.

MICHIGAN, Michigan State College, East Lansing, March 24, 1951.

MINNESOTA, Duluth Branch of University of of Minnesota, October 7, 1950.

MISSOURI, CENTRAL College, Fayette, Spring, 1951.

NEBRASKA, University of Nebraska, Lincoln, May 5, 1951.

NORTHERN CALIFORNIA, University of San Francisco, January 27, 1951.

OHIO, April 21, 1951.

OKLAHOMA, Oklahoma City, November 14, 1950.

PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.

PHILADELPHIA, Lehigh University, Bethlehem, Pennsylvania, November 25, 1950.

ROCKY MOUNTAIN, Colorado State College of Education, Greeley, April, 1951.

SOUTHEASTERN, Vanderbilt University and Peabody College, Nashville, Tennessee, March 16–17, 1951.

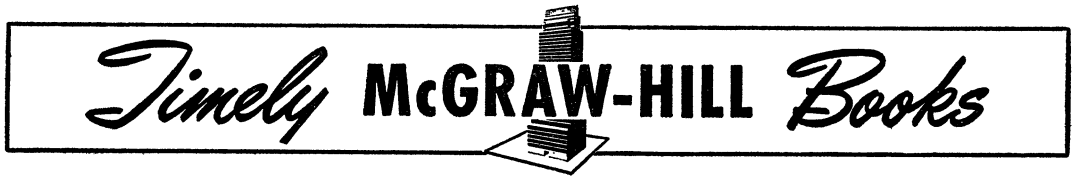
SOUTHERN CALIFORNIA, Whittier College, Whittier, March 10, 1951.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1951.

TEXAS, Southern Methodist University, Dallas, Spring, 1951.

UPPER NEW YORK STATE, Hamilton College, Clinton, May 5, 1951.

WISCONSIN, Carroll College, Waukesha, May, 1951.



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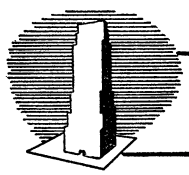
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THE NUMBER e

J. L. COOLIDGE, Harvard University

1. The Greek beginning. The distinguished American mathematician, Benjamin Peirce, was wont to find all of analysis in the equation

$$i^{-i} = \sqrt{e^{\pi}}.$$

In fact, he had his picture taken in front of a blackboard on which this mystic formula, in somewhat different shape, was inscribed. He would say to his hearers, "Gentlemen, we have not the slightest idea of what this equation means, but we may be sure that it means something very important."

With regard to the symbols which appear in this charm, there is a vast literature connected with π; and i, when written $\sqrt{-1}$, leads into the broad field of analysis in the complex domain; but it seems surprisingly difficult to find a connected account of e.

I think we may make a fair beginning with the twelfth proposition of the Second Book of Apollonius, *Conics*, which tells us that if from a point on a hyperbola lines be drawn in given directions to meet the asymptotes, the product of the two distances is independent of the position of the point chosen on the curve. This theorem is more general than we shall need to arrive at the number e, and it is not original with Apollonius. Let us confine ourselves to the very special case where the hyperbola is rectangular, and we draw to each asymptote a line parallel to the other. When x and y are distances, we may write

$$xy = 1. \tag{1}$$

It is intriguing to inquire who first discovered the theorem which leads to this equation. In the commentary of Eutocius on the Sphere and Cylinder of Archimedes [1], we come to a discussion of the classical problem of inserting two mean proportionals between two given lengths. In one solution, which he labels "ut Menaechmus," we have what amounts to the equations

$$\begin{aligned} &a/x = x/y = y/b; \\ &y^2 = bx; \qquad xy = ab. \end{aligned} \tag{2}$$

He goes on to seek the intersection of a parabola and a hyperbola.

Eutocius' statement would place the theorem very early in the history of the conics, for Menaechmus is usually regarded as the discoverer or inventor of these curves, although this ascription is by no means certain. Allman writes [2], "It is much to be regretted that the two solutions of Menaechmus have not been transmitted to us in their original form. That they have been altered either by Eutocius or by some author whom he followed appears not only in the employment in these solutions of the terms parabola and hyperbola, as has frequently been pointed out, but more from the fact that the language used in

them is, in character, altogether that of Apollonius." A similar doubt is shown in Loria [3]. On the other hand, Heath is perfectly definite on this point; he states, "This property in the particular case of the rectangular hyperbola was known to Menaechmus" [4].

But there is another reason for doubting the ascription to Menaechmus, aside from the linguistic objection. The classical Greek discussion of the conics always corresponds to our analysis when the axes are a tangent and the diameter through the point of contact, and with these data proofs are not simple. Heath, following Zeuthen, shows the fact that the hyperbola can be written immediately in the form (1) if we start with a technique like ours, that is, when the axes are a pair of conjugate diameters [5]. That is perfectly true, but the Greeks made surprisingly little study of the conics when expressed in this form more familiar to us; Apollonius comes to it quite late. It seems to me altogether doubtful that the first discoverer of the curves should have been able to make the transition.

2. Grégoire de St. Vincent. If we grant that the Greek mathematicians, perhaps Menaechmus, were familiar with the fundamental property of the rectangular hyperbola expressed in (1), what has this to do with e? We must look ahead some two thousand years to that original writer whose name appears at the head of this paragraph. In 1647, he published his fundamental *Prologomena a Santo Vincento, Opus geometricum quadraturae circuli et sectionum coni.* This I have not seen in its original form, but the content is given at great length by Bopp in [6]. Here is the general scheme. We take the hyperbola

$$xy = 1. \tag{1}$$

On the x axis we take n equivalent rectangles whose bases are

$$P_0P_1,\ P_1P_2,\ \cdots,\ P_{n-1}P_n,$$

while each has an upper vertex on the curve Q_i. Then,

$$P_0P_1 \cdot P_0Q_0 = P_1P_2 \cdot P_1Q_1 = P_2P_3 \cdot P_2Q_2 = \cdots,$$

and

$$\frac{P_0Q_0}{P_1Q_1} = \frac{P_1P_2}{P_0P_1}; \qquad \frac{P_1Q_1}{P_2Q_2} = \frac{P_2P_3}{P_1P_2} \tag{2}$$

but

$$OP_0 \cdot P_0Q_0 = OP_1 \cdot P_1Q_1 = OP_2 \cdot P_2Q_2 \cdots,$$

so that

$$\frac{OP_0}{OP_1} = \frac{P_1Q_1}{P_0Q_0} = \frac{P_0P_1}{P_1P_2} = \frac{OP_1}{OP_2}, \quad \text{by composition.}$$

If

$$OP_1 = \rho OP_0 \qquad \text{then} \qquad OP_j = \rho^j OP_0. \tag{3}$$

St. Vincent even treats the case where OP_0 and OP_j are incommensurable, but we need not follow him here.

The importance of this equation was early recognized, because of its connection with logarithms which were based on the relation of arithmetical and geometrical series. There is a good deal to be said in favor of the thesis that the credit for relating the rectangular hyperbola with logarithms is due to Sarasa. I have not seen his work, but like Cantor, I rely on Kästner. In 1649, Sarasa published *Solutio Problematis a R. P. Marino Mersenno propositi.* This was concerned with the problem: Given three positive quantities and the logarithms of two of them, find the logarithm of the third. Kästner writes [7], "Zu ihrer Beanwortung brang Sarasa drey Saetze aus des Gregorius Buche von der Hyperbel bey, die betreffen Flaechen der Hyperbel an der Aysmptoten, Sarasa erinnert wie das mit Logarithmen zusammenhangt." Cantor's view is similar [8]; he states, "Mit andern Worten, Gregorius hatte das Auftreten von Logarithmen bei der erhähnten Flächenraumen erkannt, wen auch nicht mit Namen genannt. Letzteres that Sarasa, und darin liegt das wirkliche Verdienst seiner Stratschrift."

A contrary view is expressed by Charles Hutton [9] in the words, "As to the first remarks on the analogy between logarithms and hyperbolic spaces, it having been shown by Gregory St. Vincent . . . that if an asymptote be divided into parts in geometrical progression, and from the points of division ordinates be drawn parallel to the other asymptote, they will divide the space between the asymptote and the curve into equal portions, from hence it was shown by Mersenne, that by taking continual sums of these parts there would be obtained areas in arithmetical progression which therefore were analogous to a system of logarithms."

This may be true, but I must point out that whereas St. Vincent published the work referred to above in 1647, Mersenne died in the middle of 1648, and the dates of all of his mathematical writings which I have seen were much earlier. However, St. Vincent's work was certainly well observed. We find Wallis writing in 1658 to Lord Brouncker [10], "Sumptis (in Asymptoto) rectis NH, NI, NK, NQ, NL, NM geometrice proportionalibus, in punctis H, I, K, Q, L, M, ducantur rectae parallelae alteri Asymptoto, spatium Hyperbolicum $A\ B\ H\ M$ in quinque partes dividi ostendit Gregor de Sancto Vincento (si memini) decimo."

3. The introduction of logarithms. The actual word logarithm occurs again in an account of Gregory's *Vera circuli et hyperbolae quadratura*, which was published in Padua in 1667 and laid before the Royal Society [11]. Here we read, "And lastly by the same method he calculates both the logarithm of any natural number, and, vice versa, the natural number of any given logarithm." Perhaps

the wisest word on the subject has been pronounced by the kindly old writer Montucla [12], "Au reste la découverte de cette propriété est revindiquée par divers autres géomètres." Among these I surely must mention Christian Huygens, who acknowledges the work of St. Vincent, even though he does not claim for himself the discovery of the relation between the hyperbola and logarithms. This is admirably set forth in [13], first in a French account, then Huygens' own Latin. He finds the areas bounded by the x axis, which is an asymptote, the curve and ordinates. Two such areas terminating by the same ordinate of 1 are

$$\frac{\text{Area } FGDE}{\text{Area } ABDE} = \frac{\log_e FG}{\log_e 10} = \log_{10} FG.$$

Huygens divides numerator and denominator by 32, which amounts to finding the 32nd root of each area, but this has the effect of so far closing up the figure that we may safely replace the hyperbola by a parabola whose outside area is known. He checks by finding a very good value for $\log_{10} 2$.

In the same year, 1661, Huygens finds another curve which he calls logarithmic but we should probably call it exponential. This curve has the property that the ordinate corresponding to the point mid-way between two given points of the x-axis is the mean proportional between their ordinates. The equation of the curve is $y = ka^x$. Huygens takes

$$y = 2^{x/x_0}; \qquad x = \frac{\log y}{\log 2} x_0. \tag{4}$$

The constant subtangent is

$$\frac{ydx}{dy} = \frac{x_0}{\log_e 2}. \tag{5}$$

Huygens takes

$$x_0 = 10^n \log_{10} 2.$$

This gives for the constant subtangent

$$\log_{10} e = .43429448190325180,$$

"qualium logarithmus binarij est"

$$.30102995663981195.$$

These numbers had long been known as they had appeared, for instance, in Briggs' *Arithmetica logarithmica* of 1624, pages 10 and 14. As a matter of fact, there appeared in 1618 a second edition of Wright's translation of Napier's *Mirifici Logarithmorum Canonis Descriptio* which contained an appendix, probably written by Oughtred, giving the natural logarithms of various numbers

from 100,000 to 900,000. This is probably the earliest table of natural logarithms, although a very similar table by John Spidell appeared in 1619 [14].

The astonishing thing about all of those writers who connected logarithms with hyperbolic areas is their lack of interest in what we should call the base. Napier began by considering the relation between an arithmetical and a geometric series. A geometric series consists in successive powers of one number. What is that number? Or given a set of logarithms, what number has the logarithm 1? I mentioned that Briggs gave the logarithm of e, to the base 10 but I find no mention of e itself. Of course, we might write

$$10^n \log_{10} \frac{10^n + \Delta x}{10^n} = 10^n \log_e \left(1 + \frac{\Delta x}{10^n}\right) \log_{10} e = \Delta x \log_{10} e + \cdots,$$

but e itself does not appear. The fact is that there was no comprehension that a logarithm was essentially an exponent. Tropfke is very explicit in this point; he writes, "Freilich dürfen wir nicht an die moderne Erklärung der Logarithmen denken, die in ihnen Potenzexponenten einer bestimmten Grundzahl erkennt. Diese Auffassung machte sich erst um die Mitte des achtenten Jahrhunderts geltend" [15]. This is perhaps too strong a statement, for in a note on the same page he quotes James Gregory (whom he calls David Gregory) as saying in his *Exercitationes Geometricae* of 1684, p. 14, "Exponentes sunt ut logarithmi." I have not been able to verify this, but we find in [16], "Si seriei Termonorum in Progressione geometrica ab 1 continue proportionalium, puta

1, 2, 4, 8, 16, 32, 64, etc.

accomedetur series Indicum, sive Exponentium, in progressione ab o continue procedentium, puta

0, 1, 2, 3, 4, 5, 6, etc.

Hos exponentes appelabant Logarithmos." We could not well ask for anything clearer or more explicit.

If most writers did not look on logarithms as exponents, how did they consider them? I think we find the clue in St. Vincent's identification of logarithms with hyperbolic areas, remembering that these were the days of Cavalieiri and Roberval, when an area was looked upon as the same thing as an infinite number of line segments, a very helpful if dangerous definition. We find Halley writing [17], "They may more properly be said to be numeri rationum exponentes, wherein we consider ratio as a quantity sui generis, beginning from the ratio of equality, or 1 to $1=0$, $\cdots$ and the rationes we suppose to be measured by the number of ratiunculae in each. Now these ratiunculae are in a continued scale of proportionals, infinite in number, between the two terms of the ratio, which infinite number of mean proportionals is to that infinite number of the like and equal ratiunculae between any other two terms as the logarithm of one ratio is to the logarithm of the other. Thus if we suppose there to be between 1 and 10

an infinite scale of mean proportionals whose number is 100000 ad infinitum, between 1 and 2 there shall be 30102 of said proportionals and between 1 and 3, 47712 of them which numbers therefore are the logarithms of the ratio of 1 to 10 1 to 2, and 1 to 3, and so properly called the logarithms of 10, 2, and 3."

It is hard to see how there could be a much worse explanation of logarithms for those who "make constant use of logarithms without having an adequate notion of them." The one certain thing seems to be that a logarithm is an infinite number. I suppose we might translate this into the form

$$\frac{b}{a} = \frac{a+r_1}{a}\cdot\frac{a+r_2}{a+r_1}\cdot\frac{a+r_3}{a+r_2}, \cdots, \frac{a+r_n}{a+r_{n-1}}\cdot\frac{b}{a+r_n}.$$

If

$$\frac{a+r_j}{a+r_{j-1}} = r, \qquad \frac{b}{a} = r^n.$$

Then n would be the logarithm.

4. Mercator, Newton, Leibniz. It is fair to say that such a definition of a logarithm was not original with Halley. We find Mercator writing in 1668, [18] "Est enim Logarithmus nihil aliud, quam numerus ratiuncularum contentarum in ratione quam absolutus quisque ad unitatem obtinet." I may mention also that this seems the first place where the words "logarithmus naturalis" are used. But the real significance of the article comes from the fact that instead of studying log x he takes up log $(1+x)$, which enables him to start from 0. The article is not clearly written, so I follow the much clearer exposition in Wallis [19], which was published in the same year.

We study the area under the curve (1) from $x=1$ to $x=1+X$. We divide the length on the x-axis into n equal parts, each of length Δx. The abscissas are $1, 1+\Delta x, 1+2\Delta x, \cdots, 1+X$ and the corresponding ordinates are

$$1, \frac{1}{1+\Delta x}, \frac{1}{1+2\Delta x}, \cdots, \frac{1}{1+(n-1)\Delta x}.$$

The infinitesimal, rectangular areas are

$$\Delta x, \Delta x[1-\Delta x+\Delta x^2-\Delta x^3+\cdots],$$
$$\Delta x[1-(2\Delta x)+(2\Delta x)^2-(2\Delta x)^3+\cdots], \cdots.$$

Such infinite expansions were common in Wallis' work. The sum of these rectangular areas may be written

$$\Delta x[1+1+1\cdots] - \Delta x[\Delta x+2\Delta x+3\Delta x+\cdots] + \Delta x[(\Delta x)^2+(2\Delta x)^2+(3\Delta x)^2+\cdots] - \cdots.$$

Now $n\Delta x = X$, so we have

$$(6) \qquad X - \Delta x^2[1 + 2 + 3 \cdots] + \Delta x^3[1^2 + 2^2 + 3^2 \cdots] - \cdots.$$

With regard to these sums, Wallace says [19], page 222, "quod ostendit ille prop XVI etque a me alibi demonstratum." A reference he makes to Mercator is not conclusive as the statement is sketchy; as to his own work I will follow [20], as I shall need that again. Here he is seeking the area under the curve $y=x^m$ from $x=0$ to $x=X$. His method is not perfectly clear, as he seems merely to generalize by analogy from cases worked out earlier, but what he does is essentially the following:

We take N equal lengths from 0 to $N\Delta x=X$. We have a set of rectangles whose combined areas are

$$\Delta x[0^m + (\Delta x)^m + (2\Delta x)^m + (3\Delta x)^m + \cdots].$$

Let us assume that $0^m+1^m+\cdots+(N-1)^m=\alpha N^{m+1}+\beta N^m+\gamma N^{m-1}+\cdots$. Replacing N by $N+1$, and subtracting, we obtain

$$N^m = (m + 1)\alpha N^m + bN^{m-1} + cN^{m-2} \cdots,$$

so

$$\alpha = \frac{1}{m + 1}.$$

Substituting, and remembering that $N\Delta x=X$, there results

$$\text{Area} = \frac{X^{m+1}}{m + 1} + \beta\Delta x X^m + \gamma\Delta x^2 X^{m-1}.$$

The limit of this as $N\to\infty$ is $X^{m+1}/m+1$, since $\Delta x\to 0$. We thus can substitute this result in (6), when $m=1, 2, 3, \cdots$, to obtain Mercator's famous formula:

$$(7) \qquad \log (1 + X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \frac{X^4}{4} + \cdots.$$

A good deal has been written about this series, as we see from Mazeres and elsewhere. The obvious way to obtain the equation is to apply the calculus, so we now turn to see how this instrument was brought to bear. In 1669, a year after Mercator had published his work on logarithms [18], Newton sent to Collins his article, *De Analysi per aequationes numero terminorum Infinitas* [21]. This represents his first studies of areas under curves, which he had been working at for a year or two, but had not published. In fact, publication did not occur for a goodly number of years to come; there is, however, no question of giving his results precedence over those of Mercator. It begins as shown below:

Curvarum simplicium Quadratura

Reg. 1 Si $ax^{m/n} = y$, erit $\dfrac{an}{m+n} x^{(m+n)/n} =$ Area ABD.

I must speak further of this. In [22] we read on p. 176, "Dr. Wallis published in his Arithmetica infinitorum in the year 1655 and in the 59th Proposition of that Book, if the Abscissa of any curvilinear figure be called x and m and n be two Numbers, the ordinates erected at right Angles be $x^{n/m}$ the area of the Figure shall be $(n/m+n)x^{m+n/n}$. And this is assumed by Mr. Newton, upon which he founds his Quadrature of Curves. Dr. Wallis demonstrated this by steps in many particular Propositions and then connected all the Propositions into one by a Table of Cases. Mr. Newton reduced all Cases to One, with an indefinite Index, and at the end of his Compendium demonstrated it at once by his method of moments, he being the first who introduced indefinite Indices of Dignites into the Operations of Analysis." This is Newton's own statement of the case and must be taken as final. It is true that Wallis worked out a number of special cases in a manner not exactly like the method followed here, and did not use a literal exponent. The greater generality of Newton's formula is found by replacing x by $x^{1/n}$. Newton's proof by "the method of moments" we should call differentiation, and consisted in showing that if

$$z = \frac{n}{m+n} x^{(m+n)/n}; \quad \text{then} \quad \frac{dz}{dx} = x^{m/n}.$$

It is fair to say also that although he gives Mercator's formula, he gives it as the area under the hyperbola, with no mention of Mercator or of logarithms.

It is time to turn for a moment to the other inventor of the calculus, Gottfried Leibniz. We find him writing in 1677 or 1678 [23],

"In Hyperbol sit $AB = 1$, $BM = x$, $ML = \dfrac{1}{1+x}$,

$$CBMLC = \frac{1}{1}x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots."$$

This is proved by the straight expansion of $1/1+x$, after which there is integration term by term. We find something more interesting a dozen years later, when he writes to Huygens, who is said never to have understood Leibniz calculus of differences [24], "Soit donc x l'abscisse et y l'ordonnée de la courbe, et l'équation comme je vous ay dit

$$\frac{x^3 y}{h} = b^{2xy}.$$

Je désignerai le logarithme de x par log x et nous aurons

$$3 \log x + \log y - \log h = 2xy.$$

supposant que le log de l'unite soit 0 et le log $b=1$. Donc par la quadrature de

l'hyperbole nous aurons

$$3\int\frac{dx}{x}+\int\frac{dy}{y}-\log h=2xy$$

$$3\frac{dx}{x}+\frac{dy}{y}=2xdy+2ydx,$$

dx sera à dy, on bien DB sera à y comme $2x^2y-x$ est à $3y-2xy$ c'est à dire DB sera

$$\frac{2x^2y-xa^2}{3a^2-2xy}$$

comme vous le demandées, a estant l'unité."

5. Leonhard Euler. It is now time to turn to the man who pulled all this together and who put the number e definitely on the map, Leonhard Euler. This he did in [**25**], beginning in "Caput VII" with the base a. His argument is outlined below:

Since $a^0=1$, we may put $a^w=1+kw$; $w=\log(1+kw)$. Assume w to be very small, and write

$$a^{iw}=(1+kw)^i=1+\frac{i}{1}kw+\frac{i(i-1)}{1\cdot2}k^2w^2+\frac{i(i-1)(i-2)}{1\cdot2\cdot3}k^3w^3+\cdots.$$

Since w is infinitesimally small, and i is infinitely large, we write $iw=z$

$$a^z=\left(1-\frac{kz}{i}\right)^i=1+kz+\frac{(i-1)}{i\cdot1\cdot2}k^2z^2+\frac{(i-1)(i-2)}{i\cdot1\cdot2\cdot3}k^3k^3+\cdots.$$

Since i is very large, we may assume $(i-n)/i=1$, then

$$a^z=1+kz+\frac{k^2z^2}{1\cdot2}+\frac{k^3z^3}{1\cdot2\cdot3}+\cdots.$$

If $z=1$,

$$a=1+k+\frac{k^2}{1\cdot2}+\frac{k^3}{1\cdot2\cdot3}\cdot$$

If we take $a=10$, the base in the logarithm system of Briggs, Euler gives

$$k=2.30238,\quad\text{approximately.}$$

For a natural logarithm we take $k=1$; $a=e$; and

$$e=1+\frac{1}{1}+\frac{1}{1\cdot2}+\frac{1}{1\cdot2\cdot3}+\cdots. \tag{8}$$

Euler gives this value to 18 places, without naming the source, namely,

$$e = \lim_{i \to \infty} \left(1 + \frac{1}{i}\right)^i. \tag{9}$$

With regard to the use of the letter e, Euler had long employed it, for we find him writing [26], page 80, "scribitur pro numero cujus logarithmus est unitas e, qui est 2.7182817. . . . " Note that this is Leibniz' b.

I pass to Ch. VII of the *Introductio*. Euler assumes for small values of z,

$$\sin z = z; \qquad \cos z = 1.$$

He then, following DeMoivre, writes,

$$\cos nz = \frac{(\cos z + \sqrt{-1}\sin z)^n + (\cos z - \sqrt{-1}\sin z)^n}{2},$$

$$\sin nz = \frac{(\cos z + \sqrt{-1}\sin z)^n - (\cos z - \sqrt{-1}\sin z)^n}{2\sqrt{-1}}.$$

Putting $nz = v$, and remembering that z is small,

$$\cos v = 1 - \frac{v^2}{1\cdot 2} + \frac{v^4}{1\cdot 2\cdot 3\cdot 4} - \cdots,$$

$$\sin v = v - \frac{v^3}{1\cdot 2\cdot 3} + \frac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \cdots.$$

Comparing these with the value given previously for a^z, one obtains

$$\cos v = \frac{e^{v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2}; \qquad \sin v = \frac{e^{v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}; \tag{10}$$

and

$$v\sqrt{-1} = \log\left[\cos v + \sqrt{-1}\sin v\right]. \tag{11}$$

This last formula was not, strictly speaking, original. Roger Cotes in [27] sought the area of an ellipsoid of revolution. When the rotation is about the minor axis there is no trouble, but when the motion is about the major axis we find him writing "Posset hujus etiam superficiei per Logometriam designari, sed modo inexplicabili . . . arcus erit rationis inter

$$EX + XC\sqrt{-1}\ a\overline{CE}\ \text{mensura ducta in}\ \sqrt{-1}."$$

I will leave Euler for a moment to speak of the numerical value of e. William Shanks, who, until quite recently, held the world's record of 707 places for π, had a try at e [28]. Glaisher found an error in this, but Shanks corrected it,

and calculated a value which he was sure was right to 205 places. Glaisher verified 137 of them. Boorman [29] calculated e to 346 places. He acknowledged that he and Shanks agreed only up to 187 places. "One is wrong, which one?" Boorman gives the impression of being a rather amateurish mathematician. Adams [30] calculated $\log_{10} e$ to 272 places, probably all correct. Many years ago I knew a youthful teacher of mathematics who had the vaulting ambition to calculate e by long-hand methods to 1,000 places. I lost sight of him over fifty years ago, probably he died early of heart failure.

I return to Euler. In Caput XVIII of *De Fractionibus continuis* [25], he describes methods of expansion into a continued fraction. When it is a question of turning a rational fraction into a continued one, the process is essentially that of finding a highest common factor, and can be done in only one way. Euler writes

$$e = 2.718281828459\cdots,$$

$$\frac{e-1}{2} - 0.8591409142295.$$

He writes this in the form,

$$\frac{e-1}{2} = \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{\text{etc.}}}}}}},$$

and remarks [25], page 388, "Cuius fractio ex Calculo infinitesimali dari potest."

Euler assumes that the quotients will increase by 4 each time, so that the fraction goes on indefinitely. Hence e is not a rational fraction.

As for finding this "ex Calculo infinitesimali" he returns to this very much later in life, "Summatio fractionis continuae cujus indices progressionem arithmeticam constituunt" in Vol. 23 of his Opera mentioned in [25]. The method consists in establishing contact with a Riccati differential equation. For a fuller discussion see [31]. Euler did not complete all the details with modern rigor, but what I have just shown is the first attempt to demonstrate the irrationality of e.

We must wait a whole century for anything really new and startling in this line. This came in 1874 with Hermite's proof that e is not an algebraic number [32], that is, not the root of any equation with integral coefficients. A much simpler demonstration is given by Klein in [33].

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"CLOSING-UP" OF SEQUENCE SPACES

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1. Introduction. We consider the linear space $\mathcal{L}$ of sequences of real numbers $(u_1, u_2, \cdots)$ having only a finite number of non-zero terms. On it, we define, for instance, the norm $(\sum_i |u_i|^p)^{1/p}$ where $p>1$. Since the obtained normed linear space is not complete we close it up, that is, imbed it in a complete linear space. That last it is customary to derive either as (1) the Banach space of all those sequences $(u_1, u_2, \cdots)$ having perhaps an infinite number of non-zero terms, for which $(\sum_i |u_i|^p)^{1/p} < +\infty$, or as (2) the Banach space of all fundamental sequences of elements in $\mathcal{L}$. For the above norm both (1) and (2) furnish the space l_p [**1**, p. 12]. The closures derived by (1) and (2) which obviously depend on the norm on $\mathcal{L}$, may be obtained also for a significant and broad class of norms on $\mathcal{L}$, which, however, do not differ much in character from the norm stated above. It is interesting to note that already for these well-behaving norms the methods (1) and (2) may furnish different closures.

It is the purpose of this note to present these details.

2. Linear spaces. We begin with a brief review of a few elementary notions.

A set $\mathfrak{A}$ of elements $x, y, \cdots$ is termed a linear space if to every ordered couple x, y of $\mathfrak{A}$ corresponds an element of $\mathfrak{A}$ which we term their sum and denote by $x+y$, and to every number α and x in $\mathfrak{A}$ corresponds an element of $\mathfrak{A}$ which we term their product and denote by αx.

Furthermore it is assumed that the operation of addition of elements and the multiplication of numbers by elements are such that the following conditions hold:

(i). $x+y=y+x$.
(ii). $x+(y+z)=(z+y)+z$.
(iii). $x+y=x+z$ implies $y=z$.
(iv). $\alpha(x+y)=\alpha x+\alpha y$.
(v). $(\alpha+\beta)x=\alpha x+\beta x$.
(vi). $\alpha(\beta x)=(\alpha\beta)x$.
(vii). $1x=x$.

It can be verified readily that these conditions imply the existence of a unique element θ in $\mathfrak{A}$ such that $x+\theta=x$ for every x in $\mathfrak{A}$.

By definition, we also put

$$-x = (-1)x; \qquad x - y = x + (-y).$$

A linear space $\mathfrak{A}$ is termed finite dimensional if it contains a finite number of linearly independent elements $x_1, \cdots, x_n$ with the property that every element of $\mathfrak{A}$ can be written as their linear combination. In the last case we term, $x_1, \cdots, x_n$ a basis for $\mathfrak{A}$. If $\mathfrak{A}$ is finite dimensional then every basis for $\mathfrak{A}$ consists of the same number of elements. That number is defined as the dimension of $\mathfrak{A}$.

3. Normed linear spaces; completeness. A linear space $\mathfrak{A}$ is termed normed, if on it is defined a norm, that is, a non-negative function of its elements x which we shall denote by $\|x\|$, such that the following conditions hold:

(1). $\|\theta\| = 0$; $\|x\| > 0$ for $x \neq \theta$.
(2). $\|\alpha x\| = |\alpha| \, \|x\|$ for any number α.
(3). $\|x+y\| \leqq \|x\| + \|y\|$.

In a normed linear space we define the number

$$\|x - y\|$$

as the distance of the elements x and y.

It is a consequence of (1), (2), (3), that in a normed linear space the distance satisfies the following conditions:

(4). $\|x-y\| > 0$, if $x \neq y$; $\|x-x\| = 0$.
(5). $\|x-y\| = \|y-x\|$.
(6). $\|x-z\| \leqq \|x-y\| + \|y-z\|$.

Thus, a normed linear space is at the same time a metric space.

A sequence $(x_1, x_2, \cdots)$ of elements of a normed linear space $\mathfrak{A}$ is said to be fundamental (or a Cauchy sequence) if,

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \|x_m - x_n\| = 0,$$

while a sequence $(x_1, x_2, \cdots)$ of elements of $\mathfrak{A}$ is said to be convergent to an element x of $\mathfrak{A}$ if,

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Clearly, a sequence of elements which is convergent to an element of the space is always fundamental. The converse is not necessarily true.

A normed linear space where every fundamental sequence of elements of the space is also convergent to an element of that space is termed complete.

A normed linear space which is complete is termed a Banach space.

4. The Cantor-Méray closure of a normed linear space. For a normed linear space $\mathfrak{A}$ its Cantor-Méray closure $[\mathfrak{A}]$ is constructed in the following manner:

Let $[\mathfrak{A}]$ denote the set of all fundamental sequences of elements in $\mathfrak{A}$. For $(x_1, x_2, \cdots)$ and $(y_1, y_2, \cdots)$ in $[\mathfrak{A}]$ and a constant number α we put:

$$(x_1, x_2, \cdots) + (y_1, y_2, \cdots) = (x_1 + y_1, x_2 + y_2, \cdots)$$

and

$$\alpha(x_1, x_2, \cdots) = (\alpha x_1, \alpha x_2, \cdots).$$

It is seen readily that with the above operations of addition of elements and

multiplication of numbers by elements, $[\mathfrak{A}]$ becomes a linear space.

Next we introduce the following conventions:

(a). Two fundamental sequences of elements of $\mathfrak{A}$ are considered identical if and only if the norm of their difference tends towards zero.

(b). The norm of a fundamental sequence of elements in $\mathfrak{A}$ is defined as the limit of the norms of the elements in that sequence.

It is not difficult to see that with the last two conventions $[\mathfrak{A}]$ becomes a normed linear space which is complete [3, p. 106], that is, a Banach space.

Clearly $\mathfrak{A}$ may be considered a linear subspace of $[\mathfrak{A}]$ if for x in $\mathfrak{A}$, $(x, x, \cdots)$ is identified with x. The last identification also preserves the norm.

We express this fact by saying that $\mathfrak{A}$ may be imbedded always in its Cantor-Méray closure $[\mathfrak{A}]$. Furthermore, it is not difficult to prove that $[\mathfrak{A}]$ is the smallest Banach space in which the normed linear space $\mathfrak{A}$ can be imbedded, in the sense that whenever $\mathfrak{A}$ can be imbedded in a Banach space $\mathfrak{B}$ then also $[\mathfrak{A}]$ can be imbedded in $\mathfrak{B}$.

5. Banach spaces and their conjugate spaces. For a given Banach space $\mathfrak{B}$ we are about to define the first and second conjugate space, in symbols $\mathfrak{B}^*$ and $\mathfrak{B}^{**}$.

We say that a functional F is defined on $\mathfrak{B}$ if to every point x in $\mathfrak{B}$ corresponds a number $F(x)$. Functional F is termed additive if $F(x+y)=F(x)+F(y)$ for x, y in $\mathfrak{B}$; F is termed homogeneous if $F(\alpha x)=\alpha F(x)$ for x in $\mathfrak{B}$ and any number α.

An additive and continuous functional F on $\mathfrak{B}$ is termed linear. For additive functionals linearity is equivalent to boundedness [1, p. 54]. We mean hereby that F is linear on $\mathfrak{B}$ if and only if there exists a finite number α such that $|F(x)| \leqq \alpha\|x\|$ for all x in $\mathfrak{B}$.

The least of such numbers α we term the bound of F and denote by $|||F|||$. Thus, by definition:

$$|||F||| = \text{l.u.b.}_{\theta \neq x} \frac{|F(x)|}{\|x\|}.$$

Since a linear functional is necessarily homogeneous, the last expression may be given the following form:

$$|||F||| = \text{l.u.b.}_{\|x\| \leqq 1} |F(x)|.$$

Let $\mathfrak{B}^*$ denote the set of all linear functionals $F, G, \cdots$ on $\mathfrak{B}$. We define addition of elements in $\mathfrak{B}^*$ and multiplication of numbers by elements in $\mathfrak{B}^*$ by means of the obvious relations: $(F+G)(x)=F(x)+G(x)$, $\alpha(F)(x)=F(x)$. Thus, $\mathfrak{B}^*$ becomes a linear space.

It is clear that the bound of a linear functional satisfies the conditions (1), (2), (3) imposed upon a norm. Thus, $\mathfrak{B}^*$ will be normed if for F in $\mathfrak{B}^*$ we put $\|F\| = |||F|||$. Moreover, with the last norm, $\mathfrak{B}^*$ will be complete, that is, a Banach

space. We term $\mathfrak{B}^*$ the conjugate space of $\mathfrak{B}$.

Similarly, for $\mathfrak{B}^*$ we may construct its conjugate space $(\mathfrak{B}^*)^* = \mathfrak{B}^{**}$.

There is a natural relationship combining $\mathfrak{B}$ and $\mathfrak{B}^{**}$ which can be derived by the following reasoning:

Holding x_0 fixed in $\mathfrak{B}$ and varying F in $\mathfrak{B}^*$, $F(x_0)$ represents a linear functional $\mathcal{J}_0$ on $\mathfrak{B}^*$, that is, an element of $\mathfrak{B}^{**}$. Furthermore, $|||\mathcal{J}_0||| = ||x_0||$.

Thus, the Banach space $\mathfrak{B}$ can be imbedded always in $\mathfrak{B}^{**}$ in a natural way. We express this by writing $\mathfrak{B} \subset \mathfrak{B}^{**}$. The space $\mathfrak{B}$ is termed reflexive whenever in the above imbedding $\mathfrak{B}$ coincides with $\mathfrak{B}^{**}$.

In the future we shall make use of the following:

THEOREM: *A finite dimensional normed linear space is always a reflexive Banach space.*

For a further study of Banach spaces the reader is referred to [1].

6. Symmetric norms on a finite dimensional space.

Definition 1: A real function $\phi = \phi(u_1, \cdots, u_n)$ on the n-dimensional space $\mathfrak{L}_n$ of n tuples $(u_1, \cdots, u_n)$ of real numbers is termed a norm* if it satisfies the following conditions:

(i). $\phi(u_1, \cdots, u_n) > 0$ unless $u_1 = \cdots = u_n = 0$.

(ii). $\phi(\alpha u_1, \cdots, \alpha u_n) = |\alpha| \phi(u_1, \cdots, u_n)$ for any constant α.

(iii). $\phi(u_1 + u_1', \cdots, u_n + u_n') \leqq \phi(u_1, \cdots, u_n) + \phi(u_1', \cdots, u_n')$.

A norm will be termed symmetric if in addition to (i), (ii), (iii), it satisfies the following condition:

(iv). $\phi(u_1, \cdots, u_n) = \phi(\epsilon_1 u_{1'}, \cdots, \epsilon_n u_{n'})$

where $\epsilon_i = \pm 1$ and $1', 2', \cdots n$ denotes any permutation on the first n natural numbers.

To simplify our formulas we shall always assume that the norm also satisfies the following (non-essential for our discussion) condition of "cross-property":

(v). $\phi(1, 0, \cdots, 0) = 1$.

A symmetric norm is a non-decreasing function in the absolute value of each variable u_i; that is, $|u_i| \leqq |u_i'|$ for $i = 1, \cdots, n$ implies $\phi(u_1, \cdots, u_n) \leqq \phi(u_1', \cdots, u_n')$. This is precisely the content of the following:

LEMMA 1. *Let $\phi(u_1, \cdots, u_n)$ denote a symmetric norm on $\mathfrak{L}_n$. Then, for $0 \leqq p_i \leqq 1$ we have,*

$$\phi(p_1 u_1, \cdots, p_n u_n) \leqq \phi(u_1, \cdots, u_n).$$

Proof. By virtue of (iv) we may suppose that all the u_i's are $\geqq 0$. It is clear by induction that it is sufficient to establish the last relation when $0 \leqq p_i \leqq 1$ occurs only for one i, that is,

* Some properties of norms $\phi(u_1, \cdots, u_n)$ on a finite dimensional space have been considered in [2], while symmetric norms (gauge functions) on a finite dimensional space have been discussed in [5].

$$\phi(u_1, \cdots, u_{i-1}, pu_i, u_{i+1}, \cdots, u_n) \leqq \phi(u_1, \cdots, u_{i-1}, u_i, u_{i+1}, \cdots, u_n)$$

for $0 \leqq p \leqq 1$. The last assertion follows from the following simple direct calculation:

$$\begin{aligned}
&\phi(u_1, u_2, \cdots, pu_i, \cdots, u_n) \\
&\quad = \phi\left(\frac{1+p}{2}u_1 + \frac{1-p}{2}u_1, \frac{1+p}{2}u_2 + \frac{1-p}{2}u_2, \cdots, \frac{1+p}{2}u_i\right. \\
&\qquad \left. + \frac{1-p}{2}(-u_i), \cdots, \frac{1+p}{2}u_n + \frac{1-p}{2}u_n\right) \\
&\quad \leqq \phi\left(\frac{1+p}{2}u_1, \frac{1+p}{2}u_2, \cdots, \frac{1+p}{2}u_i, \cdots, \frac{1+p}{2}u_n\right) \\
&\quad + \phi\left(\frac{1-p}{2}u_1, \frac{1-p}{2}u_2, \cdots, \frac{1-p}{2}(-u_i), \cdots, \frac{1-p}{2}u_n\right) \\
&\quad = \frac{1+p}{2}\phi(u_1, u_2, \cdots, u_i, \cdots, u_n) \\
&\qquad + \frac{1-p}{2}\phi(u_1, u_2, \cdots, (-u_i), \cdots, u_n) \\
&\quad = \frac{1+p}{2}\phi(u_1, u_2, \cdots, u_i, \cdots, u_n) \\
&\qquad + \frac{1-p}{2}\phi(u_1, u_2, \cdots, u_i, \cdots, u_n) \\
&\quad = \phi(u_1, u_2, \cdots, u_i, \cdots, u_n).
\end{aligned}$$

Remark. In the proof of our Lemma we have not assumed at all that $\phi(u_1, \cdots, u_n)$ satisfies condition (v) of Definition 1.

LEMMA 2. *Let $\phi(u_1, \cdots, u_n)$ denote a symmetric norm on $\mathfrak{X}_n$. Then,*

$$\max_i |u_i| \leqq \phi(u_1, \cdots, u_n).$$

Proof. Lemma 1, gives

$$\phi(0, \cdots, 0, u_i, 0, \cdots, 0) \leqq \phi(u_1, \cdots, u_{i-1}, u_i, u_{i+1}, \cdots, u_n).$$

By (v) and (ii) of Definition 1 for ϕ the left side of the last inequality equals $|u_i|$, that is,

$$|u_i| \leqq \phi(u_1, \cdots, u_n) \qquad \text{for } i = 1, 2, \cdots, n.$$

This concludes the proof.

LEMMA 3. *For a symmetric norm* $\phi(u_1, \cdots, u_n)$ on $\mathfrak{L}_n$ we have

$$\phi(u_1, \cdots, u_n) \leqq \sum_i |u_i|.$$

Proof. The proof is a simple consequence of conditions (iii), (iv) and (v) for $\phi(u_1, \cdots, u_n)$.

LEMMA 4. *A norm* $\phi(u_1, \cdots, u_n)$ *is a continuous function of the* u_i*'s in the sense that* $\lim_{s\to\infty} u_i^{(s)} = u_i$, $i = 1, 2, \cdots, n$, implies

$$\lim_{s\to\infty} \phi(u_1^{(s)}, \cdots, u_n^{(s)}) = \phi(u_1, \cdots, u_n).$$

Proof. Clearly,

$$|\phi(u_1^{(s)}, \cdots, u_n^{(s)}) - \phi(u_1, \cdots, u_n)|$$
$$\leqq \phi(u_1^{(s)} - u_1, \cdots, u_n^{(s)} - u_n) \leqq \sum_i |u_i^{(s)} - u_i|.$$

This concludes the proof.

We are about to define a conjugate of a given norm $\phi(u_1, \cdots, u_n)$ on $\mathfrak{L}_n$. For a fixed n tuple $(v_1, \cdots, v_n)$ consider the totality of numbers of the form

$$\frac{u_1v_1 + \cdots + u_nv_n}{\phi(u_1, \cdots, u_n)} = f(u_1, \cdots, u_n)$$

taken over the compact (that is, closed and bounded) set of all n tuples, $(u_1, \cdots, u_n)$ for which $|u_1| + \cdots + |u_n| = 1$. Clearly, $f(u_1, \cdots, u_n)$ represents a continuous function on a compact set; hence it assumes its maximum [3, p. 196], which we shall denote by $\psi(v_1, \cdots, v_n)$.

It is seen readily that $\psi(v_1, \cdots, v_n)$ may be defined also as the maximum of the number $f(u_1, \cdots, u_n)$ taken over the set of all n tuples $(u_1, \cdots, u_n) \neq (0, \cdots, 0)$.

The proof of the following two lemmas is immediate:

LEMMA 5. $\psi(v_1, \cdots, v_n)$ *is a norm whenever* $\phi(u_1, \cdots, u_n)$ *is such. If* ϕ *is symmetric, the same holds for* ψ.

LEMMA 6. $u_1v_1 + \cdots + u_nv_n \leqq \phi(u_1, \cdots, u_n)\psi(v_1, \cdots, v_n)$.

The n-dimensional space of n tuples $(u_1, \cdots, u_n)$ on which there is defined the norm $\phi(u_1, \cdots, u_n)$ will be denoted by $\mathfrak{L}_n(\phi)$.

LEMMA 7. *Every linear functional F on $\mathfrak{L}_n(\phi)$ determines an n tuple of numbers $(v_1, \cdots, v_n)$ such that,*

$$F(u_1, \cdots, u_n) = u_1v_1 + \cdots + u_nv_n \qquad \text{for } (u_1, \cdots, u_n) \text{ in } \mathfrak{L}_n(\phi). \tag{1}$$

Furthermore,

$$|||F||| = \text{l.u.b.} \frac{u_1v_1 + \cdots + u_nv_n}{\phi(u_1, \cdots, u_n)}, \tag{2}$$

where the last l.u.b. *is extended over all n tuples $(u_1, \cdots, u_n) \neq (0, \cdots, 0)$.*

CONVERSELY. *Given an n tuple of numbers $(v_1, \cdots, v_n)$, then relationship* (1) *above determines a linear functional F. Its bound is given by* (2).

Proof. Let F be a given linear functional on $\mathfrak{L}_n(\phi)$. We put

$$F(1, 0, \cdots, 0) = v_1, F(0, 1, \cdots, 0) = v_2, \cdots, F(0, 0, \cdots, 1) = v_n.$$

Then, (1) above is a consequence of additivity and homogeneity for F. Clearly, (2) is a consequence of (1) and is the definition of a bound for a linear functional. That the converse holds is immediate. This concludes the proof.

LEMMA 8. *The conjugate space of $\mathfrak{L}_n(\phi)$ may be characterized as $\mathfrak{L}_n(\psi)$.*

Proof. This is a consequence of Lemmas 5 and 7.

The previous considerations justify the following

Definition 2: $\psi(v_1, \cdots, v_n)$ is termed the conjugate norm of the norm $\phi(u_1, \cdots, u_n)$.

LEMMA 9. *Any norm $\phi(u_1, \cdots, u_n)$ on $\mathfrak{L}_n$ is at the same time the conjugate of its conjugate norm $\psi(v_1, \cdots, v_n)$.*

Proof. By Lemma 8, the conjugate space of $\mathfrak{L}_n(\phi)$ may be characterized as $\mathfrak{L}_n(\psi)$. But every finite dimensional space is reflexive. Therefore, the conjugate of $\mathfrak{L}_n(\psi)$ may be characterized as $\mathfrak{L}_n(\phi)$. This concludes the proof.

We conclude our section by pointing out on $\mathfrak{L}_n$ an important class of symmetric norms and their conjugate norms. Our arguments are based on the following well known two inequalities which hold for any pair of n tuples of real numbers $(u_1, \cdots, u_n)$, $(v_1, \cdots, v_n)$.

(**a**). Hölder's inequality:

Let $p>1$ and $1/p+1/q=1$. Then,

$$\left| \sum_{i=1}^{n} u_iv_i \right| \leqq \sum_{i=1}^{n} |u_i||v_i| \leqq \left(\sum_{i=1}^{n} |u_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |v_i|^q \right)^{1/q}.$$

(**b**). Minkowski's inequality:

Let $p>1$. Then,

$$\left(\sum_{i=1}^{n}|u_i+v_i|^p\right)^{1/p} \leqq \left(\sum_{i=1}^{n}|u_i|^p\right)^{1/p}+\left(\sum_{i=1}^{n}|v_i|^p\right)^{1/p}.$$

Due to the significance of these inequalities we sketch their elementary proof.

Proof. (a). Consider the curve $y=x^p$ where $p>1$, $x\geqq 0$. On it we choose a system of n material points with masses $m_1, \cdots, m_n$ and coordinates (x_1, y_1), $\cdots$, (x_n, y_n), respectively. Since our curve is continuously convex downward, the center of gravity (ξ, η) of the system lies above the curve, that is,

$$\eta \geqq \xi^p.$$

Substituting in the last inequality the elementary formulas for the coordinates of the center of gravity, namely,

$$\xi = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}, \qquad \eta = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i},$$

a rearrangement furnishes

$$\left(\sum_{i=1}^{n} m_i x_i\right) \leqq \left(\sum_{i=1}^{n} m_i y_i\right)^{1/p}\left(\sum_{i=1}^{n} m_i\right)^{1/q}.$$

The last inequality holds for arbitrary m_i's and x_i's.

Choosing in it $m_i=|v_i|^q$, $m_i x_i^p = m_i y_i = |u_i|^p$, we get $m_i x_i = |u_i||v_i|$, and hence conclude the proof of Hölder's inequality.

(b). We may assume

$$\sum_{i=1}^{n}|u_i+v_i|^p \neq 0;$$

otherwise the proof is trivial. We have

$$\begin{aligned}\sum_{i=1}^{n}|u_i+v_i|^p &= \sum_{i=1}^{n}|u_i+v_i|^{p-1}|u_i+v_i| \\ &\leqq \sum_{i=1}^{n}|u_i+v_i|^{p-1}|u_i| + \sum_{i=1}^{n}|u_i+v_i|^{p-1}|v_i|.\end{aligned}$$

Applying Hölder's inequality to each of the last two sums and observing that $(p-1)q=p$, we get

$$\begin{aligned}\sum_{i=1}^{n}|u_i+v_i|^p \leqq &\left(\sum_{i=1}^{n}|u_i|^p\right)^{1/p}\left(\sum_{i=1}^{n}|u_i+v_i|^p\right)^{1/q} \\ &+\left(\sum_{i=1}^{n}|v_i|^p\right)^{1/p}\left(\sum_{i=1}^{n}|u_i+v_i|^p\right)^{1/q}.\end{aligned}$$

Dividing the last inequality by

$$\left(\sum_{i=1}^{n} |u_i + v_i|^p\right)^{1/q}$$

we conclude the proof of Minkowski's inequality.

It follows therefore that for any $p>1$,

$$\phi(u_1, \cdots, u_n) = \left(\sum_{i=1}^{n} |u_i|^p\right)^{1/p}$$

represents a symmetric norm on $\mathfrak{L}_n$. That conditions (i), (ii), (iv), and (v) of Definition 1 are satisfied is clear; (iii) is given by Minkowski's inequality.

To determine the conjugate $\psi(v_1, \cdots, v_n)$ of the last norm we remark that for any non-zero n tuple $(u_1, \cdots, u_n)$, Hölder's inequality furnishes

$$\frac{u_1v_1 + \cdots + u_nv_n}{\left(\sum_{i=1}^{n} |u_i|^p\right)^{1/p}} \leqq \left(\sum_{i=1}^{n} |v_i|^q\right)^{1/q}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

It is seen readily that equality is actually assumed when we put $u_i = \epsilon_i |v_i|^{q-1}$ for $i=1, \cdots, n$, where $\epsilon_i = 1$ when $v_i > 0$, and $\epsilon_i = -1$ when $v_i < 0$. Thus, we may express the following statement:

For any $p>1$, the conjugate of the norm

$$\phi(u_1, \cdots, u_n) = \left(\sum_{i=1}^{n} |u_i|^p\right)^{1/p}$$

is given by

$$\psi(v_1, \cdots, v_n) = \left(\sum_{i=1}^{n} |v_i|^q\right)^{1/q}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

7. Symmetric norms on the linear space of finite sequences. Consider the set whose elements are infinite sequences of real numbers $(u_1, u_2, \cdots)$, having only a finite number of non-zero terms. Defining addition of elements and multiplication of scalars by elements in the obvious fashion we obtain a linear set $\mathfrak{L}$.

Definition 1′: Under a symmetric norm ϕ on $\mathfrak{L}$ we shall understand any function $\phi(u_1, u_2, \cdots)$ defined on $\mathfrak{L}$, subject to conditions (i), $\cdots$, (v) of Definition 1, in which the n tuples are replaced by elements of $\mathfrak{L}$, that is, infinite sequences having only a finite number of non-zero terms.

Clearly, the set of all elements $(u_1, \cdots, u_n, 0, 0, \cdots)$ of $\mathfrak{L}$ with $u_i = 0$ for $i>n$, may be identified with the n-dimensional linear space $\mathfrak{L}_n$.

A given norm $\phi(u_1, u_2, \cdots)$ on $\mathfrak{L}$ defines a norm $\phi_n(u_1, \cdots, u_n) = \phi(u_1, \cdots, u_n, 0, 0, \cdots)$ on each $\mathfrak{L}_n$. Each $\phi_n(u_1, \cdots, u_n)$ determines on $\mathfrak{L}_n$ a conjugate norm $\psi_n(v_1, \cdots, v_n)$.

LEMMA 10. *For a given n tuple* $(v_1, \cdots, v_n)$ *we have*

$$\psi_n(v_1, \cdots, v_n) = \psi_{n+1}(v_1, \cdots, v_n, 0) = \psi_{n+2}(v_1, \cdots, v_n, 0, 0) = \cdots.$$

Their common value we denote by $\psi(v_1, \cdots, v_n, 0, 0, \cdots)$.

Proof. By definition, we have

$$\begin{aligned}\psi_n(v_1, \cdots, v_n) &= \underset{(u_1,\cdots,u_n)}{\text{l.u.b.}} \frac{u_1v_1 + \cdots + u_nv_n}{\phi_n(u_1, \cdots, u_n)} = \underset{(u_1,\cdots,u_n)}{\text{l.u.b.}} \frac{u_1v_1 + \cdots + u_nv_n}{\phi_{n+1}(u_1, \cdots, u_n, 0)} \\ &\leqq \underset{(u_1,\cdots,u_n,u_{n+1})}{\text{l.u.b.}} \frac{u_1v_1 + \cdots + u_nv_n + u_{n+1}0}{\phi_{n+1}(u_1, \cdots, u_n, u_{n+1})} \\ &= \psi_{n+1}(v_1, \cdots, v_n, 0).\end{aligned}$$

To prove the converse inequality, we remark, by Lemma 1, that for any $n+1$ values $u_1, \cdots, u_n, u_{n+1}$ we have

$$\phi_n(u_1, \cdots, u_n) = \phi_{n+1}(u_1, \cdots, u_n, 0) \leqq \phi_{n+1}(u_1, \cdots, u_n, u_{n+1}).$$

Thus,

$$\frac{u_1v_1 + \cdots + u_nv_n + u_{n+1}0}{\phi_{n+1}(u_1, \cdots, u_n, u_{n+1})} \leqq \frac{u_1v_1 + \cdots + u_nv_n}{\phi_n(u_1, \cdots, u_n)}.$$

Definition 2 implies that $\psi_{n+1}(v_1, \cdots, v_n, 0) \leqq \psi_n(v_1, \cdots, v_n)$. Thus,

$$\psi_{n+1}(v_1, \cdots, v_n, 0) = \psi_n(v_1, \cdots, v_n).$$

The rest of the proof is clear.

LEMMA 11. *For a given symmetric norm* $\phi(u_1, u_2, \cdots)$ *on* $\mathfrak{X}$ *the function* $\psi(v_1, v_2, \cdots)$ *represents also a symmetric norm on* $\mathfrak{X}$.

Proof. Clearly, ψ satisfies properties (i), (ii), (iv), and (v). Thus, it is sufficient to verify (iii). Given two sequences representing elements of $\mathfrak{X}$, there exists a number N such that for both sequences the terms with the subscripts greater than N are equal to zero. This means that the sequences may be written in the form $v_1, \cdots, v_N, 0, 0, \cdots$ and $v_1', \cdots, v_N^1, 0, 0, \cdots$. We have

$$\begin{aligned}&\psi(v_1 + v_1', \cdots, v_N + v_N', 0, 0, \cdots) \\ &\qquad = \psi_N(v_1 + v_1', \cdots, v_N + v_N') \\ &\qquad \leqq \psi_N(v_1, \cdots, v_N) + \psi_N(v_1', \cdots, v_N') \\ &\qquad = \psi(v_1, \cdots, v_N, 0, 0, \cdots) + \psi(v_1', \cdots, v_N', 0, 0, \cdots).\end{aligned}$$

This concludes the proof.

Definition 2′: We term $\psi(v_1, v_2, \cdots)$ the conjugate norm of $\phi(u_1, u_2, \cdots)$ on $\mathfrak{X}$.

LEMMA 12. *Any norm* $\phi(u_1, u_2, \cdots)$ *on* $\mathfrak{L}$ *is at the same time the conjugate of its conjugate norm* $\psi(v_1, v_2, \cdots)$ *on* $\mathfrak{L}$.

Proof. Denote by $\bar{\psi}$ the conjugate of ψ. For $(u_1, u_2, \cdots)$ in $\mathfrak{L}$ there exists an N, such that $u_i = 0$ for $i > N$. By definition,

$$\bar{\psi}(u_1, \cdots, u_N, 0, 0, \cdots) = \bar{\psi}_N(u_1, \cdots, u_N)$$

represents the conjugate of $\psi_N(v_1, \cdots, v_N)$. Hence, by Lemma 9, it must be equal to $\phi_N(u_1, \cdots, u_N) = \phi(u_1, \cdots, u_N, 0, 0, \cdots)$.

This concludes the proof.

Once a norm $\phi(u_1, u_2, \cdots)$ is defined on $\mathfrak{L}$ we obtain a normed linear space $\mathfrak{L}(\phi)$ which, in general, will not be complete. In that case $\mathfrak{L}(\phi)$ can always be imbedded in a complete space as outlined by procedure (1) or (2) of the introduction. The Banach spaces furnished by these procedures will in general be different. The details form the content of the following two sections.

8. Strong closure. This term will be attributed to an imbedding method outlined in (1) of the introduction. Its feasibility in the general case is based essentially on Lemma 1, which clearly carries over to norms $\phi(u_1, u_2, \cdots)$ on $\mathfrak{L}$. This Lemma in the required form was stated and proved in the joint paper of John von Neumann and the author [4, p. 568].

Let ϕ be a norm on $\mathfrak{L}$. For a given infinite sequence of real numbers $(u_1, u_2, \cdots)$, of which an infinite number may be different from zero, we form the corresponding sequence of elements: $(u_1, 0, 0, 0, \cdots)$, $(u_1, u_2, 0, 0, \cdots)$, $(u_1, u_2, u_3, 0, \cdots)$, $\cdots$. By Lemma 1, $\phi(u_1, \cdots, u_n, 0, 0, \cdots)$ represents a non-decreasing function of n.

THEOREM 1. *The set of all sequences of real numbers* $(u_1, u_2, \cdots)$ *for which*

$$\lim_{n\to\infty} \phi(u_1, \cdots, u_n, 0, 0, \cdots) < +\infty$$

forms a Banach space $[[\mathfrak{L}(\phi)]]$; *the number on the left side of the last inequality represents the norm of* $(u_1, u_2, \cdots)$.

Proof. Lemma 1 and conditions (ii), (iii) for the norm ϕ prove that the set of sequences $(u_1, u_2, \cdots)$ satisfying the last inequality is a normed linear space. Thus, it is sufficient to prove completeness. Let, therefore, $(u_1^{(i)}, u_2^{(i)}, \cdots)$, $i = 1, 2, \cdots$, form a fundamental sequence of elements of $[[\mathfrak{L}(\phi)]]$. This means that for a given $\epsilon > 0$ we can find a natural $N = N(\epsilon)$, such that

$$\lim_{n\to\infty} \phi(u_1^{(i)} - u_1^{(j)}, \cdots, u_n^{(i)} - u_n^{(j)}, 0, 0, \cdots) < \epsilon$$

whenever both i and j are greater than or at most equal to N. Thus, Lemma 1 gives

(a) $\phi(u_1^{(i)} - u_1^{(j)}, \cdots, u_n^{(i)} - u_n^{(j)}, 0, 0, \cdots) < \epsilon,$

for $i, j \geqq N$; $n = 1, 2, \cdots$.

This, by Lemma 2 implies

$$|u_n^{(i)} - u_n^{(j)}| < \epsilon \quad \text{for } i, j \geqq N;\ n = 1, 2, \cdots .$$

We put $\lim_{i\to\infty} u_{n_i}^{(i)} = u_n$ for $n = 1, 2, \cdots$. By Lemma 4 and (a) above we get

$$\phi(u_1^{(i)} - u_1, \cdots, u_n^{(i)} - u_n, 0, 0, \cdots) \leqq \epsilon \qquad \text{for } i \geqq N;\ n = 1, 2, \cdots .$$

Applying again Lemma 1, we obtain

$$\lim_{n\to\infty} \phi(u_1^{(i)} - u_1, \cdots, u_n^{(i)} - u_n, 0, 0, \cdots) \leqq \epsilon \qquad \text{for } i \geqq N.$$

This proves that with $(u_1^{(i)}, u_2^{(i)}, \cdots)$ also $(u_1^{(i)} - u_1, u_2^{(i)} - u_2, \cdots)$ and therefore $(u_1, u_2, \cdots)$ are elements of $[[\mathfrak{L}(\phi)]]$; the sequence $(u_1^{(i)}, u_2^{(i)}, \cdots)$ is convergent in the norm ϕ to $(u_1, u_2, \cdots)$. This concludes the proof.

9. The Cantor-Méray closure. In the present section we shall discuss the quite frequently used method of imbedding any metric space, hence also $\mathfrak{L}(\phi)$, in its Cantor-Méray closure $[\mathfrak{L}(\phi)]$.

THEOREM 2. *The Cantor-Méray closure of $\mathfrak{L}(\phi)$ is always included in its strong-closure. The sequences $(u_1, u_2, \cdots)$ forming the elements of the Cantor-Méray closure of $\mathfrak{L}(\phi)$ must all converge towards zero.*

Proof. Clearly, every element in $\mathfrak{L}(\phi)$ also belongs to $[[\mathfrak{L}(\phi)]]$. Thus, precisely as in the proof of Theorem 1, a fundamental sequence $(u_1^{(i)}, u_2^{(i)}, \cdots)$, $i = 1, 2, \cdots$, of elements of $\mathfrak{L}(\phi)$ determines a sequence $(u_1, u_2, \cdots)$ in $[[\mathfrak{L}(\phi)]]$ to which it converges. This means that for every $\epsilon > 0$, we can find a natural number $N = N(\epsilon)$, such that, for all $i \geqq N$, we have

$$\lim_{n\to\infty} \phi(u_1^{(i)} - u_1, \cdots, u_n^{(i)} - u_n, 0, 0, \cdots) < \epsilon.$$

Thus, again Lemma 1 gives

$$|u_n^{(i)} - u_n| < \epsilon \qquad \text{for } i \geqq N;\ n = 1, 2, \cdots .$$

In particular

$$|u_n^{(N)} - u_n| < \epsilon \qquad \text{for } n = 1, 2, \cdots .$$

Since $(u_1^{(N)}, u_2^{(N)}, \cdots)$ is an element of $\mathfrak{L}(\phi)$, all its terms, say for $n \geqq n_0$, are equal to zero. Hence

$$|u_n| = |u_n^{(N)} - u_n| < \epsilon \qquad \text{for } n \geqq n_0.$$

Therefore, $\lim_{n\to\infty} u_n = 0$. This concludes the proof.

THEOREM 3. *The strong closure coincides with the Cantor-Méray closure of* $\mathfrak{L}(\phi)$ *if and only if for every sequence of real numbers* $(u_1, u_2, \cdots)$ *the condition*

$$\lim_{n\to\infty} \phi(u_1, \cdots, u_n, 0, 0, \cdots) < +\infty$$

implies

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \phi(u_m, u_{m+1}, \cdots, u_n, 0, 0, \cdots) = 0.$$

Proof. The proof is an immediate consequence of the preceding discussion. Remark: In particular, when for $(u_1, u_2, \cdots)$ in $\mathfrak{L}$,

$$\phi(u_1, u_2, \cdots) = \left(\sum_i |u_i|^p \right)^{1/p}; \qquad p > 1,$$

the strong closure of $\mathfrak{L}(\phi)$ coincides with its Cantor-Méray closure.

Example. We supplement Theorems 1 and 2 by showing that the Cantor-Méray closure may actually represent a proper subset of the strong closure for $\mathfrak{L}(\phi)$. For this purpose we define

$$\phi(u_1, u_2, \cdots) = \max_i |u_i| \qquad \text{for } (u_1, u_2, \cdots) \text{ in } \mathfrak{L}.$$

Clearly, ϕ is a norm. It is seen readily that the strong closure of $\mathfrak{L}(\phi)$ represents the Banach space of all bounded sequences $(u_1, u_2, \cdots)$ of real numbers, where l.u.b. $|u_i|$ represents the norm [1, p. 11]. The Cantor-Méray closure of $\mathfrak{L}(\phi)$ represents the Banach space of all sequences of real numbers converging towards 0, with the least upper bound stated above as its norm [1, p. 181].

10. Strong closures as conjugate spaces of Cantor-Méray closures. In the present section we indicate another relationship between the strong and Cantor-Méray closure of $\mathfrak{L}(\phi)$.

THEOREM 4. *Let* $\phi(u_1, u_2, \cdots)$ *denote a norm on* $\mathfrak{L}$ *and* $\psi(v_1, v_2, \cdots)$ *its conjugate norm. Then, the conjugate space of the Cantor-Méray closure of* $\mathfrak{L}(\phi)$ *may be characterized as the space of all those sequences of real number* $(v_1, v_2, \cdots)$ *(of which an infinite number of* v_i*'s may be* $\neq 0$*) for which*

$$\lim_{n\to\infty} \psi(v_1, \cdots, v_n, 0, 0, \cdots) < +\infty.$$

The number on the left side of the last inequality represents the norm of $(v_1, v_2, \cdots)$.

Thus, the conjugate space of the Cantor-Méray closure of $\mathfrak{L}(\phi)$ *may be interpreted as the strong closure of* $\mathfrak{L}(\psi)$. *Conversely, the strong closure of* $\mathfrak{L}(\phi)$ *may be interpreted as the conjugate space of the Cantor-Méray closure of* $\mathfrak{L}(\psi)$.

Proof. Clearly, a linear functional F on $\mathfrak{L}(\phi)$ can be extended in a unique manner without changing the value of its bound to a linear functional on $[\mathfrak{L}(\phi)]$. Thus, the conjugate space of $[\mathfrak{L}(\phi)]$ may be identified with the space of linear functionals on $\mathfrak{L}(\phi)$.

So let F denote a linear functional on $\mathfrak{L}(\phi)$ and $|||F|||$ represent its bound. As for the finite-dimensional case we put $F(1, 0, 0,0, \cdots)=v_1$, $F(0, 1, 0, 0 \cdots)=v_2$, $F(0, 0, 1, 0, \cdots)=v_3, \cdots$. Thus, F determines a sequence of numbers $(v_1, v_2, \cdots)$. Clearly,

$$\frac{u_1v_1 + \cdots + u_nv_n}{\phi_n(u_1, \cdots, u_n)} = \frac{F(u_1, \cdots, u_n, 0, 0, \cdots)}{\phi(u_1, \cdots, u_n, 0, 0, \cdots)} \leqq |||F|||.$$

The last inequality holds for any n tuple $(u_1, \cdots, u_n)$. Thus,

$$\psi(v_1, \cdots, v_n, 0, 0, \cdots) = \psi_n(v_1, \cdots, v_n) \leqq |||F|||.$$

By Lemma 1, $\psi(v_1, \cdots, v_n, 0, 0, \cdots)$ is a nondecreasing function of n. Thus,

$$\lim_{n\to\infty} \psi(v_1, \cdots, v_n, 0, 0, \cdots) \leqq |||F|||.$$

On the other hand, using Lemma 6 for an element $(u_1, \cdots, u_n, 0, 0, \cdots)$ in $\mathfrak{L}(\phi)$, we have

$$\begin{aligned}\frac{F(u_1, \cdots, u_n, 0, 0, \cdots)}{\phi(u_1, \cdots, u_n, 0, 0, \cdots)} = \frac{u_1v_1 + \cdots + u_nv_n}{\phi_n(u_1, \cdots, u_n)} &\leqq \frac{\phi_n(u_1, \cdots, u_n)\psi_n(v_1, \cdots, v_n)}{\phi_n(u_1, \cdots, u_n)} \\ &= \psi_n(v_1, \cdots, v_n) = \psi(v_1, \cdots, v_n, 0, 0, \cdots) \\ &\leqq \lim_{n\to\infty} \psi(v_1, \cdots, v_n, 0, 0, \cdots).\end{aligned}$$

Thus,

$$|||F||| \leqq \lim_{n\to\infty} \psi(v_1, \cdots, v_n, 0, 0, \cdots).$$

We have shown, therefore, so far, that a linear functional F on $\mathfrak{L}(\phi)$ determines a unique sequence $(v_1, v_2, \cdots)$ of real numbers, for which

$$|||F||| = \lim_{n\to\infty} \psi(v_1, \cdots, v_n, 0, 0, \cdots) < +\infty.$$

Conversely, it is readily seen that every sequence of numbers $(v_1, v_2, \cdots)$, for which the limit on the left side of the last inequality is finite, corresponds to some linear functional F on $\mathfrak{L}(\phi)$.

This proves the first part of our theorem. Using this and Lemma 12 we establish the remaining part. This concludes the proof.

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THE BEST POLYNOMIAL APPROXIMATION OF FUNCTIONS

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1. Introduction. One of the most interesting mathematical studies is that of finding the "best" polynomial approximation of a given function. The solution of the problem depends, of course, on the meaning we attach to the expression "the best polynomial approximation." In this expository paper we will discuss some of the aspects of best approximation in the classical Tchebycheff sense.

2. Statement of the problem. Let $f(x)$ and $\theta(x)$ be given functions which are continuous in the finite closed interval (a, b). The problem is to find values of the parameters $p_0, p_1, \cdots, p_n$ such that the polynomial $P_n(x) = p_0x^n + p_1x^{n-1} + \cdots + p_n$, of degree not exceeding n, will differ in absolute value as little as possible from $f(x)$ in the sense that the deviation

$$D_n(p) = D_n(p_0, p_1, \cdots, p_n) = \max_{a \leqq x \leqq b} \left| f(x) - P_n(x)/\theta(x) \right| , \tag{1}$$

shall be as small as possible for all x in (a, b). If $p_0, p_1, \cdots, p_n$ are values of the parameters which give $D_n(p)$ its minimum value then we call the function

$$\phi_n(x; p) = f(x) - P_n(x)/\theta(x) \tag{2}$$

a function of least deviation from zero corresponding to all functions of the given type $f(x)$. Let $x_1 < x_2 < \cdots < x_m$, $m \geqq 1$, $(x_i = x_{i,n})$ be the points in (a, b) where $\phi_n(x; p)$ attains its extreme value $\pm D_n(p)$. That is, $\phi_n(x_i; p) = \epsilon_i D_n(p)$, $\epsilon_i = \pm 1$, $(i = 1, 2, \cdots, m)$. We call each x_i a point of deviation of $\phi_n(x; p)$.

3. Fundamental theorems. The following basic theorems are necessary for the discussion which follows. Limitations of space forbid the presentation of proofs. For these the reader may consult [11]. At the outset we assume that a function of least deviation from zero exists, and that it is unique in the sense that there cannot be two sets of parameters which correspond to two different $\phi_n(x; p)$-functions having the same minimum deviation.

(i) The function $\phi_n(x; p)$ has at least $n+2$ points of deviation $x_1 < x_2 < \cdots < x_{n+2}$; $\phi_n(x_i; p) = \epsilon_i D_n(p)$, $\epsilon_i = \pm 1$, $(i = 1, 2, \cdots, n+2)$.

(ii) If the parameters $p_0, p_1, \cdots, p_n$ are so chosen that the sequence $\epsilon_i D_n(p)$, $(i = 1, 2, \cdots, m)$, has at least $n+1$ variations of signs, then the corresponding function $\phi_n(x; p)$ is a function of least deviation.

4. Applications. We will now solve two typical problems in the theory of best approximation.

Problem 1. In (2) take $\theta(x) = 1$, $f(x) = -x^{n+1}$, and seek the polynomial $F(x) = x^{n+1} + p_0x^n + \cdots + p_n$ which deviates the least from zero in $(-1, 1)$.

Solution. By *(i)*, $F(x)$ has at least $n+2$ points of deviation. The only possible points of deviation are the end points $x = \pm 1$ and the points within $(-1, 1)$ at which $F'(x) = 0$. But $F'(x) = 0$ has at most n roots, hence $F(x)$ has exactly $n+2$ points of deviation, namely, the points $x = 1$, $x = -1$, and the n roots of $F'(x) = 0$.

We denote the latter by $x_1, x_2, \cdots, x_n$, and write D for $D_n(p)$. Clearly the points $x_1, x_2, \cdots, x_n$ are double roots of $F(x)^2-D^2=0$, so that we may write

$$F(x)^2 - D^2 = c(x-1)(x+1)\prod_{i=1}^{n}(x-x_i)^2. \tag{3}$$

Since $F(x)=x^{n+1}+\cdots$ we see at once that $c=1$. Again

$$F'(x) = c_1(x-x_1)(x-x_2)\cdots(x-x_n) = (n+1)x^n+\cdots$$

so that $c_1=n+1$. Thus (3) may be written in the form

$$F(x)^2 - D^2 = (x^2-1)[F'(x)/(n+1)]^2,$$

or, as a differential equation,

$$\frac{F'(x)dx}{[F(x)^2-D^2]^{1/2}} = \frac{(n+1)dx}{(x^2-1)^{1/2}}.$$

Integrating we obtain

$$F(x) + [F(x)^2 - D^2]^{1/2} = C[x+(x^2-1)^{1/2}]^{n+1}, \tag{4}$$

and hence

$$F(x) - [F(x)^2 - D^2]^{1/2} = D^2C^{-1}[x-(x^2-1)^{1/2}]^{n+1}. \tag{5}$$

To determine C notice that $F(1)=\pm D$, and that $x=1$ in (4) gives $C=\pm D$, with the same sign as $F(1)$. Substituting this value of C into (4) and (5), and adding, we obtain

$$F(x) = \pm 2^{-1}D\{[x+(x^2-1)^{1/2}]^{n+1} + [x-(x^2-1)^{1/2}]^{n+1}\}. \tag{6}$$

Equation (6) enables us to evaluate D. Expanding the right-hand side in powers of x, and recalling that $F(x)=x^{n+1}+\cdots$, we conclude that $D=2^{-n}$. If we now let $x=\cos\phi$ in (6), we get

$$F(x) = D\cos((n+1)\phi) = 2^{-(n+1)}\cos((n+1)\arccos x). \tag{7}$$

Notice that as x varies from -1 to $+1$, $D\cos((n+1)\phi)$ passes alternately through the values $D, -D, D, \cdots$ with $n+1$ variations of signs, hence, by (*ii*) we conclude that this trigonometric polynomial is the required function of least deviation from zero.

The transformation $2x=(b+a)+(b-a)t$, $-1\leqq t\leqq 1$, $a\leqq x\leqq b$, applied to (7) shows immediately that the polynomial

$$2\left(\frac{b-a}{4}\right)^{n+1}\cos\left((n+1)\arccos\left(\frac{2x-b-a}{b-a}\right)\right) = x^{n+1}+\cdots$$

deviates the least from zero on (a, b) among all polynomials of the form, $x^{n+1}+\cdots$, a conclusion which is often used in the general theory of polynomials. Tchebycheff [9] used this and similar results to prove many beautiful and useful theorems in the theory of equations.

Problem 2. (Poncelet's problem.) Our second illustration is the celebrated problem of Poncelet which in the masterly hands of Tchebycheff became the cornerstone of the theory of best approximation. Poncelet sought an approximation of $(x^2+y^2)^{1/2}$ by a linear function $\alpha x+\beta y$ such that the maximum of the relative error

$$\left| \frac{\alpha x + \beta y}{(x^2 + y^2)^{1/2}} - 1 \right|$$

should be as small as possible when $ax \leqq y \leqq bx$, $0 \leqq a < b$, $x > 0$. If we set $z = y/x$, then the problem is to determine α and β so that the deviation D of the function $\phi(z) = (\alpha+\beta z)/(1+z^2)^{1/2} - 1$ shall be a minimum in $a \leqq z \leqq b$. This is a problem of the type we have been studying with $\theta(x) = (1+z^2)^{1/2}$, $f(z) = 1$.

Exactly as in Problem 1, the points of deviation are $z = a$, $z = b$, and $z = \beta/\alpha$, the point at which $\phi'(z) = 0$. Since there must be at least two variations of signs, we write

$$\phi(a) = \pm D, \tag{8}$$

$$\phi(\beta/\alpha) = \mp D, \tag{9}$$

$$\phi(b) = \pm D. \tag{10}$$

These three equations enable us to determine α, β, and D. To this end, introduce $a = \tan \delta_0$, $b = \tan \delta_1$, then from (8) and (10) we have

$$\alpha \cos \delta_0 + \beta \sin \delta_0 - 1 = \pm D, \tag{11}$$

$$\alpha \cos \delta_1 + \beta \sin \delta_1 - 1 = \pm D; \tag{12}$$

hence, by subtraction,

$$\alpha \sin \frac{\delta_0 + \delta_1}{2} = \beta \cos \frac{\delta_0 + \delta_1}{2},$$

so that

$$\alpha = C \cos \frac{\delta_0 + \delta_1}{2}, \qquad \beta = C \sin \frac{\delta_0 + \delta_1}{2},$$

where $C \neq 0$ is a constant to be determined. Substituting these values of α and β into (12) and (9) we obtain, in turn,

$$C \cos \frac{\delta_0 - \delta_1}{2} - 1 = \pm D, \qquad C - 1 = \mp D,$$

and, by addition, we find that

$$C = \sec^2 \frac{\delta_0 - \delta_1}{4},$$

so that $C>1$. Hence $D=C-1=\sec^2(\delta_0-\delta_1)/4-1=\tan^2(\delta_0-\delta_1)/4$, and $\phi(a)=-D, \phi(\beta/\alpha)=+D, \phi(b)=-D$. Thus by (*ii*), we have discovered the required function of least deviation and the problem is completely solved.

It remains to give a numerical example. Suppose, for instance, that we take $0\leqq y\leqq x$ so that $a=0$, $b=1$, $\delta_0=0°$, $\delta_1=45°$. Then $\alpha=0.9605$, $\beta=0.3978$, and $D=0.0396$. Hence

$$(x^2+y^2)^{1/2}=0.9605x+0.3978y, \qquad x\geqq y\geqq 0,$$

with a relative error less than 4%.

The following result may be derived in a similar manner.

$$(x^2+y^2+z^2)^{1/2}\cong 0.940x+0.328y+0.299z, \quad x\geqq y\geqq z\geqq 0,$$

with a relative error less than 6%. We may prove that

$$e^x\cong 0.8459x+0.9549x+1.0007, \text{ error } <0.088, \qquad 0\leqq x\leqq 1;$$

$$e^{(x+y)/2}\cong 0.946+0.649(x+y)+0.421xy, \text{ error } <0.054, \qquad 0\leqq x, y\leqq 1;$$

$$e^{(x+y)/2}\cong 0.908+0.638x+1.050y, \text{ error } <0.054, \qquad 0\leqq x, y\leqq 1;$$

$$|x|\cong 0.0675+1.9313x^2-1.0665x^4, \text{ error } <0.0677, \qquad -1\leqq x\leqq 1.$$

5. Conclusion. The problem of Poncelet stands at the beginning of the theory of best approximation. G. Zolotareff [12], W. Markoff [5], S. Bernstein [2], to mention only a few names, have greatly extended the theory. Present day mathematicians are still finding this a fertile field for further research. We mention, for example, the work of N. Achyeser [1] on polynomials of least deviation in two disjoint intervals, and that of J. Shohat [8] who uses the close connection between the theory of interpolation and that of best approximation. A problem similar to Poncelet's problem was solved by J. V. Uspensky [10]. As examples of problems in a closely related field we list the work of I. Schur [7], and that of P. Erdös and G. Szegö [4]. There is still much to be done in extending the theory to functions of several variables. In 1889 Tchebycheff studied rational approximations of the function $x^{-1/2}$ and applied his theory to the approximate evaluation of elliptic integrals. The theory of best approximation can be used to give approximate solutions of differential equations. The recent book by E. Remes [6] is recommended for its success in applying the classical theory of best approximation to practical engineering problems.

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MATHEMATICAL NOTES

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ON THE METHOD OF CHARACTERISTICS*

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1. Introduction. In problems in applied mathematics, it is frequently necessary to consider systems of equations which contain both finite and differential relations.[1] As an example, let us consider a set of $m+n$ unknown functions which are assumed to satisfy n first order quasi-linear partial differential equations and m finite relations. The straightforward method of solution (method A) is to solve the finite relations for m of the variables, substitute in the differential equations, and solve the resulting set of n first order equations in n unknowns. However, it may prove more convenient to differentiate each of the finite relations with repect to one of the independent variables and solve the resulting set of $m+n$ first-order equations in $m+n$ unknowns (method B).

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[1] See, for example, P. S. Symonds, "On the general equations of problems of axial symmetry in the theory of plasticity," Q. Appl. Math. *6*, 448–452 (1949). It was a consideration of this problem which suggested the present note.

Since a system of first order equations is often most easily solved by the method of characteristics, the question arises as to the relation between the characteristics obtained by the two methods. It is intuitively plausible that the $m+n$ families of characteristics obtained by method B will consist of the n families obtained by method A, plus families of total multiplicity m parallel to one or the other coordinate axis. In the present note, a proof of this statement will be given.

2. Discussion. Let the given set of equations be

$$\sum_{j=1}^{n}\left[A_{ij}\frac{\partial u_j}{\partial x}+B_{ij}\frac{\partial u_j}{\partial y}\right]+\sum_{\alpha=1}^{m}\left[C_{i\alpha}\frac{\partial v_j}{\partial x}+D_{i\alpha}\frac{\partial v_j}{\partial y}\right]=E_i, \tag{1}$$

$$v_\alpha = f_\alpha(x, y, u_1, \cdots, u_n) ;$$

here Latin indices range from 1 to n, and Greek indices from 1 to m. The functions $A_{ij}, \cdots, E_i$ may depend upon x, y, u_j, v_α, but are independent of the derivatives of u_j and v_α. In order to save space, we shall consider the case where $n=2$, $m=1$, although the method is perfectly general. In this case (1) consists of three equations of the form

$$\begin{aligned} A_1u_x + B_1u_y + C_1v_x + D_1v_y + E_1w_x + F_1w_y &= G_1, \\ A_2u_x + B_2u_y + C_2v_x + D_2v_y + E_2w_x + F_2w_y &= G_2, \end{aligned} \tag{2}$$

$$w = f(x, y, u, v), \tag{3}$$

where the subscripts x and y indicate differentiation.

To solve the problem by method A, we substitute (3) into (2), obtaining

$$\begin{aligned} (A_k + E_kf_u)u_x + (B_k + F_kf_u)u_y + (C_k + E_kf_v)v_x \\ + (D_k + F_kf_v)v_y = G_k - E_kf_x - F_kf_y, \qquad k = 1, 2. \end{aligned} \tag{4}$$

The characteristics of (4) are those curves along which the determinant

$$\Delta_A = \begin{vmatrix} A_1 + E_1f_u & B_1 + F_1f_u & C_1 + E_1f_v & D_1 + F_1f_v \\ A_2 + E_2f_u & B_2 + F_2f_u & C_2 + E_2f_v & D_2 + F_2f_v \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{vmatrix}$$

vanishes.

To solve the problem by method B we differentiate (3) with respect to x, obtaining

$$f_uu_x + f_vv_x - w_x = f_x. \tag{5}$$

The characteristics of the three equations (2) and (5) are the curves along which the determinant

$$(6)\qquad \Delta_B = \begin{vmatrix} A_1 & B_1 & C_1 & D_1 & E_1 & F_1 \\ A_2 & B_2 & C_2 & D_2 & E_2 & F_2 \\ f_u & 0 & f_v & 0 & -1 & 0 \\ dx & dy & 0 & 0 & 0 & 0 \\ 0 & 0 & dx & dy & 0 & 0 \\ 0 & 0 & 0 & 0 & dx & dy \end{vmatrix}$$

vanishes. We perform elementary operations on the columns and rows respectively to obtain

$$(7)\qquad \Delta_B = \left| C_1 + f_u C_5 \quad C_2 + f_u C_6 \quad C_3 + f_v C_5 \quad C_4 + f_v C_6 \quad C_5 \quad C_6 \right|,$$

$$(8)\qquad \Delta_B = \left| R_1 \quad R_2 \quad R_3 \quad R_4 \quad R_5 \quad R_6 - f_u R_4 - f_v R_5 \right|;$$

here C_i represents the ith column of (6), and R_j the jth column of (7). Finally we may expand (8) in terms of the cofactors of the third and last rows respectively to obtain

$$\Delta_B = (-1)(dy)(\Delta_A).$$

Therefore, the characteristics according to method B are the four families of curves obtained as the characteristics by method A, plus the family of curves $dy = 0$.

3. Conclusion. In the general case, an analogous procedure applied to (1) leads to the relation

$$\Delta_B = \pm (dy)^p (dx)^{m-p} (\Delta_A);$$

here p represents the number of finite relations differentiated with respect to x, and $m-p$ the number differentiated with respect to y.

The same conclusion can be drawn in the case where some of the equations are of the second or higher order. Thus, in general, it is permissible to differentiate any or all of a set of equations before obtaining the characteristics.

A REMARK ON INFINITE GROUPS

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1. Introduction. L. J. Paige [1] has shown that if G is a finite abelian group such that the number of elements of order two is different from unity, then there exists a one-to-one mapping $x \to \theta(x)$ of G onto itself such that $x \to \eta(x) = x \cdot \theta(x)$ is also a one-to-one mapping of G onto itself. This result was essentially proved also by Vijayaraghavan and Chowla [2] (independently of Paige) by explicitly constructing such mappings. Later Paige in effect proved (cf. [3], pp. 50–51) that mappings θ exist for finitely generated infinite abelian groups in which the number of elements of order two is different from unity. It is the purpose of this note to remark that the well-ordering principle implies immediately

the existence of a mapping θ for any infinite group whatever, in fact for any infinite quasigroup. (A quasigroup is a set G in which is defined a product xy for every ordered pair x, y of elements of G such that if, in the equation $xy=z$, any two of the symbols x, y, z are assigned as elements of G, the third is uniquely determined as an element of G.)

2. Theorem. We shall establish the following result.

THEOREM. *For any infinite quasigroup G there exists a one-to-one mapping $x\to\theta(x)$ of G onto itself such that $x\to\eta(x)=x\cdot\theta(x)$ is also a one-to-one mapping of G onto itself. Moreover if a and b are any given elements of G, we can choose θ so that $\theta(a)=b$.*

3. Proof. We may assume that $G=\{g\alpha\}$, where α runs over the ordinal numbers less than some fixed initial ordinal δ (*cf.* [4], pp. 71–72). We want to show that for each α less than δ we can define x_α, $y_\alpha=\theta(x_\alpha)$, and $z_\alpha=\eta(x_\alpha)$, such that $x_\alpha y_\alpha=z_\alpha$, and such that each element of G occurs once and only once in $\{x_\alpha\}$, in $\{y_\alpha\}$, and in $\{z_\alpha\}$. First we choose $x_1=a$, $y_1=b$, $z_1=ab$. Then suppose β is any ordinal less than δ and assume that we have defined x_α, y_α, z_α for $\alpha<\beta$. Put $X_\beta=\{x_\alpha\}_{\alpha<\beta}$, $Y_\beta=\{y_\alpha\}_{\alpha<\beta}$, $Z_\beta=\{z_\alpha\}_{\alpha<\beta}$. Suppose t_β is the first element of G not in $X_\beta\cap Y_\beta\cap Z_\beta$. If $t_\beta\notin X_\beta$, we choose x_β as t_β and y_β as the first element of G not in $Y_\beta\cup\{y\,|\,t_\beta y\in Z_\beta\}$. (The set $Y_\beta\cup\{y\,|\,t_\beta y\in Z_\beta\}$ is a proper subset of G since δ is an initial ordinal.) If $t_\beta\in X_\beta$ but $t_\beta\notin Y_\beta$, we choose y_β as t_β and z_β as the first element of G not in $Z_\beta\cup\{z\,|\,z\in X_\beta t_\beta\}$. If $t_\beta\in X_\beta$ and $t_\beta\in Y_\beta$ but $t_\beta\notin Z_\beta$, we choose z_β as t_β and x_β as the first element of G not in $X_\beta\cup\{x\,|\,t_\beta\in x Y_\beta\}$. This completes the proof.

4. Remarks. It is known (cf. [3], pp. 40–42 and the first part of the proof of Theorem II.3 on p. 50) that if G is an abelian group, then the existence of a mapping θ (of the above type) with $\theta(1)=1$ ensures the existence of a neofield for which G is the multiplicative group. Thus as a corollary of the above theorem we see that any infinite abelian group can be the multiplicative group of a neofield. Our theorem also implies (cf. [1], §1) that (i) every infinite quasigroup is isotopic to an idempotent quasigroup, (ii) every infinite group is a subgroup of the automorphism group of some loop.

The question of the existence of mappings θ for finite non-abelian groups appears difficult.

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A NOTE ON SUMMATION

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The use of arithmetic means in summing nonconvergent series suggests that perhaps the harmonic and geometric means may yield results of interest. Proceeding in a manner similar to that of Hölder, we consider the series

$$U = \sum_{k=1}^{\infty} u_k. \quad \text{Let } s_n = \sum_{k=1}^{n} u_k, \quad \text{and let } {}_1H_n = n \Big/ \sum_{k=1}^{n} (1/s_k).$$

Then ${}_1H_n$ is merely the harmonic mean of the first n partial sums of U. In general, let ${}_rH_n = n/\sum_{k=1}^{n}(1/{}_{r-1}H_k)$. If r is the smallest integer for which $\lim_{n\to\infty} {}_rH_n$ exists, we shall call the series summable (r, H) by harmonic means* of order r. This, of course, is not a regular method of summation, for it is not applicable to series having any partial sum zero, nor to series converging to zero.

As an example of a series summable $(1, H)$, consider the series whose partial sums are recurrently $2/3, 1, 2$. Then $\lim_{n\to\infty} {}_1H_n = 1$. Hence the series is summable $(1, H)$ to the value 1. Similarly, we can easily construct a series for which the harmonic means of the partial sums are recurrently $2/3$, 1, 2. That is, for six terms, $U = 2/3 + 4/3 - 4 + 20/9 - 11/9 + 1/2 + \cdots$. Then $s_1 = 2/3$, $s_2 = 2$, $s_3 = -2$, $s_4 = 2/9$, $s_5 = -1$, $s_6 = -1/2$, $\cdots$, and ${}_1H_{3n} = 2/3$, ${}_1H_{3n+1} = 1$, ${}_1H_{3n+2} = 2$ for $n = 1, 2, \cdots$. Clearly, a series constructed in such a manner is summable $(2, H)$ to the value 1.

The harmonic mean has the desired summation property that if U is a convergent series with the ordinary sum S, satisfying the special conditions above, then U is summable $(1, H)$ to S. For let $p_n = 1/s_n$ and $P = 1/S$. Then $\lim_{n\to\infty} p_n = P$. The p_n may be considered as the partial sums of a convergent series. Hence, $\lim_{n\to\infty} (\sum_{k=1}^{n} p_k)/n = P$, or $\lim_{n\to\infty} n/\sum_{k=1}^{n}(1/s_k) = S$.

A series may be summable (r, H) when it is not summable (C, r), and the converse is true. Consider the series whose partial sums occur in pairs, either 1, 1 or 10, 10/19. The harmonic mean of either pair is 1. The arithmetic mean of the first pair is 1 and that of the second is 100/19. Let us generate a series in the following manner: For the first pair of partial sums, take 1, 1. Then take 10, 10/19 enough times so that the arithmetic mean exceeds 4. Then take 1, 1 enough times so that the arithmetic mean is less than 2. If this process is continued, we obtain a series that is summable $(1, H)$ to the value 1 but is not summable $(C, 1)$. Obviously, a similar series could be constructed that would be summable $(C, 1)$ but not $(1, H)$.

In view of the arithmetic-geometric-harmonic mean inequality, it is apparent that the geometric mean of the partial sums is also applicable to series having only positive partial sums; and so, indeed, are means of other orders.

*Not to be confused with

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=0}^{n} \frac{s_{n-k}}{k+1}. \quad \text{Ed.}$$

CLASSROOM NOTES

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$C_{n,r}$, WHERE THE n THINGS INCLUDE IDENTITIES

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1. Introduction. Given the set of objects:

$$A,\ A,\ A,\ A,\ B,\ B,\ B,\ C,\ C,\ D,$$

in which the first 4 are identical, the next three, and so forth, it is desired to find the number of different combinations of those 10 things taken 5 at a time. This number is quite different from $C_{10,5}$, where the 10 things are considered to be all different.

2. Notation. $[a, b, c, \cdots]$ is a finite set of objects having a subset of a identical objects, a second subset of b identical objects, and so forth. The set mentioned in Section 1 would be $[4, 3, 2, 1]$, since there are 4 identical A's making up the first subset, 3 B's in the second, and so forth.

The number of objects in a set is always called n.

$$n = a + b + c + \cdots . \tag{1}$$

All the letters used are supposed to be non-negative integers.

If two or more of the subsets of identical objects have the same number of elements, the notation can be abbreviated by the use of exponents. Thus, the set $[5, 5, 1, 1, 1, 1, 1]$ may be written $[5^2, 1^5]$.

$C_r[a, b, \cdots]$ is the number of different combinations of the objects of the set taken r at a time. For example, the problem proposed in Section 1 in notation is to evaluate $C_5[4, 3, 2, 1]$.

3. The ϕ-function for a Set. Associated with each set is a function ϕ which is a polynomial in x, and which serves as a generating function.

$$\phi[a] = 1 + x + x^2 + \cdots + x^a. \tag{2}$$

$$\phi[a, b, c \cdots] = \phi[a]\cdot\phi[b]\cdot\phi[c]\cdots . \tag{3}$$

The degree of ϕ is always n. In fact:

$$\begin{aligned}\phi[a, b, c] &= (1 + x + \cdots + x^a)(1 + \cdots + x^b)(1 + \cdots + x^c),\\ &= A_0 + A_1\cdot x + A_2\cdot x^2 + \cdots + A_n\cdot x^n.\end{aligned} \tag{4}$$

The value of any coefficient, A_r, can be found by multiplying out the factors of $\phi[a, b, c]$, and collecting terms of like powers. Before collecting the terms, their number will be $(1+a)(1+b)(1+c)$, and each coefficient is 1. Each term in ϕ is formed by selecting one term in each of the factors.

For example, suppose the term $x^{a'}$ is selected from the first factor, $x^{b'}$ from the second, and $x^{c'}$ from the third. Their product is x to the power $a'+b'+c'$. Now if we are trying to evaluate A_r, we shall select a', b', and c' so that their sum will be r. Also a' is at most a, b' is at most b, and c' is at most c. The value of A_r will be precisely the number of ways of choosing a set of values for a', b', and c'.

In trying to evaluate $C_r[a, b, c]$, one combination will result from choosing a' things from the first subset of a identical things, b' from the second, and c' from the third. The number of possible choices for a', b', and c' is the same as when we were forming terms of $\phi[a, b, c]$. Therefore,

$$C_r[a, b, c] = A_r,$$

the typical coefficient in Equation (4). Moreover, the reasoning may be extended from the set $[a, b, c]$ with only 3 subsets to the general set $[a, b, c, \cdots]$. Therefore,

$$\phi[a, b, c, \cdots] = \sum_{r=0}^{r=n} C_r[a, b, c, \cdots .] \cdot x^r. \tag{5}$$

As a check, apply Equation (5) in the case of the set $[1^n]$, having n distinct elements. $C_r[1^n]$ is the familiar $C_{n,r}$, and

$$\phi[1^n] = (\phi[1])^n = (1 + x)^n. \tag{6}$$

Equation (6), is the familiar relation between the values of $C_{n,r}$ and the binomial coefficients.

Another familiar case is obtained by setting $x=1$ in Equation (5). The sum of the values of $C_r[a, b, c, \cdots]$ for all values of r is $\phi(1)=(1+a)(1+b)(1+c)\cdots$.

4. A Recursion Formula. Next a recursion formula for the determination of $C_r[a, b, c, \cdots .]$ will be derived.

$$\begin{aligned}\phi[a, \cdots, b, c] &= \phi[c] \cdot \phi[a, \cdots, b], \\ &= (1 + x + x^2 + \cdots + x^c) \sum_{j=0}^{j=n-c} C_j[a, \cdots, b] \cdot x^j.\end{aligned}$$

This is an identity, so the coefficients of x^r may be equated.

$$C_r[a, \cdots, b, c] = \sum_{j=r-c}^{j=r} C_j[a, \cdots, b], \tag{7}$$

subject to the following conventions:

1. [] is the empty set, and $n=0$.
2. $\phi[\]=1$.
3. $C_0(S)=1$, for any set S.
4. $C_r(S)=0$, if $r<0$, or if $r>n$.

EXAMPLE 1. Find $C_5[4, 3, 2, 1]$.

Solution. The computation is arranged in the following table:

r	$C_r[\]$	$C_r[4]$	$C_r[4, 3]$	$C_r[4, 3, 2]$	$C_r[4, 3, 2, 1]$
0	1	1	1	1	1
1	0	1	2	3	4
2	0	1	3	6	9
3	0	1	4	9	15
4	0	1	4	11	20
5	0	0	3	11	22

The value of $C_0[\]$ is 1, and $C_r[\]=0$ when $r>0$. So the values of $C_r[\]$ are quickly filled out. The values in each following column are found by applying Equation (7). The numbers in each column would be the coefficients of ϕ for the same set up to the term in x^5. That is, $\phi[4, 3]=1+2x+3x^2+4x^3+4x^4+3x^5+\cdots$.

EXAMPLE 2. The number of different Bridge hands is $C_{52,13}=635{,}013{,}559{,}600$. From a practical standpoint, a hand is not essentially changed if one card is replaced by another in the same suit, provided both cards are below the ten-spot. Under this practical point of view, how many essentially different Bridge hands are possible?

Solution. The problem, stated in our notation, is to evaluate $C_{13}[8^4, 1^{20}]$. Following is the solution in tabular form for $C_r[8^4]$ for values of r up to 13. The last column of the table is explained later.

r	$C_r[\]$	$C_r[8]$	$C_r[8, 8]$	$C_r[8^3]$	$C_r[8^4]$	$C_{13-r}[1^{20}]$
0	1	1	1	1	1	77,520
1	0	1	2	3	4	125,970
2	0	1	3	6	10	167,960
3	0	1	4	10	20	184,756
4	0	1	5	15	35	167,069
5	0	1	6	21	56	125,970
6	0	1	7	28	84	77,520
7	0	1	8	36	120	38,760
8	0	1	9	45	165	15,504
9	0	0	8	52	216	4,845
10	0	0	7	57	270	1,140
11	0	0	6	60	324	190
12	0	0	5	61	375	20
13	0	0	4	60	420	1

It would be out of the question to extend the table for 20 more columns, to bring in the 20 remaining subsets of 1 element each. So now we use Equation (5), and form $\phi[8^4, 1^{20}]$. This is now easy, since the coefficients of one factor, $\phi[8^4]$, are known, and the coefficients of the other factor, $\phi[1^{20}]$, are the easily found binomial coefficients, which we now enter in the last column. Since only the coefficient of x^{13} in $\phi[8^4, 1^{20}]$ is wanted, this is found as the sum:

$$C_{13}[8^4, 1^{20}] = \sum_{r=0}^{r=13} C_r[8^4]\cdot C_{11-r}[1^{20}] = 34{,}033{,}880.$$

Each term of this sum has a special meaning. Take for example the term in which $r=8$. Then $C_8[8^4]=165$, and this is the number of different ways of choosing 8 non-honor cards. $C_{13-8}[1^{20}]=15{,}504$ is the number of different ways of choosing the 5 honor cards. Their product, (165)(15,504), is the number of essentially different hands containing 8 non-honors and 5 honors.

A NOTE ON LEAST SQUARES

D. B. HOUGHTON, Franklin Institute

In the usual treatment of the method of least squares it is recommended that the so-called normal equations be computed explicitly using, for instance, Crelle's multiplication table and a table of squares, and that these equations then be solved. In this age of computing machinery a slightly different form of the result which gives the solution of the normal equations directly in terms of the coefficients of the given equations of condition may be helpful. The method is easily remembered, is similar to Cramer's Rule in form, and may be thought of as a generalization of it in that when the number of equations of condition is equal to the number of unknowns the method reduces to Cramer's Rule. The treatment in this form may be of interest to teachers to furnish exercises as it involves the multiplication of matrices, the evaluation of determinants, Cramer's Rule, and least squares theory.*

The equations of condition are a set of linear equations whose number is greater than or equal to the number of unknowns, and the equations are not strictly compatible with each other. The solution desired is a set of values of the unknowns which satisfies the equations as nearly as possible; that is, in the least squares sense. The equations of condition are:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (i = 1, 2, 3, \cdots, m),\ m \geqq n, \tag{1}$$

and may be written as:

$$AX = B, \tag{2}$$

where A is the m by n matrix of the coefficients; X, the one by n column matrix of the unknowns; and B the one by m column matrix of the right-hand constants.

Transpose the b's of (1) to the left sides leaving the right members zero. Since for independent equations with $m>n$ no solution that satisfies all the equations may be found, any approximate solution—in this case, the least squares solution—will not make the right members zero but small. Call these small quantities ϵ_i.

* For similar discussions, see: Holzinger and Harmon, Factor Analysis, University of Chicago Press, 1941, p. 290 ff.; and Arley and Buch, Probability and Statistics, John Wiley and Sons, 1950, p. 185 ff.

$$\sum_{j=1}^{n} a_{ij}x_j - b_i = \epsilon_i, \qquad i = 1, 2, 3, \cdots, m. \tag{3}$$

Let

$$\epsilon = \sum_{i=1}^{m} \epsilon_i^2 \tag{4}$$

then, for the least squares solution, ϵ is a minimum, and

$$\frac{\partial \epsilon}{\partial x_j} = 0, \qquad j = 1, 2, 3, \cdots, n. \tag{5}$$

This constitutes a set of n linear equations in n unknowns (the normal equations):

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ik}a_{ij}\right)x_j = \sum_{i=1}^{m} a_{ik}b_i, \qquad k = 1, 2, \cdots, n. \tag{6}$$

These may be written in matrix notation:

$$A^TAX = A^TB. \tag{7}$$

If A^TA is non-singular the solution is

$$X = (A^TA)^{-1}A^TB. \tag{8}$$

By writing M_j for the matrix A with the jth column replaced by the column matrix B, we may write (8) in the form:

$$x_j = \frac{|A^TM_j|}{|A^TA|} \qquad j = 1, 2, \cdots, n \tag{9}$$

Note that $|A^TA|$ is always symmetric.

When $m = n$ the matrices are square and since the determinant of a product of *square* matrices is equal to the product of the determinants of the matrices,

$$x_j = \frac{|A^T||M_j|}{|A^T||A|} = \frac{|M_j|}{|A|} \tag{10}$$

which is Cramer's Rule.

The following examples illustrate the use of (9) above.

EXAMPLE 1. (Taken from *The Calculus of Observations*, Whittaker and Robinson, The Method of Least Squares, p. 212.)

$$
\begin{aligned}
4.91x - 59.0\,y &= -339.8\\
2.72x - 2.70y &= -47.5\\
0.05x + 32.4y &= 262.5\\
-2.91x + 27.7\,y &= 152.9\\
-4.77x + 1.50y &= -27.9
\end{aligned}
$$

$$
x = \frac{\begin{vmatrix} \begin{bmatrix} 4.91 & 2.72 & 0.05 & -2.91 & -4.77 \\ -59.00 & -2.70 & 32.40 & 27.70 & 1.40 \end{bmatrix} \begin{bmatrix} -339.8 & -59.0 \\ -47.5 & -2.70 \\ 262.5 & 32.4 \\ 152.9 & 27.7 \\ -27.9 & 1.40 \end{bmatrix} \end{vmatrix}}{\begin{vmatrix} \begin{bmatrix} 4.91 & 2.72 & 0.05 & -2.91 & -4.77 \\ -59.00 & -2.70 & 32.40 & 27.70 & 1.40 \end{bmatrix} \begin{bmatrix} 4.91 & -59.0 \\ 2.72 & -2.70 \\ 0.05 & 32.4 \\ -2.91 & 27.7 \\ -4.77 & 1.40 \end{bmatrix} \end{vmatrix}}
$$

$$
x = \frac{\begin{vmatrix} -2096.349 & -382.699 \\ 32877.72 & 5307.3 \end{vmatrix}}{\begin{vmatrix} 62.73 & -382.699 \\ -382.699 & 5307.3 \end{vmatrix}} = \frac{1456317.5}{186468.404} = 7.81
$$

Similarly, $y = 6.76$

EXAMPLE 2. (Taken from *Numerical Mathematical Analysis*, J. B. Scarborough, p. 366.)

Find a formula of the form

$$y = a + bx + cx^2$$

which will fit the following data:

x.	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
y	3.1950	3.2299	3.2532	3.2611	3.2516	3.2282	3.1807	3.1266	3.0594	2.9756

Substituting these values of x and y into the quadratic equation $y = a + bx + cx^2$ gives ten linear equations of condition in a, b, and c whose solution is obtained by the above process: $a = 3.1951$; $b = .44253$; $c = -.76530$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 936. *Proposed by R. E. Horton, Los Angeles City College*

A wheel of radius a with a flange of radius b is rolling without slipping along a straight rail. On the corner of the rail a piece projects which scratches a mark on the face of the flange. Identify the curve of the mark so formed. What conditions on a and b are necessary in order that the mark on the flange will be a simple closed curve?

E 937. *Proposed by I. N. Herstein, University of Kansas*

If p is prime and $n \geqq p$, then

$$n! \sum_{pi+j=n} 1/p^i i!j! \equiv 0, \text{ mod } p.$$

E 938. *Proposed by H. F. Sandham, Trinity College, Ireland*

If $x = \sum_{i=1}^{n} 1/i$, prove that $n = [e^{x-\gamma}]$, where γ is Euler's constant, and $[a]$ denotes the integral part of a.

E 939. *Proposed by P. J. Schillo, University of Buffalo*

Circles C_0, C_1, C_2, $\cdots$, having radii r_0, r_1, r_2, $\cdots$, are tangent to a circle C, of radius $2r$, and to a diameter of C. Circle C_0 passes through the center of C, and C_n is tangent externally to C_{n-1}. Show that

$$r_n = 4w^n r/(w^n + 1)^2,$$

where $w = 3+2\sqrt{2}$, and that r/r_n is an integer.

E 940. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Being given a tetrahedron $ABCD$, an arbitrary point P, and four coplanar points A', B', C', D' lying in the planes of the faces BCD, CDA, DAB, ABC, show that the lines PA', PB', PC', PD' intersect the spheres $PBCD$, $PCDA$, $PDAB$, $PABC$ in four points cospherical with P.

SOLUTIONS

A Polygonal Path of Arbitrarily Small Length

E 906 [1950, 184]. *Proposed by Vern Hoggatt, Oregon State College*

Let S be a bounded infinite point set. Show that there exists a polygonal

path of arbitrarily small length passing through an infinite number of points of S.

Solution by Norman Miller, Queen's University, Ontario. The Bolzano-Weierstrass theorem guarantees the existence of a limit point P of S. With P as center draw a circle of diameter $\epsilon/2$, and within it select a point of S. Within a second circle with center P and diameter $\epsilon/4$ select another point of S. Continue this process with circles of diameters $\epsilon/8$, $\epsilon/16$, $\cdots$. The polygonal path formed by consecutively joining the infinite set of selected points has length $l<\epsilon/2+\epsilon/4+\cdots=\epsilon$.

Also solved by F. Bagemihl, David Ellis, J. B. Feidner, M. D. Montgomery, C. S. Ogilvy, C. F. Pinzka, L. A. Ringenberg, R. M. Simons, O. D. Smith, Charles Solky, G. H. M. Thomas, S. S. Walters, and the proposer.

As pointed out by Ellis, the above solution can be generalized to solve the problem where S is any bounded infinite subset of a finitely compact metric space (metric space in which bounded subsets are compact). It is to be noted that the proof assumes Zermelo's axiom of choice.

Sum of Two Squares in Three Different Ways

E 909 [1950, 184]. *Proposed by W. E. Buker, Perry High School, Pittsburgh, Pa.*

Find positive integers satisfying

$$N = p^2 + k_1^2 = (p+1)^2 + k_2^2 = (p+2)^2 + k_3^2.$$

I. *Solution by P. Somanadham, Vizianagram, India.* Clearly $k_1>k_2>k_3$ and k_1 and k_3 are of the same parity. Setting $a=(k_1+k_3)/2$ and $b=(k_1-k_3)/2$ we have $k_1=a+b$ and $k_3=a-b$, whence

$$p^2 + (a+b)^2 = (p+2)^2 + (a-b)^2$$

or

$$p = ab - 1.$$

Since

$$p^2 + (a+b)^2 = (p+1)^2 + k_2^2.$$

we have

$$(ab-1)^2 + (a+b)^2 = (ab)^2 + k_2^2$$

or

$$k_2^2 = a^2 + b^2 + 1.$$

The right side will be a perfect square if $a = 2t^2$ and $b = 2t$. We then have the parametric solution

$$k_2 = 2t^2 + 1, \qquad p = 4t^3 - 1, \qquad k_1 = 2t(t+1), \qquad k_3 = 2t(t-1),$$

$$N = 16t^6 + 4t^4 + 4t^2 + 1, \qquad t > 1.$$

This does not give all solutions, however, since the particular solution

$$(539)^2 + (48)^2 = (540)^2 + (35)^2 = (541)^2 + (12)^2$$

cannot arise from an integral value of t.

II. *Solution by Elijah Swift, University of Vermont.* We easily find that

$$k_1^2 + k_3^2 = 2(k_2^2 - 1). \tag{1}$$

Noting that the square of an odd number is of the form $8a+1$ we see that k_2 must be odd and k_1 and k_3 both even. Consequently, any solution of (1) will determine p as an integer and yield a solution of the given equations. For $2(k_2^2-1)$ to be the sum of two squares it must be of the form $2^r MQ^2$, where $r \geqq 0$, the factors of M are primes of the form $4x+1$, and the factors of Q are primes of the form $4x+3$. Substituting odd numbers for k_2 we find eight values of $k_2 < 100$ such that $2(k_2^2-1)$ satisfies the above conditions. These give the following eleven solutions.

k_1	k_2	k_3	p	N
12	9	4	31	1105
24	19	12	107	12025
40	33	24	255	66625
48	35	12	539	292825
60	51	40	499	252601
68	51	24	1011	1026745
72	51	4	1291	1671865
84	73	60	863	751825
104	81	48	2127	4534945
112	81	24	2991	8738625
112	99	84	1371	1892185

Also solved by D. H. Browne, R. L. Caskey, Monte Dernham, W. A. Ferguson, Daniel Finkel, G. B. Huff, Free Jamison, Sam Kravitz, Roger Lessard, Leonard Morgenstern, Prasert Na Nagara, C. S. Ogilvy, K. S. Rao, L. A. Ringenberg, Azriel Rosenfeld, C. M. Sandwick, R. M. Simons, O. D. Smith, G. W. Walker, and the proposer. Most of these gave the parametric solution in I above. A few provided more or less tentative processes for finding all solutions. Walker listed all solutions (twenty-three of them) for $N < 10^8$. Jamison found some two-parameter solutions.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4412. *Proposed by Jacques Dutka, Rutgers University*

Of the freshmen admitted to a certain college last fall it was found that:
74% were seventeen years of age or older,
89% weighed 125 pounds or more,
78% measured $5\frac{1}{2}$ feet tall or more.
Let p be the probability that a freshman chosen at random had all three characteristics. Determine bounds for the value of p, and generalize the problem.

4413. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Let $a_1=2\cdot 3$, $a_2=3\cdot 5$, $a_3=5\cdot 7$, $\cdots$, $a_k=p_k\cdot p_{k+1}$, $\cdots$, where p_k is the kth prime. Denote by $f(n)$ the number of integers $\leqq n$ composed entirely of the a's (i.e., the integers of the form $\prod a_i^{\alpha_i}$, $0\leqq\alpha_i$). Prove that

$$f(n) = cn^{1/2} + o(n^{1/2}),$$

where $\frac{1}{2}<c<1$.

4414. *Proposed by Lucien Droussent, Clermont Ferrand, France*

The trilinear polar, with respect to triangle T, of any point P on the circumscribed Steiner ellipse of T is tangent to the conjugate conic of T that has P for center. (The circumscribed Steiner ellipse of a triangle ABC is the circumscribed ellipse whose tangents at A, B, C are parallel to the opposite sides BC, CA, AB.)

SOLUTIONS

Root of Maximum Modulus

4331 [1949, 111]. *Proposed by V. F. Ivanoff, San Francisco*

Let the equation, $x^n+p_1x^{n-1}+\cdots+p_n=0$, possess a root x_1 whose modulus exceeds that of every other root of the equation. Prove that

$$\lim_{k\to\infty}\frac{\Delta_k}{\Delta_{k-1}} = -x_1,$$

where Δ_k is the k-rowed determinant $(k>n)$

$$f(x) = 1 + p_1x + p_2x^2 + \cdots + p_nx^n = \prod_{i=1}^{n} (1 - x_ix),$$

then

$$\frac{f'(x)}{f(x)} = -\sum_{i=1}^{n} \frac{x_i}{1 - x_ix} = -\sum_{i=1}^{n}\sum_{j=0}^{\infty} x_i^{j+1}x^j = -\sum_{j=0}^{\infty} x^j \sum_{i=1}^{n} x_i^{j+1}.$$

Thus $a_{k-1} = -\sum_1^n x_i^k$. Hence $\Delta_k = (-1)^k \sum_1^n x_i^k$ and

$$\lim \frac{\Delta_k}{\Delta_{k-1}} = -\lim \frac{\sum_{i=1}^{n} x_i^k}{\sum_{i=1}^{n} x_i^{k-1}} = -x_1.$$

Also solved by the Proposer.

Editorial Note. The Proposer's solution proceeds by first showing that $\Delta_k = (-1)^k S_k$, where S_k is the sum of the kth powers of all the roots. C. R. Phelps points out the intimate connection with Graeffe's method of approximating roots. A. W. Goodman refers to similar problems in F. Riesz, *Les Systemes d'Equations a une Infinite d'Inconnues*, Paris 1913, pp. 9–11. D. H. Browne compares the present result with an earlier editorial note, [1942, 463].

Sequences of pth Powers

4333 [1949, 112]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In every system of numeration whose base B has the form am^p+1, each of the p-digit numbers

$$(am^{p-1} + 1)^p, (2am^{p-1} + 1)^p, (3am^{p-1} + 1)^p, \cdots, [(m - 1)am^{p-1} + 1]^p,$$

gives rise to an infinite sequence of pth powers. The law of formation is very simple and is sufficiently clear from the following example:

If $B = 97 = 6\cdot4^2+1 = 12\cdot2^3+1 = 6\cdot2^4+1 = 3\cdot2^5+1$, then we have

$\overline{25}^2 = 6\ \overline{43}$, $\overline{24}\ \overline{25}^2 = 6\ 6\ \overline{42}\ \overline{43}$, $\overline{24}\ \overline{24}\ \overline{25}^2 = 6\ 6\ 6\ \overline{42}\ \overline{42}\ \overline{43}, \cdots$

$\overline{49}^2 = \overline{24}\ \overline{73}$, $\overline{48}\ \overline{49}^2 = \overline{24}\ \overline{24}\ \overline{72}\ \overline{73}$,

$\overline{48}\ \overline{48}\ \overline{49}^2 = \overline{24}\ \overline{24}\ \overline{24}\ \overline{72}\ \overline{72}\ \overline{73}, \cdots$

$\overline{73}^2 = \overline{54}\ \overline{91}$, $\overline{72}\ \overline{73}^2 = \overline{54}\ \overline{54}\ \overline{90}\ \overline{91}$,

$\overline{72}\ \overline{72}\ \overline{73}^2 = \overline{54}\ \overline{54}\ \overline{54}\ \overline{90}\ \overline{90}\ \overline{91}, \cdots$

$\overline{49}^3 = \overline{12}\ \overline{48}\ \overline{85}$, $\overline{48}\ \overline{49}^3 = \overline{12}\ \overline{12}\ \overline{48}\ \overline{48}\ \overline{84}\ \overline{85}$,

$\overline{48}\ \overline{48}\ \overline{49}^3 = \overline{12}\ \overline{12}\ \overline{12}\ \overline{48}\ \overline{48}\ \overline{48}\ \overline{84}\ \overline{84}\ \overline{85}, \cdots$

$\overline{49}^4 = 6\ \overline{30}\ \overline{66}\ \overline{91}$, $\overline{48}\ \overline{49}^4 = 6\ 6\ \overline{30}\ \overline{30}\ \overline{66}\ \overline{66}\ \overline{90}\ \overline{91}$,

$\overline{48}\ \overline{48}\ \overline{49}^4 = 6\ 6\ 6\ \overline{30}\ \overline{30}\ \overline{30}\ \overline{66}\ \overline{66}\ \overline{66}\ \overline{90}\ \overline{90}\ \overline{91}, \cdots$

$\overline{49}^5 = 3\ \overline{18}\ \overline{48}\ \overline{78}\ \overline{94}$, $\overline{48}\ \overline{49}^5 = 3\ 3\ \overline{18}\ \overline{18}\ \overline{48}\ \overline{48}\ \overline{78}\ \overline{78}\ \overline{93}\ \overline{94}$,

$\overline{48}\ \overline{48}\ \overline{49}^5 = 3\ 3\ 3\ \overline{18}\ \overline{18}\ \overline{18}\ \overline{48}\ \overline{48}\ \overline{48}\ \overline{78}\ \overline{78}\ \overline{78}\ \overline{93}\ \overline{93}\ \overline{94}, \cdots$.

Establish the general rule.

Solution by E. P. Starke, Rutgers University. Put

$$n_0 = kam^{p-1} + 1, \qquad k < m,$$

$$n_s = 1 + (n_0 - 1)(B^s + B^{s-1} + \cdots + B + 1) = \frac{n_0 - 1}{B - 1}(B^{s+1} - 1) + 1.$$

Then we have

$$a = \frac{B - 1}{m^p}, \qquad \frac{n_0 - 1}{k} = \frac{B - 1}{m}, \qquad \frac{n_s - 1}{k} = \frac{B^{s+1} - 1}{m}.$$

The following formulas are easily computed:

$$n_0^p = \left[1 + \frac{k}{m}(B - 1)\right]^p = 1 + a\sum_{j=0}^{p-1} m^j \binom{p}{j} k^{p-j}(B - 1)^{p-j-1},$$

$$n_s^p = \left[1 + \frac{k}{m}(B^{s+1} - 1)\right]^p$$

$$= 1 + a(B^s + B^{s-1} + \cdots + B + 1)\sum_{j=0}^{p-1} m^j \binom{p}{j} k^{p-j}(B^{s+1} - 1)^{p-j-1}.$$

Now suppose n_0^p, written in the scale of B, is $d_1d_2d_3 \cdots d_p$. Then $n_s^p - 1$ can be obtained from $n_0^p - 1$ by replacing each B by B^{s+1} and multiplying the result by $B^s + B^{s-1} + \cdots + B + 1$. The result is clearly a number $d_1d_1 \cdots d_1d_2 \cdots d_2d_3 \cdots d_3 \cdots d_p \cdots d_p$ with each digit now written $s+1$ times.

Trigonometric Sum and Product

4335 [1949, 112]. *Proposed by D. D. Wall, Cambridge, Massachusetts*

If n is a prime greater than 3, show that

$$\prod_{x=1}^{n}\left(1 + 2\cos\frac{2\pi x}{n}\right) = 3, \tag{1}$$

$$\sum_{x=1}^{n} \sec\frac{2\pi x}{n} = (-1)^{(n-1)/2}n. \tag{2}$$

The restrictions on n can be lessened somewhat in each case.

Solution by N. J. Fine, University of Pennsylvania. Let n be a positive integer. If n is a multiple of 3, the product (1) is zero since it contains the factor

$$1 + 2\cos 2\pi\frac{(n/3)}{n} = 0.$$

Otherwise, set $w = e^{2\pi i/n}$. Then

$$\prod_{k=1}^{n}\left(1+2\cos\frac{2\pi k}{n}\right)=\prod_{k=1}^{n}(1+w^k+w^{-k})=w^{-n(n+1)/2}\prod_{k=1}^{n}(1+w^k+w^{2k})$$

$$=(-1)^{n+1}3\,\frac{\prod_{k=1}^{n-1}(1-w^{3k})}{\prod_{k=1}^{n-1}(1-w^k)}=(-1)^{n+1}3,$$

since $3k$ runs over all residues (mod n) except 0 as k does.

For the second part, let n be odd and $w=e^{2\pi i/n}$. Then

$$\sum_{k=1}^{n}\sec\frac{2\pi k}{n}=2\sum_{k=1}^{n}\frac{1}{w^k+w^{-k}}=2\sum_{k=1}^{n}\frac{w^k}{1+w^{2k}}$$

$$=\sum_{k=1}^{n}w^k\left(\frac{1+w^{2kn}}{1+w^{2k}}\right)=\sum_{k=1}^{n}w^k\sum_{r=0}^{n-1}(-1)^r w^{2kr}$$

$$=\sum_{r=0}^{n-1}(-1)^r\sum_{k=1}^{n}(w^{2r+1})^k=(-1)^{(n-1)/2}n,$$

since the inner sum is 0 if n is not a divisor of $2r+1$, and is n if n is a divisor of $2r+1$, and the latter is true for $0\leqq r\leqq n-1$ only when $r=(n-1)/2$.

Also solved by N. C. Scholomiti and the Proposer.

Motion Through a Planet of Constant Density

4342 [1949, 268]. *Proposed by R. J. Walker, Cornell University*

A spherical planet whose density at any point P is a function only of the distance of P from the center of the planet has the following property. If a straight frictionless tunnel is bored between two points on the planet's surface the time required for an object to slide from one of these points to the other is independent of the positions of the points. Prove that the planet has constant density.

Solution by M. Kac, H. Pollard, and the proposer, Cornell University. Let a be the radius of the sphere, $\rho(r)$ the density at a point P, r units from the center, and $f(r)$ the acceleration due to gravity at P. Then

$$f(r)=\frac{k}{r^2}\int_0^r 4\pi\rho(s)s^2ds. \tag{1}$$

If a unit mass falls along any frictionless path from the surface to P its velocity at P is given by

$$\frac{1}{2}[V(r)]^2=\int_r^a f(s)ds. \tag{2}$$

Consider a straight frictionless tunnel whose distance from the center is x. If y is the distance from P to the center of the tunnel we have

$$y^2 = r^2 - x^2,$$

$$V(r) = \frac{dy}{dt} = \frac{r}{\sqrt{r^2 - x^2}} \frac{dr}{dt}.$$

Hence if T is the time required to slide through the tunnel,

$$\int_x^a \frac{rdr}{V(r)\sqrt{r^2 - x^2}} = \frac{T}{2}. \tag{3}$$

To solve this for $V(r)$ we use a variation of the Abel inversion formula. (See Bôcher, *An Introduction to the Study of Integral Equations.*) Multiply (3) by $x/\sqrt{x^2-y^2}$ and integrate with respect to x from y to a, y being an arbitrary number between 0 and a.

$$\int_y^a dx \int_x^a \frac{xrdr}{V(r)\sqrt{x^2 - y^2}\sqrt{r^2 - x^2}} = \int_y^a \frac{T}{2} \frac{xdx}{\sqrt{x^2 - y^2}}$$

$$= \frac{T}{2}\sqrt{a^2 - y^2}.$$

Interchanging the order of integration gives

$$\frac{T}{2}\sqrt{a^2 - y^2} = \int_y^a \frac{rdr}{V(r)} \int_y^r \frac{xdx}{\sqrt{x^2 - y^2}\sqrt{r^2 - x^2}}$$

$$= \frac{\pi}{2} \int_y^a \frac{rdr}{V(r)}.$$

On differentiating with respect to y we get

$$V(y) = \frac{\pi}{T}\sqrt{a^2 - y^2}.$$

Differentiating (2) now gives

$$f(r) = \frac{\pi^2}{T^2} r,$$

and putting this in (1) and differentiating gives

$$\rho(r) = \frac{3\pi}{4kT^2}.$$

Also solved by R. N. Haskell, and Fritz Herzog.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Sampling Methods for Censuses and Surveys. By Frank Yates. London, Charles Griffin & Co. Ltd., 1949. 14+318 pages. 24 s.

"This book has been written primarily for those who have little or no previous training in mathematical statistics, but who have some training or experience in the presentation and handling of statistical data. It is consequently not written in the form of a mathematical treatise, and mathematical proofs have not been included. On the other hand an attempt has been made to cover all the modern developments of sampling theory which are of importance in census and survey work, and to give an adequate discussion of the complexities that are encountered in their practical application." This statement, with which the author begins his preface, accurately describes the book.

The character of the work is influenced by its origin in a request of the United Nations Sub-Commission on Statistical Sampling that a manual be prepared for use in the projected 1950 world censuses of agriculture and population. It is concerned primarily with the techniques of planning, execution, and analysis of censuses and surveys. The first half of the book is devoted to a description of the different methods of sampling and to various non-mathematical procedural matters. The second half consists of recipes for estimating population values, sampling errors, and efficiencies. Numerous examples (almost entirely agricultural) are worked out with clarity and completeness, often being carried along from one section of the book to another. There is a classified bibliography on sampling, which is based on a more complete one prepared by the Statistical Office of the United Nations.

It is a regrettable fact that many workers in statistics "have little or no previous training in mathematical statistics." The large and growing demand for statistical work implies that the situation is not likely to mend automatically. Of course the only satisfactory long run solution is to insist that statistical workers secure the necessary training in mathematics. A "statistician" without this basic equipment is as much a public hazard as the quack doctor. But it will certainly be some time before professional standards can be fully enforced in the field of statistics, and meanwhile there is a real need to increase the effectiveness of the mathematically untrained statistician. One solution is the statistical "cook book." Here the formulas and techniques are presented categorically with exact instructions as to what to do, but practically without justification. The reader with "little or no previous training" is still in this category after mastering such a book, although he is acquainted with new terminology and "tricks of the trade." Moreover, he is likely to be confirmed in his impression

that "mathematical statistics" is a mystical religion understandable only to its high priests. Another and, it seems to this writer, a better solution is to try by means of precise statement, heuristic arguments, and intuitively appealing examples to increase the reader's appreciation of the meaning and justification of statistical procedures. He may thus gain a "feel" for the theory which can be a helpful prelude or complement to the rigorous mathematics essential to a thorough understanding. Since the book under review is largely in the "cook book" category, the writer would be more inclined to recommend it as an up-to-date compilation of techniques for the mathematically trained statistician than as a text for those who stand in need of basic training in mathematical statistics.

KENNETH MAY

Higher Algebra for the Undergraduate. By Marie J. Weiss. New York, John Wiley, 1949. 8+165 pp. $3.75.

Higher Algebra for the Undergraduate is designed for a one year course in algebra for students who have completed a year of calculus. The author's purpose has been to give an introduction to simpler concepts of algebra at a level of difficulty about the same as that of the traditional courses in theory of equations. To keep the presentation elementary the author has placed emphasis on careful exposition with many illustrative examples, on exercises for homework, and on keeping the number of new ideas introduced at any one time to a minimum. The smoothness of the exposition and careful choice of subject matter achieved by the author result at least in part from her testing of preliminary versions for a number of years in her own classes.

The first chapter treats the integers. In line with the elementary character of the text there is no attempt at an axiomatic development of the properties of the natural numbers, but a complete set of postulates is set down. The extension from natural numbers to integers is treated quite fully by the introduction of ordered pairs with a suitable equivalence relation. The frequent use of equivalence relations is an interesting feature of the text. Although the connection between equivalence relations and partitions of sets is never stated as a proposition in set theory, it is sufficiently emphasized in specific cases to give the student an excellent background for a more complete treatment in some later course. The second chapter begins with the extensions from integers to rationals to reals to complex numbers. The first and third of these extensions are made by the use of ordered pairs. The extension from rationals to reals is a mixture of Kronecker, Dedekind, and geometric intuition and is in the reviewer's opinion the weakest section in the entire book. The chapter closes with a discussion of roots of unity.

Groups are introduced in the third chapter. The treatment begins with the group axioms and develops some of the elementary theorems on abstract groups and permutation groups. This first treatment includes the concepts of isomorphism, subgroup, coset, cyclic group, and regular permutation group. The fourth chapter brings in rings, integral domains, and fields making considerable use of concrete examples from the number systems already discussed in the first two

chapters. The definition of "indeterminate" is not too satisfactory logically although it is unlikely to lead any but the unusually gifted student into difficulty. The fifth chapter treats polynomials over a field, with primary emphasis on divisibility theory. There is a brief discussion of the determination of rational roots of an equation with integer coefficients; the elementary symmetric functions are defined (but not symmetric functions in general); and the idea of formal derivative is introduced and applied to get the usual criterion for multiple factors of a polynomial and to get Taylor's Theorem for polynomials.

In chapter six matrices and some matrix algebra are introduced. The definition of rank is approached via the concept of row (column) equivalence and thus makes no use of determinants. The general theory for systems of linear equations is handled in matrix language. Then in chapter seven determinants are introduced as functions of square matrices. The author carefully limits the use of the word "determinant" to this function and never uses "determinant" where "matrix" is meant. The proof of the theorem on the determinant of a product is made to rest on the earlier results about "elementary" matrices. Then the determinant criterion for rank of a matrix and Cramer's Rule are given. The chapter concludes with brief treatments of similarity of matrices and of polynomials with matrix coefficients which include the Cayley-Hamilton Theorem and the diagonalization theorem for matrices without repeated eigenvalues.

The final chapter returns to abstract algebra with primary attention given to the fundamental homomorphism theorems for groups, rings and immediately related concepts and topics. Automorphisms of groups are briefly treated.

The reviewer agrees with the author's position that much of the content of the traditional course in theory of equations is more analysis than algebra. However, he feels that the balance of the text would have been improved if enough of the traditional theory of equations (in particular cubic and quartic equations) had been retained so that the prospective high school teacher could find in a one year course the necessary background material for school algebra as well as an introduction to abstract concepts.

This book by Professor Weiss is a welcome and important addition to the list of advanced undergraduate texts in algebra. To the reviewer it seems to be the simplest introduction to abstract algebra yet available. It is recommended for courses which attempt to bridge the gap between college algebra and modern abstract algebra, especially in situations where some of the other modern algebra texts have been found too advanced.

R. M. THRALL

An Introduction to the Algebra of Vectors and Matrices. By T. L. Wade. Tallahassee, Florida, 1949. 6+97 pages. $2.25.

To quote from the preface: "the mathematical background of the student using this book is assumed to be quite limited, namely, a first course in analytic geometry and acquaintanceship with the usual theorems on determinants."

The book presents a fairly extensive view of vectors and matrices, as the following chapter headings will indicate: IV. Vectors of N Dimensions; VII. Groups, Matrices and Transformations; VIII. The Characteristic Equation of a Matrix; X. Some Applications of Matrix Algebras. There are frequent exercises.

I feel that this textbook offers the beginning student a useful body of information, unencumbered by the (at first bewildering) abstractions of modern algebra. In the hands of a well-informed and skillful teacher, it should prove valuable. At the same time it should be stated that the text would benefit from a thorough revision. The typographical errors are covered by three mimeographed sheets issued by the author, but there are other defects as well. For example, the definition of a field is incorrect as it stands. Again, the definition of the product of two permutations is given two paragraphs later than a discussion of the representation of a permutations as a product of cycles.

R. H. BRUCK

NEW BOOKS RECEIVED

Phenomena, Atoms and Molecules. By I. Langmuir. New York, Philosophical Library, 1950. 12+436 pp. $10.00.

Introduction to the Theory of Probability and Statistics. By N. Arley and K. R. Buch. New York, John Wiley and Sons, 1950. 12+236 pp. $4.00.

Giant Brains or Machines That Think. By E. C. Berkeley. New York, John Wiley and Sons, 1949. 16+270 pp. $4.00.

Plane and Spherical Trigonometry. Fifth Edition. By C. I. Palmer, C. W. Leigh and S. H. Kimball. New York, McGraw-Hill, 1950. 14+102 pp. $3.25.

L'analyse Mathematique. ("Que sais-je? " no. 378). By A. Delachet. Paris, Presses Univerisiaires de France, 1949. 118 pp.

Vorlesungen über höhere Mathematik. Vol. 1. By A. Duschek. Vienna, Springer, 1949. 10+395 pp. $8.70.

Fourier Methods. By P. Franklin. New York, McGraw-Hill, 1949. 10+289 pp. $4.00.

Grundzuge der theoretischen Logik. Third Revised Edition. By D. Hilbert and W. Ackerman. Berlin, Springer, 1949. 8+155 pp. DM 19.80.

Matrix Analysis of Electric Networks (Harvard Monographs in Applied Science, no. 1). By P. Le Corbeiller. Harvard University Press, 1950. 12+112 pp. $3.00.

Determinanten und matrizen. Third Revised Edition. By F. Neiss. Berlin, Springer, 1948. 8+111 pp. DM 6.

Problemi di secondo grado risolti e discussi. By A. Sartori. Turin, Paravia, 1949. 5+163 pp. Lire 600.

Principles of Finance and Investment. By L. G. Whyte. Cambridge University Press, 1950, xi+192 pp. $2.50.

Statistics, Volume I. By N. L. Johnson and H. Tetley. Cambridge University Press, 1950, xii+294 pp. $4.00.

Out of My Later Years. By Albert Einstein. New York, Philosophical Library, 1950. 8+282 pp. $4.75.

Elements of Analytic Geometry. Third Edition. By C. E. Love. New York, MacMillan, 1950. 8+218 pp. $2.75.

Plane and Spherical Trigonometry. Revised Edition. By J. A. Northcott. New York, Rinehart, 1950. 10+94 pp. $3.50.

Transcendental Numbers. By C. L. Siegel. Princeton University Press, 1949. 8+102 pp. $2.00.

Theory of Sets. Translated by F. Bagemihl (from the Second German Edition by E. Kamke). Dover Publications, 1950. x+150 pp. $2.45.

Numerical Solutions of Differential Equations. By H. Levy and E. A. Baggott. Dover Publications, 1950. viii+238 pp. $3.00.

Apercu Historique de la Trigonometrie Rectiligne et Spherique. By A. Gloden. Luxembourg, Worre-Mertens, 1949. 1+18 pp.

Apercu Historique des Multigrades. By A. Gloden. Luxembourg, Worre-Mertens, 1949. 18+23 pp.

Contributions to the Theory of Nonlinear Oscillations. By S. Lefschetz. (Annals of Mathematics Studies, no. 20). Princeton University Press, 1950. 10+350 pp. $4.00.

L'analysis situs et la Geômetrie Algébrique. By S. Lefschetz. Paris, Gauthier-Villars, 1950. 6+154 pp.

Albert Einstein: Philosopher-scientist. (Library of Living Philosophers, vol. 7). Edited by P. A. Schilpp. Evanston, Library of Living Philosophers, 1949. 16+781 pp. $8.50.

Teorijska fizika i struktura materije. By I. Supek. Zagreb, Nakladni Zavod Hrvatske, 1949. 688 pp.

Elements of Calculus. T. S. Peterson. New York, Harper, 1950. 10+369 pp. $3.50.

Tables of the Bessel Functions of the First Kind of Orders Sixty-four through Seventy-eight. The Annals of the Computation Laboratory of Harvard University, volume XIII. Cambridge, Harvard University Press, 1949, 2+566 pp. $8.00.

Science and the Goals of Man. By A. Rapoport. New York, Harper, 1950. 29+262 pp. $3.50.

Electromagnetic Theory. (Proceedings of symposia in applied mathematics, volume 2). New York, American Mathematical Society, 1950. 2+91 pp. $3.00.

Algebraic Curves. By R. J. Walker. Princeton University Press, 1950. 11+201 pp. $4.00.

Geometry for Advanced Pupils. By E. A. Maxwell. Oxford, England, Clarendon Press, 1949. 176 pp.

Plane Trigonometry—with Tables. By H. M. Dadourian. Cambridge, Addison-Wesley, 1950. 11+84 pp.

Elements of Analytic Geometry. By W. L. Hart. Boston, Heath, 1950. 9+264 pp. $2.75.

The Science of Chance. By H. C. Levinson. New York, Rinehart, 1950. 8+348 pp. $2.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

EDITOR'S NOTE. During the period from January, 1948, to December, 1949, inclusive, this section of this MONTHLY listed 854 titles of papers presented to over one-hundred different clubs or societies. An attempt has been made to catalogue these titles but the variety of topics reported was so great that a summary appeared valueless.

Letters have been sent to the Chairman of Mathematics Departments urging that mathematics groups report their activities for 1949–50. Many clubs have responded and the reports will be published in the order received.

CLUB REPORTS, 1949–50

Kappa Mu Epsilon, Chicago Teachers College

The following papers were presented at meetings of the *Illinois Gamma* Chapter of *Kappa Mu Epsilon*:

Concepts of infinity and infinite sets, by Norman Stein.

Discriminants of polynomials, by Howard James.

In addition, a film on *The Einstein theory of relativity* was shown at a regular meeting.

Illinois Gamma Chapter was invited to sponsor a meeting in connection with the Twenty-Eighth Annual Convention of the National Council of Teachers of Mathematics. This meeting was held on April 14, 1950 at the Congress Hotel in Chicago. The National President, Prof. H. Van Engen, was present and opened the meeting with a brief speech. The program consisted of the following student papers:

A step forward, by Mary Lou Hodor and Nan Hutchings of the College of St. Francis, Joliet, Illinois.

Plane geometry without compasses, by Robert Allender of Iowa State Teachers College, Cedar Falls, Iowa.

Topology—modern supplement to Euclid, by Patricia Mitchell of Chicago Teachers College.

Some little known constructions with straight edge and compass, by Frances Donlon of Mount St. Scholastica College.

Inversion, by Dorothy Karner, of Mount Mary College, Milwaukee, Wisconsin.

The mathematical method in the light of a philosophical system, by Jeanne Culivan of Mount St. Scholastica College.

Providing for individual differences in arithmetic, by Dorothy Dahlberg of Chicago Teachers College.

The meeting was well attended and served the dual purposes of acquainting

members of Kappa Mu Epsilon with the National Council of Teachers of Mathematics and the reverse.

A party for entering freshmen, the annual banquet, and a spring picnic were the social events of the year.

The officers for 1949–50 were: President, Marie Marciante; Vice-President, William Woods; Secretary, Delphine Szulakiewicz; Treasurer, Manuel Sanchez; Sponsor, Dr. J. M. Sacks.

Pi Mu Epsilon, University of Oklahoma

The years activities of the *Oklahoma Alpha* Chapter of *Pi Mu Epsilon* included business, social, and regular meetings at which the following talks were given:

Lion chasing, by Donald Dubois

Mathematics and astonomy, by Prof. J. O. Hassler

The construction of the real number system, by Gene Levy

The application of mathematics to engineering problems, by Prof. J. Ray Matlock, School of Civil Engineering

Some aspects of mathematics related to naval sciences, by Forrest Grogan.

Ronald Smith was first place winner and T. L. McElhaney was second place winner of the annual, university-wide competitive mathematics examination. The chapter sponsored a tea for students and teachers of the state high schools that have entered students in the Interscholastic Algebra and Geometry Examinations.

An initiation banquet was held at which the guest speaker was Dean W. H Carson of the College of Engineering. A total of 26 new members were inducted into the chapter.

The following officers were elected: Director, Don A. Gorsline; Vice-Director, Walter C. Brown; Secretary-Treasurer, Rose M. Pratt.

Mathematics Club, McMaster University

The activities of the *Mathematics Club* of McMaster University for the academic year 1949–50 include the following topics, presented by members and guests:

Scale and notation, by Robert Attridge

A biography of Archimedes, by James McCulloch

Greek mathematics, by Lorraine McDiarmid

Laplace, by Edward Pennington

Mathematical instruments, by Richard Beesack

Experimental errors, by Joseph O'Brien

Life of Fourier, by Ross McFarlane

Fourier analysis with applications, by Vern Turner

Lorentz transformations and the Minkowski four-dimensional space-time continuum, by R. J. Donnelly

"*It is obvious*," by James Eastrott

Topology, By Dr. Krieger of the University of Toronto
Why does applied mathematics work?, by Dr. J. D. Bankier.

The officers for the coming year are: President, R. J. Donnelly; Vice-President, Robert Attridge; Secretary-Treasurer, U. C. Strahlendorf; Year Representatives, Robert Stevenson, James McCulloch, and Ross McFarlane.

Pi Mu Epsilon, Louisiana State University

The *Louisiana Alpha* Chapter of *Pi Mu Epsilon* held the following meetings during the year:

Newton's law of gravity, by Dr. F. A. Rickey
Statistical analysis of data, by Dr. Donald Billings
Length and area, by Dr. H. L. Smith
Transfinite numbers, by Dr. N. E. Rutt
Conservation of systems in phase space, Mr. Ernest Ikenberry
Mathematics as applied to chemistry, by Dr. Olen A. Nance.

Thirty-eight new members were initiated during the year. The Pi Mu Epsilon Lectures were given by Dr. H. S. Householder, Head of the Mathematics Section of Oak Ridge National Laboratory. His subject was, *The relation of mathematics to the sciences*. He also spoke on the subject *Statistics and the computer's art*.

The Senior Award was won by Miss Katherine Rollins and the Freshman Award was won by Mr. Van Be Luong. Five new volumes were added to the Pi Mu Epsilon shelf in the Mathematics Library.

The officers for the year were: President, Marion Smith; Vice-President Rodger Richardson; Secretary, Alice Pecot; Treasurer, Agnes Schertz; Faculty Advisor and Corresponding Secretary, Dr. H. T. Karnes.

Mu Alpha Theta, Yeshiva College

The Yeshiva College Mathematics Club, *Mu Alpha Theta*, devoted its program to lectures, delivered by the students, on various branches of geometry as follows:

Foundations of geometry (2 lectures), by Mr. Frank
Non-Euclidean geometry (3 lectures), by Mr. Gold
Analytic projective geometry (3 lectures), by Mr. Rosenfeld
Axiomatic projective geometry (2 lectures), by Mr. Solomon
Combinatorial topology (2 lectures), by Mr. Frank
Differential geometry (2 lectures), by Mr. Hellerstein.

Miss Ester Adler of Columbia University, spoke on *Godel's system of axioms sigma*.

Joint meetings were held with the Classical Society (on Pythagoras and Greek mathematics) and with the Hebrew Club (on mathematical terms in Hebrew).

Other activities included a number of formal and stag parties; a colloquium during the Passover recess; a picnic at the end of the school year; and the publication of a journal devoted to the original researches of the members.

The present officers of the club are: Presidents, Seymour Huber (1st semester) and Azriel Rosenfeld (2nd semester); Vice-President, Gustave Solomon; Secretary, Simon Hellerstein.

Kappa Mu Epsilon, Mount St. Scholastica College

The *Kansas Gamma* Chapter of *Kappa Mu Epsilon* at Mount St. Scholastica College had an enrollment of twenty-three members. The meetings were held bi-monthly. Besides the discussion meetings, the Chapter held the annual formal Christmas Wassail party, the pledge program, and the formal initiation and banquet.

In line with the general theme of the year, *Mathematics as a system of thought*, the following papers were presented:

Status of mathematics with the Greek philosophers, by Sister Helen Sullivan, O.S.B.

Mathematics in the patristic period, by Noreen Hurter

Mathematics in the scholastic period, by Mary Alice Weir

Contemporary mathematical philosophers, by Jeanne Culivan

Demonstration of mathematical systems, by Patti Shideler

Korean mathematics, by Peter Kim, guest speaker (Korea)

Geometric constructions with straight edge and compass, by Frances Donlon

Frequency distributions, by Frances Walsh

Various number systems, by Victoria Fritton

Our mathematical heritage, panel discussion by Margot Acree, Terry Breitenbach, and Dorothy Ripley.

The Chapter sent two delegates to Chicago to attend the convention of the National Council of Mathematics Teachers. The delegates read original papers in the Kappa Mu Epsilon section of the convention. Miss Frances Donlon read the paper, *Geometrical constructions with straight edge and compass*, while Miss Jeanne Culivan's paper was *The mathematical method in the light of a philosophical system.*

The tenth anniversary of the founding of *Kansas Gamma* on this campus was celebrated by a founder's day dinner. Testimonial letters and telegrams from charter members and past presidents were read.

The Hypatian Award was granted to Jeanne Culivan and Frances Donlon. Anne Robben won the underclassman Scholarship award.

The new officers of 1950–51 are: President, Frances Donlon; Vice-President, Jeanne Culivan; Secretary, Terry Breitenbach; Treasurer, Ruth Link; Sponsor, Sister Helen Sullivan, O.S.B.

Mathematics Club, College of the Holy Cross

The Holy Cross *Mathematics Club* conducted a series of seminars during the academic year 1949–50. The meetings were held twice monthly under the direction of Professors C. G. Schilling and V. O. McBrien. The following papers were

presented:

The gamma and beta functions, a series of four lectures, by Prof. V. O. McBrien

Theory of infinite products (two seminars)

The theorem of mean value

Unusual substitutions as a method of integration

Parametric method in maximum and minimum problems

Vertex and axis of a parabola in general position

The Heine-Borel covering theorem.

Solutions to proposed problems involving infinite products were presented by C. T. Mullins, J. A. Gobes, and W. F. Reynolds.

Pi Mu Epsilon, University of Delaware

The *Delaware Alpha* chapter of *Pi Mu Epsilon* heard the following talks during 1949–50:

Topological dynamics, by Russell Remage

Partitions, by Prof. N. J. Fine of the University of Pennsylvania

The brachistochrone problem, by A. C. Nelson.

Professor Fine spoke at the annual banquet in March.

Officers for 1950–51 are: President, Patricia Reybold; Secretary, Donald Clark; Treasurer, E. K. Rees; Social Chairman, Mervine Rosen; Program Chairman, Martha Walker; Director, Russell Remage.

Kappa Mu Epsilon, Westminster College

An important event in the past year for *Delta Nabla*, Westminster honorary mathematics society, was their installation as the *Pennsylvania Alpha* chapter of *Kappa Mu Epsilon*. At this ceremony, conducted by Dr. L. F. Ollmann, National Treasurer, nineteen students and four faculty members were initiated into *Kappa Mu Epsilon*.

The outstanding papers presented to the club were:

Linkages, by Fred Cassell and Robert Gunnett

I. B. M. computation, by Mrs. Ben Bolt

Abacus, by Thomas Petrecca.

During the year, the club donated to the book drive for the library and also donated a book to the outstanding mathematics student now in the Sophomore class. The award was made on the basis of mathematics grades and general average during the Freshman year.

Banquets held at Christmas and in the Spring, served as opportunities for the initiation of new members. At these ceremonies, each new member took the name of a famous mathematician and gave a short history of that person's life.

The present officers of the Pennsylvania Alpha chapter of Kappa Mu Epsilon are: President, John Williamson; Secretary, Barbara Williams; Vice-President, John Kratz; Treasurer, Henry Barnhart; Faculty Advisor, Prof. G. W. Conway, Jr.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

ANNUAL MEETING OF THE A.A.A.S.

The Annual Meeting of the American Association for the Advancement of Science will be held on December 26–30, 1950, in Cleveland, Ohio. All of the sections and subsections of the A.A.A.S. and more than forty participating organizations have completed plans for more than two hundred sessions. An extensive series of tours to museums, laboratories, and industrial plants of the Cleveland area has been planned. The Annual Science Exposition will be held in the Arena of Cleveland's Public Auditorium.

The Cleveland Convention Bureau is in charge of housing arrangements. Announcements and coupons for reservations will be found in recent issues of *Science* and *The Scientific Monthly*. Advance registration closes on December 4, 1950.

STANFORD UNIVERSITY COMPETITIVE EXAMINATION IN MATHEMATICS

The fifth Stanford University Competitive Examination in Mathematics (see this MONTHLY, vol. 53, no. 7, pp. 406–409 (1946)) was held March 25, 1950 in 34 high schools in California; 207 senior high school students participated. The following problems were proposed:

1. Observe that

$$1 = 1$$
$$1 - 4 = -(1 + 2)$$
$$1 - 4 + 9 = 1 + 2 + 3$$
$$1 - 4 + 9 - 16 = -(1 + 2 + 3 + 4)$$

Guess the general law suggested by these examples, express it in suitable mathematical notation, and prove it.

2. Given a square. Find the locus of the points from which the square is seen under an angle (a) of 90° (b) of 45°. (Let P be a point outside the square, but in the same plane. The smallest angle with vertex P containing the square is the "angle under which the square is seen" from P.) Sketch clearly both loci, give a full description, and a proof.

3. Call "axis" of a solid a straight line joining two points of the surface of the solid and such that the solid, rotated about this line through an angle which is greater than 0° and less than 360° coincides with itself.

A cube has 13 different axes, which are of three different kinds. Describe clearly the location of these axes, find the angle of rotation associated with each. Assuming that the edge of the cube is of unit length, compute the arithmetic mean of lengths of the 13 axes. Do not use tables and compute to two decimals.

The writer of the best paper, Glen Bredon, student at Sanger Union High School, Sanger, California, received a $600 Scholarship at Stanford University. Also, John R. Berry of Berkeley, California, Richard Couchman of Santa Ana, California, and John Rosser of Oakland, California, received "Honorable Mention."

PERSONAL ITEMS

Professor R. E. Langer of the University of Wisconsin and Professor L. M. Graves, University of Chicago, were the delegates of the Association to the International Congress of Mathematicians which was held at Cambridge, Massachusetts, on August 30–September 6, 1950.

Professor Emeritus F. W. Owens of Pennsylvania State College was the representative of the Association at the inauguration of President M. S. Eisenhower of Pennsylvania State College on October 5, 1950.

Associate Professor P. C. Rosenbloom of Syracuse University has been awarded the Fréchet Prize by the French Academy of Sciences for a paper pertaining to applications in abstract spaces.

Professor John von Neumann of the Institute for Advanced Study has been awarded honorary degrees of Doctor of Science by Harvard University and the University of Pennsylvania.

Case Institute of Technology announces: Dr. R. A. Clark, Massachusetts Institute of Technology, has been appointed to an instructorship; Mr. Philip Clyne of Mid-Essex Technical College, Chelmsford, England, has been appointed Visiting Instructor for the academic year 1950–51; Dr. R. F. Rinehart, who has been serving as Executive Secretary of the Research and Development Board, Washington, D. C., has resumed the position of Professor of Mathematics.

Central Michigan College reports the following: Mr. August Arndt, formerly principal of Fenton High School, Michigan, has been appointed Visiting Assistant Professor; Mr. Dana Sudborough has been granted a year's leave of absence for the purpose of doing graduate work at Michigan State College.

Kansas State College makes the following announcements: Dr. Abraham Franck of the University of Minnesota has been appointed to an assistant professorship; Dr. Violet G. Hachmeister, formerly an instructor in the Extension Division of the University of Wisconsin, has been appointed to an assistant professorship.

At Randolph-Macon Woman's College: Associate Professor M. Gweneth Humphreys has been promoted to the position of Professor and Head of the Department of Mathematics; Professor Gillie A. Larew, formerly head of the Department of Mathematics, is continuing as Professor of Mathematics and Dean of the College.

Stanford University reports that the following persons delivered lectures in the Mathematics Department during the Summer Quarter, 1950: Dr. Paul Erdös, Aberdeen, Scotland; Professor Theodor Estermann, University College, London; Professor M. Fekete, University of Jerusalem; Professor K. A. Hirsch,

Durham University, King's College, Newcastle, England; Professor Paul Levy, Ecole Polytechnique, Paris; Professor Walter Saxer, Polytechnical School, Zurich, Switzerland.

Syracuse University announces the following: Dr. W. A. Pierce, formerly graduate student at Harvard University, has been appointed to an assistant professorship; Assistant Professor P. C. Rosenbloom has been promoted to an associate professorship; Mrs. May N. Harwood has retired with the title of Professor Emeritus.

At the University of Delaware: Dr. Russell Remage, Jr. has been promoted to an assistant professorship; Dr. Henry Teicher has been appointed to an assistant professorship.

University of Minnesota reports the following: Dr. M. D. Donsker of Cornell University has been appointed to an assistant professorship; Assistant Professor Elizabeth S. Carlson has been promoted to an associate professorship; Miss Ella Thorp has been promoted to an assistant professorship.

University of Nebraska announces the following appointments to assistant professorships: Dr. L. K. Jackson of the University of California at Los Angeles and Dr. George Seifert of Cornell University.

University of South Carolina makes the following announcements: Associate Professor E. A. Hedberg has been promoted to a professorship; Instructors C. F. Martin, Lillian Perkins, and Betty Weber have been promoted to assistant professorships; Associate Professor J. D. Novak is on leave and is doing graduate work at Michigan State College; Assistant Professor R. R. Croxton is on leave of absence and is located at the University of North Carolina; Instructor C. W. Huff is also on leave of absence and is engaged in graduate study at Tulane University.

University of Toledo reports: Professor H. L. Zeiders who has been in military service in Germany has returned to his teaching position; Professor O. L. Dustheimer and Miss Lois L. Martin have resigned.

University of Tulsa announces the following appointments to instructorships: Dr. R. N. Goss, formerly instructor at Iowa State College, and Mr. H. G. Shaklee, previously assistant at Oklahoma University.

Assistant Professor E. H. Allen of Springfield College has been appointed Registrar of Northeastern University, Springfield Extension.

Dr. R. D. Anderson, University of Pennsylvania, has been promoted to an assistant professorship.

Assistant Professor T. W. Anderson of Columbia University has been promoted to an associate professorship.

Associate Professor H. M. Bacon of Stanford University has been promoted to a professorship.

Mr. Max Beberman, formerly a graduate assistant at Teachers College, Columbia University, has been appointed to an instructorship at University High School, University of Illinois.

Associate Professor J. H. Bell of Michigan State College is serving as Acting

Head of the Department of Mathematics during the year 1950–51.

Mr. C. R. Blyth, research assistant at the University of California at Berkeley, has been appointed to an assistant professorship at the University of Illinois.

Professor Harald Bohr of the University of Copenhagen, Denmark, delivered a series of lectures at the University of Georgia on September 13–18, 1950.

Miss Marguerite A. Brown, recent graduate of Carleton College, has been appointed Teaching Assistant at the University of Minnesota.

Mr. Errol Buker, a teacher in the Springfield Public Schools, has received an appointment to an instructorship at Springfield College.

Dr. F. E. Clark of Tulane University has been appointed to an assistant professorship at University College, Rutgers University.

Instructor H. C. Cooke of North Carolina State College has been promoted to an assistant professorship.

Mr. R. B. Derflinger is teaching in the Department of Chemistry of Turtle Creek High School, Turtle Creek, Pennsylvania.

Mr. G. M. Dillon, statistician in the Treasury Department of E. I. DuPont de Nemours Company, Wilmington, Delaware, is now in military service and is located at Camp Gordon, Augusta, Georgia.

Dr. W. B. Evans of the University of Illinois has been appointed to an assistant professorship at the Georgia Institute of Technology.

Associate Professor G. M. Ewing of the University of Missouri has been promoted to a professorship.

Dr. H. M. Feldman, formerly a lecturer at Washington University, is now teaching at Beaumont High School, University City, Missouri.

Miss Ruth A. Fish of the University of Arizona is teaching at Contra Costa Junior College, Richmond, California.

Assistant Professor Evelyn Frank of the University of Illinois has been promoted to an associate professorship.

Dr. J. R. Garrett of Duke University has been appointed to an assistant professorship at Georgia Institute of Technology.

Assistant Professor R. D. Gordon of the University of Buffalo has been appointed Professor and Head of the Department of Mathematics of Washington College.

Assistant Professor J. B. Greeley has been promoted to an associate professorship and is continuing to serve as Head of the Department of Mathematics of Utica College, Syracuse University.

Assistant Professor R. E. Greenwood of the University of Texas has been recalled to active duty by the Navy Department.

Mr. W. H. Gregory is employed as a teacher at Hayward Union High School, Hayward, California.

Mr. F. D. Grogan, previously teaching assistant at the University of Oklahoma, is teaching now at Minden High School, Minden, Louisiana.

Dr. Emil Grosswald of the University of Pennsylvania has been appointed

Lecturer at the University of Saskatchewan.

Dr. A. J. Hoffman of Columbia University is now a member of the Institute for Advanced Study.

Dr. R. V. Hogg of the State University of Iowa has been promoted to an assistant professorship.

Instructor J. R. Holzinger of Franklin and Marshall College has been promoted to an assistant professorship.

Miss Marian B. Jeffries, graduate student of the University of Virginia, has been appointed to an instructorship at the University of Richmond.

Associate Professor N. D. Lane of Acadia University has been appointed Lecturer at Carleton College, Ottawa, Canada.

Mr. H. R. Leifer of the Registration and Research Section, Veterans Administration, Pittsburgh, has been promoted to the position of Chief of the Section.

Assistant Professor R. C. Luippold of Antioch College has been appointed to an instructorship at New York State College for Teachers, Albany.

Dr. Spencer Macy has been appointed Assistant Professor of Mathematical Physics at the South Dakota School of Mines and Technology.

Associate Professor W. C. McDaniel of Southern Illinois University has been promoted to the position of Professor and Chairman of the Department of Mathematics.

Mr. J. W. Mettler of Lehigh University has accepted a position with the Educational Testing Service, Princeton, New Jersey.

Mr. P. T. Mielke, industrial research fellow at Purdue University, has been appointed to an assistant professorship at Wabash College.

Associate Professor E. P. Northrop, who is also Associate Dean of the College of the University of Chicago, has been promoted to a professorship.

Dr. Margaret Owchar of the University of Minnesota has been appointed to an associate professorship at Southwest Missouri State College.

Dr. Jesse Pierce has been appointed Research Engineer at Northrop Aircraft, Hawthorne, California.

Assistant Professor A. L. Putnam has been promoted to an associate professorship in the College of the University of Chicago.

Mr. V. J. Rathbun, formerly head of Science Department of Colegio Ponceno de Varones, Ponce, Puerto Rico, is now a graduate student at Catholic University.

Mr. T. L. Reynolds, previously a part-time instructor at the University of North Carolina, has been appointed to the position of Professor and Head of the Department of Mathematics of Millsaps College.

Associate Professor H. E. Robbins of the University of North Carolina has been promoted to a professorship.

Mr. F. E. Ross of the University of North Carolina has accepted a position as auditor in the Division of Rates and Services, Department of Transportation, Frankfort, Kentucky.

Miss Augusta L. Schurrer, graduate assistant at the University of Wisconsin, has been appointed to an instructorship at Iowa State Teachers College.

Mr. W. H. Sellers, formerly teaching fellow at West Virginia University, has been appointed Head of the Department of Mathematics of Alderson-Broaddus College.

Sister Mary Ferrer, previously a graduate student at Notre Dame University, is now Head of the Department of Mathematics at St. Francis Xavier College for Women.

Associate Professor R. W. Wagner of Oberlin College has been appointed to a professorship at the University of Massachusetts.

Mr. A. M. Weitzenhoffer has accepted a position as research associate at the College of Dentistry, University of Detroit.

Associate Professor William Beverley of Lafayette College died on July 24, 1950.

Professor H. L. Smith of Louisiana State University died on June 13, 1950. He was a charter member of the Association.

Mr. T. E. Tuthill, head of the Department of Mathematics of Nichols School, Buffalo, New York, died on August 22, 1950.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

SUMMER MEETING OF THE BOARD OF GOVERNORS

Although the usual Summer Meeting of the Mathematical Association of America was omitted because of the International Congress of Mathematicians, the Board of Governors of the Association met on the evening of Thursday, August 31, 1950, in Phillips Brooks House of Harvard University, Cambridge, Massachusetts. Twenty-six of the forty-one members of the Board were present. Among the more important items of business transacted were the following:

Since the Committee on an Index of the MONTHLY has completed its task, the Committee was discharged with a vote of thanks for its labors on behalf of the Association. It was also voted to express the thanks of the Association to Professors C. B. Allendoerfer, C. O. Oakley, D. L. Thomsen, and A. H. Wilson of Haverford College, and C. P. Bruderle of Villanova College.

It was voted to accept the invitation of the Mathematics Division of the American Society for Engineering Education to hold a joint meeting at Michigan State College, East Lansing, Michigan, in June 1951.

Reports of progress were submitted by the Committee on a Guidance Pamphlet, the Committee on Slaught Memorial Papers, and the Director of the Putnam Prize Competition. It was voted to discharge the standing Committee for the Coordination of Studies in Mathematical Education and to change the name of the standing Committee on Section Meetings to the Committee on Sections. Reports were also received from the Committee on Cooperation with

Industry and from the representatives on the Policy Committee for Mathematics.

H. M. GEHMAN, *Secretary-Treasurer*

MATHEMATICS CONTEST OF THE METROPOLITAN NEW YORK SECTION

On May 11, 1950, the Metropolitan New York Section of the Association conducted a mathematical contest open to high school students in the area of the Section. A total of 690 students from 238 high schools participated in the competition.

The Bronx High School of Science, whose students scored 437 out of a possible 450 points, was awarded the Association's bronze cup. Honorable mentions were won by Brooklyn Technical High School, Abraham Lincoln High School and Marymount School.

First place individual honors were awarded to Paul H. Monsky of Brooklyn Technical High School for presenting a perfect paper. Individual honorable mentions go to Arnold M. Faden, Lee R. Abramson, and Samuel J. Klein, all of the Bronx High School of Science.

Certificates of Merit for proficiency in Secondary School Mathematics were awarded to the highest ranking student in each of the eleven regions to which the participating schools were assigned. Also the top student in each school received a gold medal in the form of a lapel button or pin bearing a facsimile of the seal of the Mathematical Association of America.

The management of the contest was undertaken by the Section's Committee on Contests and Awards consisting of Brother Bernard Alfred, W. H. Fagerstrom (Chairman), H. F. Fehr, Julius Freilich, J. J. Kinsella, V. S. Mallory, K. B. Morgan.

The contest included material only from Elementary Algebra, Plane Geometry and Intermediate Algebra. Copies of the contest questions and other material used in publicizing the contest may be obtained from Professor W. H. Fagerstrom, City College of the City of New York, New York 31, New York.

The engraved Certificate of Merit and the gold medals are available to any Section wishing to conduct a contest of this type. For information consult the Secretary-Treasurer of the Association.

ANNUAL MEETING OF THE KENTUCKY SECTION

The annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky, Lexington, Kentucky, on Saturday, April 29, 1950. Dr. H. H. Downing, Chairman of the Section, presided at the morning and afternoon sessions.

Sixty-four persons were present at the meeting, including the following thirty-two members of the Association: D. F. Atkins, P. P. Boyd, M. C. Brown, Esther A. Compton, A. E. Cook, Lake C. Cooper, V. F. Cowling, D. L. Daly, H. H. Downing, Clarence Ford, A. E. Foster, A. W. Goodman, Beulah Graham, Charles Hatfield, Jr., A. S. Householder, Aughtum S. Howard, Tadeusz

SPRING MEETING OF THE SOUTHWESTERN SECTION

The tenth annual meeting of the Southwestern Section of the Mathematical Association of America was held at Arizona State College, Flagstaff, Arizona, on April 29, 1950.

Professor E. J. Purcell, Chairman of the Section, presided at the morning and afternoon sessions. Thirty-six persons attended the sessions, including the following twenty-three members: J. W. Beach, A. W. Boldyreff, C. E. Buell, J. H. Butchart, Lee Byrne, W. W. Denton, F. C. Gentry, R. F. Graesser, M. S. Hendrickson, R. C. Hildner, Max Kramer, Lincoln La Paz, Rachael La Roe, B. C. Meyer, E. J. Purcell, B. D. Roberts, H. P. Rogers, J. L. Slechticky, R. D. Stalley, Deonisie Trifan, Earl Walden, D. L. Webb, and Charles Wexler.

At the business session the following officers were elected for the year 1950–51: Chairman, A. W. Boldyreff, University of New Mexico; Vice-Chairman, Charles Wexler, Arizona State College, Tempe, Arizona; Secretary-Treasurer, R. L. Westhafer, New Mexico College of A. and M. A.; Lecturer, one-year term, J. H. Butchart, Arizona State College, Flagstaff, Arizona; two-year term, M. S. Hendrickson, University of New Mexico. The section accepted the invitation of the University of New Mexico to hold the spring 1951 meeting in Albuquerque. An assessment of $1.00 per member was voted.

At an evening banquet session Dr. Earl C. Slipher of Lowell Observatory delivered an illustrated lecture on the planet Mars.

Fifteen papers were crowded into the one-day sessions:

1. *An extension of Eisenstein's criteria of irreducibility*, by D. L. Webb, of the University of Arizona.

The speaker presented an extension of Eisenstein's criteria of irreducibility of polynomials with regard to linear factors, treating the case when p^m is a factor of the constant term.

2. *The place of statistics in the curriculum*, by J. W. Beach and H. P. Rogers, University of New Mexico.

3. *Short formulations of Boolean algebra, using ring operations*, by Dr. Lee Byrne, Arizona State College, Tempe.

Much interest has attached to recent formulations of Boolean algebras intended to emphasize their character as rings, and thus featuring especially ring operations. Most of these are relatively long, and Dr. Byrne's note was addressed to the question whether formulations of this type could not be made as brief as those representing other modes of approach. Leaving closure (and non-emptiness) assumptions tacit, and using as undefined concepts ring addition and multiplication and a special element 1 (one), he presented four "transformation" axioms: I. $(X+Y)+Z=X+(Y+Z)$; II. $(X+Y)+X=Y$; III. $X(W+YZ)=XW+Y(ZX)$; IV. $X(X+1)=Y+Y1$. From these are derived nine theorems sufficing to show the system to be a Boolean group, a ring, a Boolean ring, and finally a Boolean algebra. He further set up an alternative set of two such axioms, II as already given and the new I′: $X+X=P_1+(P_2+P_3)$ (employing a device due to B. A. Bernstein), where P_1, P_2, and P_3 are respectively the abbreviations $P_1=[(A+B)+C]+[A+(B+C)]$, $P_2=C(F+DE)+[CF+D(EC)]$, $P_3=G(G+1(+(H+H1)$. After deducing three of the old theorems and two new ones it is then possible to deduce the previous four axioms from the two last mentioned.

4. *An inverse problem of the calculus of variations for multiple integrals*, Lincoln La Paz, University of New Mexico.

5. *On proofs of Stokes' theorem, involving Taylor's expansion*, W. W. Denton, University of Arizona.

In most text books on mathematical physics, the proofs of Stokes' formula, Gauss' formula, the relation of curl to angular velocity, the equation of continuity, and similar theorems are, in general, rather poorly presented. The proofs in question involve the expansion of functions by Taylor's formula, and integration of results over an infinitesimal curve or surface. The object of this paper is to explain why these particular proofs are weak, and to point out how they may be strengthened. The improvements suggested depend upon the following device. With each point of the given vector field there is associated an auxiliary system of coordinates having that point as origin, and the integration over the infinitesimal curve or surface is actually carried out.

6. *The boundaries of an arbitrary function*, M. S. Hendrickson, University of New Mexico.

7. *Invariant spherical harmonics*, B. C. Meyer, University of Arizona.

Let G be a finite group of orthogonal linear transformations of the variables X, Y, Z. A generating function was developed which gives the number of linearly independent spherical harmonics of a given degree M which are invariants of G.

8. *An arithmetical triangle*, R. D. Stalley, University of Arizona.

A triangular array of numbers is presented as it arises in the summation of a generalized geometric series. Properties and further applications of the triangular array are discussed.

9. *Some diophantine equations*, A. W. Boldyreff, University of New Mexico.

10. *A range of conics associated with a system of coaxial circles*, J. H. Butchart, Arizona State College, Flagstaff.

Professor Butchart considered the figure formed by drawing tangents from a variable point of a circle of a non-intersecting system of coaxial circles to two arbitrary circles of the same system so that both of the chosen circles are within the angle formed by the tangents. When the points of contact are joined, the circles determine chords on this line which have a constant ratio. The envelope of this line is an ellipse whose major auxiliary circle is another circle of the system. As other circles of the system are considered as loci of the variable point, other ellipses are formed all of which are tangent to the four common tangents to the two chosen circles.

11. *A non-involutorial Cremona transformation, in four-dimensional space*, E. J. Purcell, University of Arizona.

12. *Mathematics in meteorology*, Max Kramer, New Mexico College of A. and M. A.

Requirements for mathematical forecasting were discussed, and the curriculum pursued by the average meteorologist outlined. Seven tables compiled from a thorough analysis of the mathematical content of textbooks and other publications covering the entire meteorological field were presented with comment. Three important problems were solved to exhibit the type of mathematics required. In each case the theorems were derived, using both cartesian and vector techniques, and results applied to the following meteorological problems: (1) The Eulerian form of the equations of motion; (2) Bjerknes's circulation theorem; (3) The variation of wind in the friction layer, leading to the logarithmic spiral due to Ekman.

13. *An infinite set of formulas connecting binomial coefficients*, R. F. Graesser, University of Arizona.

14. *The dilemma in mathematics*, Charles Wexler, Arizona State College, Tempe.

15. *The plastic bending of plates*, Deonisie Trifan, University of Arizona.

In this paper Dr. Trifan formulated the basic partial differential equation governing the mechanical behavior of thin plates subjected to distributed transverse loads which produce plastic deformations. The plates were considered homogeneous and composed of isotropic materials exhibiting a gradual transition from the elastic to the plastic state. The stress-strain relation used was

$$\dot{s}_{ij}=2G\dot{e}_{ij}-(G_0-G)\dot{E}/Ee_{ij}.$$

where s_{ij} and e_{ij} are tensors of stress and strain deviation, the elastic shear modulus and the tangent shear modulus are G_0 and G respectively, the latter being assumed as a function of the strain intensity $E=2/3\ e_{ij}\ e_{ij}$, and variables with dots indicate rates of change with respect to time.

B. D. ROBERTS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December 30, 1950.

Joint Meeting with American Society for Engineering Education, Michigan State College, East Lansing, June 25–26, 1951.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May, 1951.

ILLINOIS, University of Illinois, Urbana, May 11–12, 1951.

INDIANA, May 5, 1951.

IOWA, Wartburg College, Waverly, April 20–21, 1951.

KANSAS

KENTUCKY, Eastern Kentucky State College, Richmond, April 28, 1951.

LOUISIANA-MISSISSIPPI, Mississippi State College, State College, February 16–17, 1951.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Catholic University, December 9, 1950.

METROPOLITAN NEW YORK, Spring, 1951.

MICHIGAN, Michigan State College, East Lansing, March 24, 1951.

MINNESOTA, College of St. Benedict, St. Joseph, April 28, 1951.

MISSOURI, Central College, Fayette, Spring, 1951.

NEBRASKA, University of Nebraska, Lincoln, May 5, 1951.

NORTHERN CALIFORNIA, University of San Francisco, January 27, 1951.

OHIO, Columbus, April 21, 1951.

OKLAHOMA, Oklahoma City, November 13, 1950.

PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.

PHILADELPHIA, Lehigh University, Bethlehem, Pennsylvania, November 25, 1950.

ROCKY MOUNTAIN, Colorado State College of Education, Greeley, April, 1951.

SOUTHEASTERN, Vanderbilt University and Peabody College, Nashville, Tennessee, March 16–17, 1951.

SOUTHERN CALIFORNIA, Whittier College, Whittier, March 10, 1951.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1951.

TEXAS, Southern Methodist University, Dallas, Spring, 1951.

UPPER NEW YORK STATE, Hamilton College, Clinton, May 5, 1951.

WISCONSIN, Carroll College, Waukesha, May, 1951.

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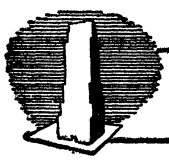
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Each member of the Mathematical Association of America is entitled to purchase one copy of each Carus Monograph at $1.75. Orders (preferably accompanied by remittance) should be sent to: Harry M. Gehman, Secretary-Treasurer, Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

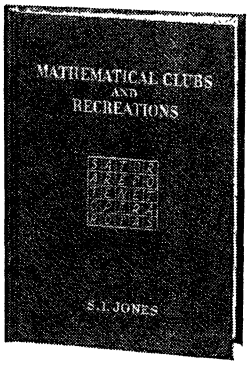
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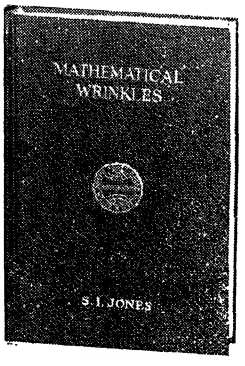


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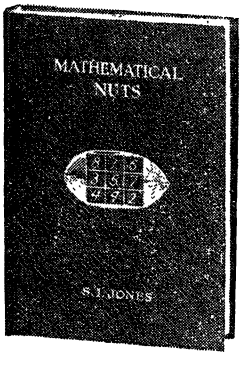


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A CURVE ASSOCIATED WITH A TETRAHEDRON*

VICTOR THÉBAULT, Tennie, France

1. Introduction. The first twelve-point sphere of an orthocentric tetrahedron passes through the extremities of the bimedians and of the bialtitudes. The bimedians meet at the centroid and the bialtitudes at the orthocenter. This suggested the following problem:** Given a tetrahedron $T \equiv ABCD$, determine the locus (Γ) of points M such that the three lines passing through M and meeting respectively two opposite edges of T intersect these edges in six points of a sphere (S).

The object of this article is to report some of the results obtained and to link them to a general theorem concerning pencils of spheres associated with a tetrahedron.

2. Notation. Let O be the center of the circumsphere (O) and G be the centroid of a tetrahedron $T \equiv ABCD$. Let the lengths of the edges of T be denoted by $BC=a$, $DA=a'$, $CA=b$, $DB=b'$, $AB=c$, $DC=C'$.

3. Analytical considerations. In barycentric coordinates, referred to T as the reference tetrahedron, the equations of the curve (Γ) may be written:†

$$\begin{aligned} (1) \qquad & a^2yz/(y+z) + a'^2xt/(x+t), \\ & = b^2xz/(x+z) + b'^2yt/(y+t), \\ & = c^2xy/(x+y) + c'^2zt/(z+t), \end{aligned}$$

where M has the coordinates (x, y, z, t). The sphere (S) cuts the edges of T in six other points. The lines joining pairs of these points on opposite edges pass through a point M'.‡ The correspondance on (Γ) between the coordinates of M and M' is given by

$$\begin{aligned} & a^2yzy'z'/(y+z)(y'+z') = a'^2xtx't'/(x+t)(x'+t'), \\ (2) \qquad & = b^2xzx'z'/(x+z)(x'+z') = b'^2yy't'/(y+t)(y'+t'), \\ & = c^2xyx'y'/(x+y)(x'+y') = c'^2ztz't'/(z+t)(z'+t'). \end{aligned}$$

The point M whose barycentric coordinates (x, y, z, t) are inversely proportional to the differences

$$\begin{aligned} (3) \qquad & b^2+c^2+a'^2-(b'^2+c'^2+a^2), \\ & c^2+a^2+b'^2-(c'^2+a'^2+b^2), \\ & a^2+b^2+c'^2-(a'^2+b'^2+c^2), \\ & a'^2+b'^2+c'^2-(a^2+b^2+c^2), \end{aligned}$$

* Translated from the French by W. E. Byrne, Virginia Military Institute.

** V. Thébault, Intermédiaire des Recherches Mathématiques (I.R.M.), Tome 1, 1945, p. 49.

† A. de Majo, I.R.M., t. 3, 1947, p. 88.

‡ This remark is still valid if sphere (S) is replaced by any quadric. J. Neuberg, Mathesis, 1887, p. 126.

between the sum of the squares of the edges drawn from a vertex and the sum of the squares of the edges of the opposite face, is situated on (Γ). We have for this point:

$$yz/(y+z) = 2(a^2 - a'^2),\ xy/(x+t) = -2(a^2 - a'^2),$$
$$xz/(x+z) = 2(b^2 - b'^2),\ yt/(y+t) = -2(b^2 - b'^2),$$
$$xy/(x+y) = 2(c^2 - c'^2),\ zt/(z+t) = -2(c^2 - c'^2).$$

The barycentric coordinates (x', y', z', t') of the point M' corresponding to M satisfy the relations

$$a^2yzy'z'/(y+z)(y'+z') + a'^2xtx't'/(x+t)(x'+t') = 0, \cdots$$

and

$$a^2/x' + a'^2/y' + a'^2/z' + a^2/t' = 0,$$
$$b'^2/x' + b^2/y' + b'^2/z' + b^2/t' = 0,$$
$$c'^2/x' + c'^2/y' + c^2/z' + c^2/t' = 0,$$

so that

$1/x', 1/y', 1/z', 1/t'$ are proportional to

$$\begin{vmatrix} c'^2 & c^2 & c^2 \\ a'^2 & a'^2 & a^2 \\ b^2 & b'^2 & b^2 \end{vmatrix}, \quad -\begin{vmatrix} c'^2 & c^2 & c^2 \\ a^2 & a'^2 & a^2 \\ b'^2 & b'^2 & b^2 \end{vmatrix},$$

$$\begin{vmatrix} c'^2 & c'^2 & c^2 \\ a^2 & a'^2 & a^2 \\ b'^2 & b^2 & b^2 \end{vmatrix}, \quad -\begin{vmatrix} c'^2 & c'^2 & c^2 \\ a^2 & a'^2 & a'^2 \\ b'^2 & b^2 & b'^2 \end{vmatrix}.$$

In other words the coordinates of M' are inversely proportional to the quantities

$$\begin{aligned} l^2 &= 2(bca')^2 + (b'c'a)^2 - [c'a'b)^2 + (a'b'c)^2 + (abc)^2] \\ m^2 &= 2(cab')^2 + (c'a'b)^2 - [(b'c'a)^2 + (a'b'c)^2 + (abc)^2], \\ n^2 &= 2(abc')^2 + (a'b'c)^2 - [(b'c'a)^2 + (c'a'b)^2 + (abc)^2], \\ p^2 &= 2(a'b'c')^2 + (abc)^2 - [(b'c'a)^2 + (c'a'b)^2 + (a'b'c)^2], \end{aligned} \tag{4}$$

and (1) is satisfied.

4. Geometric considerations. If we consider a sphere (ω), of radius σ, for which the powers of the vertices A, B, C, D of T are kl^2, km^2, kn^2, kp^2, where l^2, m^2, n^2, p^2 are given and k is an arbitrary coefficient, the radical plane (π) of (O) and (ω) satisfies

$$\overline{A\omega}^2 - \sigma^2 = kl^2 = 2O\omega \cdot A\alpha,$$
$$\overline{B\omega}^2 - \sigma^2 = km^2 = 2O\omega \cdot B\beta,$$
$$\overline{C\omega}^2 - \sigma^2 = kn^2 = 2O\omega \cdot C\gamma,$$
$$\overline{D\omega}^2 - \sigma^2 = kp^2 = 2O\omega \cdot D\delta,$$

where α, β, γ, δ are the orthogonal projections of A, B, C, D on (π). When k varies the locus of the center ω of (ω) is the line Δ drawn through O perpendicular to (π); (π) remains fixed and has as its tangential barycentric coordinates l^2, m^2, n^2, p^2.* To two values k_1, k_2 of the parameter k correspond two spheres (ω_1), (ω_2) such that

$$O\omega_1/O\omega_2 = k_1/k_2$$
$$A\alpha + B\beta + C\gamma + D\delta = 4GP$$

where P is the orthogonal projection of the centroid G of T on (π). In the special case where

$$\overline{A\omega}^2 - \overline{B\omega}^2 = k[(a^2 - a'^2) - (b^2 - b'^2)], \tag{5}$$
$$\overline{B\omega}^2 - \overline{C\omega}^2 = k[(b^2 - b^{12}) - (c^2 - c'^2)], \cdots$$

Δ coincides with the Euler line OG of T, since

$$\overline{AG}^2 - \overline{BD}^2 = -\tfrac{1}{4}[(a^2 - a'^2) - (b^2 - b'^2)],$$
$$\overline{BG}^2 - \overline{CG}^2 = -\tfrac{1}{4}[(b^2 - b'^2) - (c^2 - c'^2)], \cdots$$

as (π) is then normal to OG, we have $O\omega/OG = -4k$

$$\overline{A\alpha} = -l^2/8\overline{OG}, \quad \overline{B\beta} = -m^2/8\overline{OG}, \quad \overline{C\gamma} = -n^2/8\overline{OG}, \quad \overline{D\delta} = -p^2/8\overline{OG},$$

and

$$(\overline{A\alpha} + \overline{B\beta} + \overline{C\gamma} + \overline{D\delta})/4 = \overline{GP} = -(l^2 + m^2 + n^2 + p^2)/32\overline{OG}. \tag{6}$$

If we write

$$l^2 = b^2 + c^2 + a'^2 - (b'^2 + c'^2 + a^2),$$
$$m^2 = c^2 + a^2 + b'^2 - (c'^2 + a'^2 + b^2),$$
$$n^2 = a^2 + b^2 + c'^2 - (a'^2 + b'^2 + c^2), \tag{7}$$
$$p^2 = a'^2 + b'^2 + c'^2 - (a^2 + b^2 + c^2),$$

we find

$$\overline{A\omega}^2 - \overline{B\omega}^2 = -2[(a^2 - a'^2) - (b^2 - b'^2)], \cdots$$

of the same form as (5). For this choice of l^2, m^2, n^2, p^2 (π) is perpendicular to

* V. Thébault, Mathesis, 1932, Supplément.

OG at a point P such that

$$\overline{GP} = -(l^2 + m^2 + n^2 + p^2)/32\overline{OG} = 0,$$

so P coincides with G. The point M of (Γ) for which the barycentric coordinates are inversely proportional to the quantities (7) is the harmonic pole of the plane (π) perpendicular to OG at G.*

The point M' which corresponds to M is the harmonic pole of the plane (π_1) whose barycentric coordinates are proportional to (4). (π_1) is parallel to the radical plane (π') of (O) and of the sphere determined by

$$\begin{aligned} l'^2 &= (bca')^2 + (b'c'a)^2, \\ m'^2 &= (cab')^2 + (c'a'b)^2, \\ n'^2 &= (abc')^2 + (a'b'c)^2, \\ p'^2 &= (a'b'c')^2 + (abc)^2. \end{aligned}$$

If we take values (4) of l^2, m^2, n^2, p^2 we have

$$l^2 - m^2 = 2[(bca')^2 + (b'c'a)^2 - (cab')^2 - (c'a'b)^2] = 2(l'^2 - m'^2), \cdots$$

5. Special tetrahedrons.

A. *The orthocentric tetrahedron.* The centroid G is on (Γ) since (1) is satisfied by $x=y=z=t$. The barycentric coordinates (x', y', z', t') of the point G' which corresponds to G are:

$$1/x' = 2(b^2 + c^2 + a'^2) - (b'^2 + c'^2 + a^2), \cdots$$

G' coincides with the orthocenter H. The sphere (S) is then the first twelve-point sphere of T.

B. *Circumscriptible tetrahedron.* Besides the point M mentioned in section 2 the point M_1 whose barycentric coordinates are inversely proportional to the quantities:

$$\begin{aligned} l^2 &= 2(b + c + a') - (b' + c' + a), \\ m^2 &= 2(c + a + b') - (c' + a' + b), \\ n^2 &= 2(a + b + c') - (a' + b' + c), \\ p' &= 2(a' + b' + c') - (a + b + c), \end{aligned}$$

is on (Γ). It is the harmonic pole of the radical plane of (O) and (ω), where l^2, m^2, n^2, p^2 are proportional to the powers of A, B, C, D with respect to (ω). Let G_e be the centroid of the edges of the tetrahedron T. The barycentric coordinates of the point Q symmetric of G_e with respect to G are:

$$a + b' + c', \qquad a' + b + c', \qquad a' + b' + c, \qquad a + b + c.$$

The coordinates of the anticomplimentary point of Q are precisely the values

* A. de Majo and V. Thébault, Mathesis, t, 57, p. 75.

l^2, m^2, n^2, p^2. Furthermore, M_1 coincides with the point of intersection of the lines which join the vertices of the tetrahedron with the Gergonne points of the triangles of the opposite faces. The lines drawn through M_1 meeting opposite edges of the tetrahedron encounter the opposite edges at the points of contact of the sphere tangent to the six edges, so M_1 and M_1' coincide.

NOTE 1. A. de Majo has communicated the following results concerning the harmonic pole M of the plane normal to the Euler line OG at G:

a). *The parallels to the medians of T drawn through M meet the corresponding faces in four coplanar points.*

b). *The six isotomic planes of the planes drawn through M and the edges of T are parallel to the line joining the centroid G to the point Ω of coordinates $b^2+c^2+a'^2$, $c^2+a^2+b'^2$, $a^2+b^2+c'^2$, $a'^2+b'^2+c'^2$.*

NOTE 2. In general, the locus of the harmonic poles of the planes parallel to a fixed plane, with respect to a given tetrahedron T, is a skew cubic which passes through G. If the fixed plane is perpendicular to the Euler line OG, its harmonic pole describes the cubic whose equation in normal coördinates is:

$$\left(b^2 + c^2 + a'^2 + \frac{\lambda}{h_a}\right)x = \left(c^2 + a^2 + b'^2 + \frac{\lambda}{h_b}\right)y = \left(a^2 + b^2 + c'^2 + \frac{\lambda}{h_c}\right)z = \left(a'^2 + b^2 + c'^2 + \frac{\lambda}{h_d}\right)t,$$

where h_a, h_b, h_c, h_d indicate the lengths of the altitudes AA', BB', CC', DD' of T. This cubic is the isogonal inverse of the line joining the first point of Lemoine K to the point Q_1 of normal coördinates $b^2+c^2+a'^2$, $c^2+a^2+b'^2$, $a^2+b^2+c'^2$, $a'^2+b'^2+c'^2$.

A MINIATURE THEORY IN ILLUSTRATION OF THE CONVOLUTION TRANSFORM*

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1. Introduction. The convolution transform is defined by the equation

$$f(x) = \int_{-\infty}^{+\infty} G(x - t)\phi(t)dt. \tag{1}$$

The kernel $G(x)$ is thought of as given. Equation (1) then "transforms" functions

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$\phi(t)$ into functions $f(x)$. If the kernel $G(x)$ is suitably chosen the transform (1) becomes, after an obvious exponential change of variable, the Laplace transform

$$f(x) = \int_0^\infty e^{-xt}\phi(t)dt. \tag{2}$$

For another choice of the kernel, equation (1) becomes

$$f(x) = \int_0^\infty \frac{\phi(t)}{x+t}\,dt, \tag{3}$$

the so-called Stieltjes transform.

The present authors [1]–[4]* have given a fairly complete discussion of the transform for a very general class of kernels. The analytic development of the theory is somewhat involved, but the type of result obtained can be illustrated by use of a special kernel. By use of it we shall obtain directly, without any reference to earlier work, results that are quite analogous to the general ones. They may also serve to illustrate the general theory for the Laplace and Stieltjes transforms, and we shall preface each theorem with a number in braces, { }, referring to the analogous theorems in *The Laplace Transform*,** [6].

The simplification effected by our present choice of kernel is due to the fact that for it equation (1) can be inverted (solved for $\phi(t)$) by a linear differential operator of first order, whereas in general, and in particular for the transforms (2) and (3), the inversion operator must be of infinite order. The results of the present note, though designed primarily as illustrations of a more complex theory, are not without interest in themselves.

2. Notations. Throughout we shall assume that $\phi(t)$ is continuous and that $f(t)$ has a continuous first derivative for all t. More general classes of functions could be considered, but the main points of the theory are brought out by the present simplifying assumptions. We say that $f(t)\in B$ if it is bounded on $-\infty < t < \infty$ and that $f(t)\in L^p$ if

$$\int_{-\infty}^{\infty} |f(t)|^p dt < \infty.$$

Arrows, $\Rightarrow$, $\Leftrightarrow$, are used for implication. The kernel used throughout the remainder of this note is

$$G(t) = 0, \qquad t > 0;$$
$$G(t) = e^t, \qquad t \leqq 0.$$

* Numbers in brackets refer to the bibliography at the end of this note.

** Roman numbers refer to chapters, arabic to theorems.

3. Convergence.

{VIII 2a} THEOREM 1. *The integral* (1) *converges for some* x_0

$$\Longrightarrow \qquad \textit{the integral converges for all } x.$$

For,

$$\int_{-\infty}^{\infty} G(x-t)\phi(t)dt = \int_{x}^{\infty} e^{x-t}\phi(t)dt.$$

Then

$$\int_{x_0}^{\infty} e^{-t}\phi(t)dt$$

converges by hypothesis, from which the conclusion of the theorem is immediate. As a result of this theorem we may speak of (1) simply as convergent or divergent.

It should be observed that for other kernels the integral (1) may have other convergence properties; {compare II, 1}. The general theory distinguishes between these properties on the basis of characteristics of the kernel.

4. Inversion.

{VII 6a, VIII 9} THEOREM 2. (1) *converges*

$$\Longrightarrow \qquad f(x) - f'(x) = \phi(x), \qquad -\infty < x < \infty.$$

That is, the inversion operator, with our present choice of kernel, is the differential operator $(1-D)$, where D indicates differentiation with respect to x. The proof is immediate. We have

$$f(x) = e^{x}\int_{x}^{\infty} e^{-t}\phi(t)dt,$$

$$f'(x) = e^{x}\int_{x}^{\infty} e^{-t}\phi(t)dt - \phi(x), \qquad -\infty < x < \infty.$$

The result now follows by subtracting these two equations.

5. Representation. We wish now to investigate what classes of functions $f(x)$ can be represented in the form (1).

{VII 11a, VIII 13} THEOREM 3.

$$f(x) \sim Ae^{x}, \qquad (\textit{some constant } A,\ x \to +\infty),$$

$$\Longrightarrow \qquad f(x) = \int_{\infty}^{\infty} G(x-t)[f(t) - f'(t)]dt + Ae^{x}, \qquad -\infty < x < \infty$$

For, we have after integration by parts,

$$\int_x^R e^{x-t}f'(t)dt = f(R)e^{x-R} - f(x) + \int_x^R e^{x-t}f(t)dt,$$

$$\int_x^R e^{x-t}[f(t) - f'(t)]dt = f(x) - f(R)e^{x-R}. \tag{4}$$

Let $R\to\infty$. By hypothesis the right hand side tends to $f(x)-Ae^x$ so that

$$\int_x^\infty e^{x-t}[f(t) - f'(t)]dt = f(x) - Ae^x. \tag{5}$$

This equation is equivalent to the conclusion of the theorem.

COROLLARY 3.1 $\lim_{x\to+\infty} e^{-x}f(x) = 0$

$$\Longrightarrow \quad f(x) = \int_{-\infty}^\infty G(x - t)[f(t) - f'(t)]dt.$$

COROLLARY 3.2 $\int_0^\infty e^{-t}[f(t) - f'(t)]dt$ *converges*

$$\Longrightarrow \quad f(x) \sim Ae^x, \qquad x \to +\infty,$$

where

$$A = f(0) - \int_0^\infty e^{-t}[f(t) - f'(t)]dt. \tag{6}$$

This conclusion follows from equation (4), where now it is the left-hand side which is known to approach a limit as $R\to+\infty$. Equation (6) follows from (4) when $x=0$, $R=+\infty$. Thus we see that equation (5) is valid whenever the integral converges.

6. Bounded $\phi(t)$. We now characterize the functions $f(x)$ which have the representation (1) with bounded $\phi(t)$.

{VII 16a, VIII20a} THEOREM 4. $f(x), f'(x)\in B$

$$\Longleftrightarrow \quad f(x) = \int_{-\infty}^\infty G(x - t)\phi(t)dt,$$

where $\phi(t)\in B$.

We observe first that $G(t)\geqq 0$ for all t and that

$$\int_{-\infty}^\infty G(x - t)dt = \int_{-\infty}^0 e^t dt = 1.$$

First assume the representation (1) with $|\phi(t)| \leqq M$ for all t. Then

$$|f(x)| \leqq \int_{-\infty}^{\infty} G(x-t)|\phi(t)|\,dt \leqq M\int_{-\infty}^{\infty} G(x-t)dt = M$$

$$|f(x) - f'(x)| = |\phi(x)| \leqq M,$$

from which it follows that $f(x), f'(x) \in B$.

Conversely, from the boundedness of $f(x)$ it is clear that

$$\lim_{x\to\infty} e^{-x}f(x) = 0.$$

By Corollary 3.1

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\phi(t)dt,$$

where $\phi(t) = f(t) - f'(t)$. Since f and f' are bounded, the same is true of ϕ.

COROLLARY 4. *$f(x) - f'(x) \in B$*

$$\iff \quad f(x) = \int_{-\infty}^{\infty} G(x-t)\phi(t)dt + Ae^{x},$$

where $\phi(t) \in B$ and A is some constant.

Half of this conclusion follows from Theorem 2, the other half from Corollary 3.2.

7. Positive $\phi(t)$. Here we investigate the analogue of Bernstein's classic representation of completely monotonic functions.

{VII 14a, VIII 17c} THEOREM 5.

$$f(x) \geqq 0, \qquad f(x) - f'(x) \geqq 0, \qquad (-\infty < x < \infty),$$

$$\iff \quad f(x) = \int_{\infty}^{\infty} G(x-t)\phi(t)dt + Ae^{x},$$

where $\phi(t) \geqq 0$ and $A \geqq 0$.

If $f(x)$ has the convolution representation described in the theorem one sees by inspection that $f(x)$ and $f(x) - f'(x) = \phi(x)$ are non-negative functions.

Conversely, we have from equation (4) that

$$\int_0^R e^{-t}[f(t) - f'(t)]dt = f(0) - f(R)e^{-R}.$$

Since we are now assuming that $f(t) \geqq 0$ and $f(t) - f'(t) \geqq 0$, we have

$$\int_0^R e^{-t}[f(t) - f'(t)]dr \leqq f(0),$$

$$\int_0^\infty e^{-t}[f(t) - f'(t)]dt \leqq f(0). \tag{7}$$

Hence we may apply Corollary 3.2 to obtain

$$f(x) = \int_{-\infty}^{\infty} G(x - t)\phi(t)dt + Ae^x,$$

where $\phi(t) = f(t) - f'(t) \geqq 0$. From equation (6) and inequality (7) we see that $A \geqq 0$, and our result is proved.

In the light of Corollary 4 we might seek to alter the present theorem in such a way as to eliminate the hypothesis $f(x) \geqq 0$. That it is not redundant for the present theorem is evident from the function $f(x) = -xe^x$. Since it changes sign it cannot have the representation of Theorem 5. Yet $f(x) - f'(x) = e^x > 0$. However, we have the following alternative result.

COROLLARY 5.1

$$\begin{aligned} &1.\ f(x) - f'(x) \geqq 0, && -\infty < x < \infty, \\ &2.\ \lim_{x\to\infty} e^{-x}f(x) = 0 \end{aligned}$$

$$\Longleftrightarrow \qquad f(x) = \int_{-\infty}^{\infty} G(x - t)\phi(t)dt,$$

where $\phi(t) \geqq 0$.

This follows immediately from Theorem 2 and Corollary 3.1

COROLLARY 5.2

$$\begin{aligned} &1.\ f(x) - f'(x) \geqq 0, && -\infty < x < \infty, \\ &2.\ \lim_{x\to+\infty} e^{-x}f(x) = 0 \end{aligned}$$

$$\Longrightarrow \qquad f(x) \geqq 0, \qquad -\infty < x < \infty.$$

This is a consequence of Corollary 5.1. It is of interest, independent as it is of any representation theory.

8. Functions $\phi(t)$ in L^p. In general theory these classes must be discussed separately according as $p > 1$ or $p = 1$. Here the two cases coalesce into a single one.

{VII 15a, VIII 19a} THEOREM 6. $f(x), f'(x) \in L^p$, $(p \geqq 1)$,

$$\Longleftrightarrow \qquad f(x) = \int_{-\infty}^{\infty} G(x - t)\phi(t)dt,$$

where $\phi(t) \in L^p$.

First suppose that $f(x)$ has the integral representation described. Then if $p=1$

$$|f(x)| \leqq \int_{-\infty}^{\infty} G(x-t)|\,\phi(t)|\,dt,$$

and by Fubini's theorem*

$$\begin{aligned}\int_{-\infty}^{\infty}|f(x)|\,dx &\leqq \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}G(x-t)|\,\phi(t)|\,dt\\ &\leqq \int_{-\infty}^{\infty}|\phi(t)|\,dt\int_{-\infty}^{\infty}G(x-t)dx\\ &= \int_{-\infty}^{\infty}|\phi(t)|\,dt.\end{aligned}$$

Thus $f(x)\in L$. Since $f(x)-f'(x)=\phi(x)\in L$, it is clear that $f'(x)\in L$.

If $p>1$, we have by use of Hölder's inequality

$$f(x) = \int_{-\infty}^{\infty}[G(x-t)]^{1/2}[G(x-t)]^{1/2}\phi(t)dt,$$

$$|f(x)| \leqq \left[\int_{-\infty}^{\infty}[G(x-t)]^{q/2}dt\right]^{1/q}\left[\int_{-\infty}^{\infty}[G(x-t)]^{p/2}|\,\phi(t)|^p dt\right]^{1/p},$$

where $1/p+1/q=1$.

Evaluating the first integral on the right and again using Fubini's theorem, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}|f(x)|^p dx &\leqq \left[\frac{2}{q}\right]^{p/q}\int_{-\infty}^{\infty}|\phi(t)|^p dt\int_{-\infty}^{\infty}[G(x-t)]^{p/2}dx\\ &\leqq \frac{2}{p}\left[\frac{2}{q}\right]^{p/q}\int_{-\infty}^{\infty}|\phi(t)|^p dt.\end{aligned}$$

Thus it is again evident, from Minkowski's inequality,** that f and $f'\in L^p$.

Conversely, if $f-f'\in L^p$ it is clear from Hölder's inequality** (or from the boundedness of e^{-t} if $p=1$) that

$$\int_0^{\infty} e^{-t}[f(t)-f'(t)]dt$$

converges. Hence we may apply Corollary 3.2 to obtain

$$f(x) = \int_{\infty}^{\infty} G(x-t)\phi(t)dt + Ae^x,$$

* See, for example, E. C. Titchmarsh, The Theory of Functions, Oxford 1932, p. 392.

** See E. C. Titchmarsh, loc. cit., pp. 383–384.

where $\phi(t)=f(t)-f'(t)\in L^p$. Then the function defined by this integral $\in L^p$ by the first half of the proof. Since $e^x \notin L^p$ and $f(x)\in L^p$ we see from Minkowski's inequality (or directly if $p=1$) that $A=0$. This concludes the proof.

COROLLARY 6. $f(x)-f'(x)\in L^p$ $(p\geqq 1)$

$$\Longleftrightarrow \qquad f(x) = \int_{-\infty}^{\infty} G(x-t)\phi(t)dt + Ae^x,$$

where $\phi(t)\in L^p$ and A is some constant.

This follows from Theorem 2 and Corollary 3.2

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MATHEMATICAL NOTES

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NOTE ON MARTIN'S ERGODIC FUNCTION

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1. **Introduction.** Let M be a fixed bounded point set in the plane containing more than one point; let ϵ denote a positive number. If C is a continuous, rectifiable curve in the plane such that each point of M is within distance ϵ of some point of C, then M. H. Martin (*Ergodic curves*, American Journal of Mathematics, vol. 58 (1936); *The ergodic function*, Bulletin of the American Mathematical Society, vol. 43 (1937)) calls C *ϵ-ergodic* to M. Let $\rho>0$ be the radius of the smallest circle containing M; for a given ϵ let $\Lambda(\epsilon)$ be the greatest lower

bound of the lengths of the continuous, rectifiable curves ϵ-*ergodic* to M. Martin has shown that $\Lambda(\epsilon)=0$ if and only if $\epsilon \geqq \rho$, and that, for all positive ϵ, the function $\Lambda(\epsilon)$ is continuous, monotone non-increasing and non-negative. He proves that there is a continuous, rectifiable curve ϵ-ergodic to M and having length $\Lambda(\epsilon)$.

To show that $\Lambda(\epsilon)$ is finite for each ϵ, he exhibits a family of spirals ϵ-ergodic to M and of finite length. In this note we remark that $\Lambda(\epsilon)$ is strictly monotone decreasing and bounded above by a function more simple than the lengths of the spirals.

2. **Theorem.** We shall establish the following result.

THEOREM 1. *The function* $\Lambda(\epsilon)$ *is strictly monotone decreasing on* $0<\epsilon<\rho$. *In fact, if* $0<a<b<\rho$, *then* $\Lambda(a)-\Lambda(b) \geqq b-a>0$.

Proof. Let

$$C: \quad x = x(t), \qquad y = y(t), \qquad 0 \leqq t \leqq 1,$$

be an a-ergodic curve to M with length $\Lambda(a)$. We first observe that $\Lambda(a)>b-a$. If this were not true, an arbitrary point P of M could be joined to $(x(0), y(0))$ by a path of length not exceeding b by first joining P to C by a path of length not greater than a and then joining this point to the initial end-point of C by a path along C of length not greater than $b-a$. Thus, since $b<\rho$, the inequality $\Lambda(a) \leqq b-a$ is impossible.

Let t^* be the value of the parameter t such that the part C^* of C corresponding to $0 \leqq t \leqq t^*$ has length $b-a$. We shall now show that $C-C^*$, a non-degenerate, proper sub-curve of C given by $t^* \leqq t \leqq 1$, is b-ergodic to M. If P is a point of M within distance a of $C-C^*$, then P is within distance b of $C-C^*$ since $b>a$. If P is farther than a from $C-C^*$, then its distance to some point Q of C^* is not greater than a. Since Q is within distance $b-a$ of $(x(t^*), y(t^*))$, the distance of P from this end-point of $C-C^*$ is not more than b. Thus $\Lambda(b)$ is smaller than $\Lambda(a)$ by at least the positive number $b-a$.

3. **A further result.** We shall establish another inequality, as expressed in the following theorem.

THEOREM 2. *For all* $0<\epsilon \leqq \rho$ *we have*

$$\rho - \epsilon \leqq \Lambda(\epsilon) < \pi\rho\left(\frac{\rho}{\epsilon}+3\right).$$

Proof. The inequality on the left is an immediate consequence of the inequality in Theorem 1 and $\Lambda(\rho)=0$.

Let S denote the circle of radius ρ bounding M. For a given ϵ let n be the positive integer for which $n-1<\rho/\epsilon \leqq n$. We shall construct a continuous, rectifiable curve from circles concentric with S and held together by a radius of S. As our system of circles we take the $n-1$ circles concentric with S and having

radii $\rho - j\epsilon$, $j = 1, 2, \cdots, n-1$. Any point of the interior of S (hence any point of M) is within distance ϵ of a circle of the system. Clearly the number of circles could be decreased if $n > 3$. The sum of the lengths of circles of the system is

$$2\pi\left((n-1)\rho - \frac{n(n-1)}{2}\epsilon\right).$$

If we add a radius of S, we secure a continuous, rectifiable curve having length less than $\Lambda(\epsilon)$. In particular, we have

$$\Lambda(\epsilon) < 2\pi\left(n\rho - \frac{n(n-1)}{2}\epsilon\right).$$

Since $n \geqq \rho/\epsilon$ and $n < \rho/\epsilon + 1$ we have

$$\Lambda(\epsilon) < \pi\rho\left(\frac{\rho}{\epsilon} + 3\right).$$

4. Discussion. It would be of interest to know what the shortest ergodic curves are in case M consists of a finite number of points. In case M consists of only two points, the shortest ergodic curve, for a fixed ϵ, is a straight line segment. In case M consists of three points, the shortest ergodic curve is easily proved to be a straight line segment or two straight line segments with a common end-point. The two cases arise from considering the various possible positions of the three points (cf. Courant and Robbins, *What Is Mathematics*, Oxford, 1941, p. 336 et seq. for a lemma useful in determining the position of the curve). The question asked here could be phrased as follows: A paper boy is to throw papers at $n > 1$ given houses in a plane. If he can throw a paper ϵ feet and can have his papers delivered to him at any point in the plane, what route has a minimum length?

As is seen in the simplest case (when M consists of two points), $\Lambda = \Lambda(\epsilon)$, $0 < \epsilon \leqq \rho$, has a linear graph in the (ϵ, Λ)-plane. Professor Martin has conjectured (in a letter to the author) that for any point set M, if $\Lambda(\epsilon)$ is not linear, it is convex. That is, for any two values ϵ_1, ϵ_2 selected from the interval $0 < \epsilon \leqq \rho$ we have

$$\Lambda(\lambda\epsilon_1 + \mu\epsilon_2) \leqq \lambda\Lambda(\epsilon_1) + \mu\Lambda(\epsilon_2),$$

where $\lambda \geqq 0$, $\mu \geqq 0$ and $\lambda + \mu = 1$.

In addition to these questions one might ask when the (generally improper) Riemann integral of $\Lambda(\epsilon)$ exists on $0 \leqq \epsilon \leqq \rho$. What does the linearity of $\Lambda(\epsilon)$ on an interval of $0 < \epsilon \leqq \rho$ imply concerning the structure of M? When is $\Lambda(\epsilon)$ differentiable?

EUCLIDEAN n-SPACE ANALOGUE OF A THEOREM OF GALILEO

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1. **Introduction.** The purpose of this paper is to establish an n-space analogue of the following theorem due* to Galileo: *The area of a circle is the mean proportional between the areas of any two similar polygons, of which one is circumscribed about the circle, and the other is isoperimetric with the circle.*

2. **Theorem.** We shall establish the following result.

THEOREM: *Let V_1 denote the n-dimensional content of a sphere in Euclidean n-space, V the n-dimensional content of a circumscribed polytope, and v that of a similar polytope whose $(n-1)$-dimensional content is the same as that of the sphere. Then $Vv^{n-1}=V_1^n$.*

3. **Proof.** Let r denote the radius of the sphere, and S_1, S, and s respective $(n-1)$-dimensional contents, so that $s=S_1$ by hypothesis. Since the circumscribed polytope is composed of pyramids of altutide r, having a common apex at the center of the sphere, we have $V=Sr/n$, and similarly $V_1=S_1r/n$. Since the two polytopes are similar, it follows that

$$\left(\frac{V}{v}\right)^{1/n}=\left(\frac{S}{s}\right)^{1/n-1}.$$

Solving for v, we get $v=V(s/S)^{n/n-1}$. Therefore we have

$$Vv^{n-1}=\left(\frac{Sr}{n}\right)\left(\frac{Sr}{n}\right)^{n-1}\left(\frac{S_1}{S}\right)^n=\left(\frac{S_1r}{n}\right)^n=V_1^n.$$

4. **Conclusion.** Hence we have a direct generalization† of Galileo's theorem: *The volume of a sphere in Euclidean n-space is the geometric mean of the volume of a circumscribed polytope, and that of a similar polytope whose $(n-1)$-dimensional content is the same as that of the sphere, with weights 1 and $n-1$, respectively.*

A PROPERTY OF MINKOWSKIAN CIRCLES

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1. **Introduction.** Let a Euclidean circle C with center A be given. Move C rigidly to circle C_0 with center A_0 on C, and let A_1 be one of the intersection points of C and C_0. If now C_0 is moved to circle C_1 with center at A_1, then C and C_1 intersect at A_0 and at a second point A_2. Again, if C_1 is moved to C_2 with center at A_2, then C and C_2 intersect at A_2 and at a new point A_3. The fact that

* The writer found the above theorem entitled *Galileo's Theorem* as Ex. 135, pp. 377, in "Plane and Solid Geometry," 1899, by William J. Milne.

† The writer is greatly indebted to Professors E. F. Beckenbach and H. S. M. Coxeter for their aid in the preparation of this paper.

in this process the new circle always passes through a previous intersection point can be called the "re-entrant" property of a circle with respect to its center. The intersection sequence, of course, contains only six points, or, the circle "goes around itself six times."

2. **Theorem.** If the motions above are restricted to translations, the process can be carried out, with identical results, on Minkowskian circles. Or, with translations understood, the same thing can be stated for the Euclidean plane as follows.

THEOREM 1. *Every simple, closed, strictly convex curve which has a center is re-entrant with respect to that center and goes around itself six times.*

3. **Proof.** Let such a curve C with center A be given, and let A_0 be an arbitrary point on C. The line t determined by A and A_0 cuts C again at A_3. Take P a variable point on C and consider the chord which has P as an end-point and whose line is parallel to t. It is well known that the segment $\overline{A_0A_3}$ has the unique maximal length for all such chords, and that as P moves away from A_0 the corresponding chord length decreases monotonically until P reaches a position where its chord line is a supporting line of C and the chord length is zero. As P moves away from A_0, on one side of t, let A_1 be the first point at which the chord length is $A_0A_3/2$, and let the second end of this chord be A_2. Let the line of A_1 and A cut C in A_4, and the line of A_2 and A cut C in A_5. Join these points determined on C in the index order to form an inscribed hexagon. By the center property of C and the equality of vertical angles, it follows from congruent triangles that the opposite sides of this hexagon are equal in length and on parallel lines. Moreover, opposite sides have length half that of the chord determined by the remaining two vertices, and this third chord is parallel to the opposite sides. To show this, take T a translation which moves C to C' so that A falls, for example, on A_4, and let P be an intersection point of C and C'. Then P and $T^{-1}(P)$ determine a chord of C whose line is parallel to that of A_1 and A_4, and whose length is $A_1A_4/2$. Since C has only two such chords, namely $\overline{A_0A_5}$ and $\overline{A_2A_3}$, it follows from the position of the vertices that P is A_3 or A_5. If A is moved to any vertex of the hexagon, then, C in its new position passes through the adjacent vertices. Starting with A_0, which was arbitrary, then, C has the re-entrant property and the intersection sequence closes with A_5.

4. **Another result.** If a simple, closed, convex curve has the re-entrant property, with respect to an interior point A, and if P and Q are two consecutive points in an intersection sequence, it is easy to see that the chord through A, which is parallel to the line determined by P and Q, must be bisected by A. By the use of continuity this can be extended to give the following result.

THEOREM 2. *If a simple, closed, strictly convex curve C has the re-entrant property with respect to an interior point A, then A is the center of C.*

EQUATIONS OF COMPATIBILITY FOR FINITE DEFORMATION OF A CONTINUOUS MEDIUM*

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This note contains a concise derivation, in explicit form, of the integrability conditions for the case of finite deformations. These do not appear to be contained in the existing literature on the theory of finite deformations.**

If a continuous medium, occupying a region of three-dimensional Euclidean space, is subjected to a one-to-one continuous deformation such that the points of a line element of length ds_0 are displaced to the points of a line element of length ds, then the Eulerian strain tensor e_{ij} is defined by

$$(ds)^2 - (ds_0)^2 = 2e_{ij}dx^i dx^j, \qquad (i, j = 1, 2, 3), \tag{1}$$

where x^1, x^2, x^3 are the coordinates of a point in the deformed space. These coordinates are, in general, curvilinear. Using the same set of coordinates to designate points of the medium before and after deformation, we define the covariant metric tensors g_{ij} and h_{ij} of the deformed and undeformed spaces, respectively, by the fundamental quadratic forms:

$$\begin{cases} (ds)^2 = g_{ij}dx^i dx^j, \\ (ds_0)^2 = h_{ij}dx^i dx^j. \end{cases} \tag{2}$$

Inasmuch as the manifold in question is Euclidean, the Riemann tensor R_{ijkl}, based on either g_{ij} or h_{ij}, must vanish. If this condition is applied to the tensor h_{ij}, by using g_{ij} as the fundamental metric of the space in which the e_{ij} are defined, one obtains a set of six non-linear partial differential equations, called the *equations of compatibility.*

In the following derivation, covariant differentiation is performed with respect to the metric tensor g_{ij}, and a prime is used to distinguish Riemann tensors and Christoffel symbols based on h_{ij} from those based on g_{ij}.

Comparing (1) and (2) we get

$$h_{ij} = g_{ij} - 2e_{ij}, \tag{3}$$

so that

$$h_{ij,k} = \frac{\partial h_{ij}}{\partial x^k} - \left\{ \begin{matrix} \alpha \\ ik \end{matrix} \right\} h_{j\alpha} - \left\{ \begin{matrix} \alpha \\ jk \end{matrix} \right\} h_{i\alpha} = -2e_{ij,k}. \tag{4}$$

A calculation of Christoffel's symbols $[ij, k]'$ yields

* I am indebted for the suggestion of the method of deducing these equations to Professor I. S. Sokolnikoff.

** A paper by V. Z. Vlasov, entitled "The Equations of Continuity of Deformations in Curvilinear Coordinates," Journal of Applied Mathematics and Mechanics, vol. 8 (1944), pp. 301–306, utilizes ideas somewhat related to those of this note. However, in deriving his equations, Vlasov used variational techniques which restricted him to the consideration of infinitesimal deformations.

$$[ij, k]' = \left\{ \begin{matrix} \alpha \\ ij \end{matrix} \right\} h_{k\alpha} - e_{ijk}, \tag{5}$$

where e_{ijk} is defined by

$$e_{ijk} \equiv e_{ik,j} + e_{kj,i} - e_{ij,k}. \tag{6}$$

Since contravariant metric tensors g^{ij} and h^{ij} are

$$g^{ij} \equiv G^{ij}/g, \qquad h^{ij} \equiv H^{ij}/h, \tag{7}$$

where G^{ij} and H^{ij} are the cofactors of g_{ij} and h_{ij} in $|g_{ij}| = g$ and $|h_{ij}| = h$, respectively, it follows that

$$\left\{ \begin{matrix} \beta \\ ij \end{matrix} \right\}' = h^{k\beta}[ij, k]' = \left\{ \begin{matrix} \beta \\ ij \end{matrix} \right\} - h^{k\beta} e_{ijk}. \tag{8}$$

In order to obtain the Riemann tensor R'_{ijkl}, we compute

$$\begin{aligned} \frac{\partial}{\partial x^k}[jl, i]' &= \left(\frac{\partial}{\partial x^k} \left\{ \begin{matrix} \alpha \\ jl \end{matrix} \right\} \right) h_{\alpha i} + \left\{ \begin{matrix} \alpha \\ jl \end{matrix} \right\} \frac{\partial h_{\alpha i}}{\partial x^k} - \frac{\partial e_{jli}}{\partial x^k} \\ &= \left(\frac{\partial}{\partial x^k} \left\{ \begin{matrix} \beta \\ jl \end{matrix} \right\} + \left\{ \begin{matrix} \alpha \\ jl \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ \alpha k \end{matrix} \right\} \right) h_{\beta i} + \left\{ \begin{matrix} \alpha \\ jl \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ ik \end{matrix} \right\} h_{\alpha\beta} \\ &\quad + \left\{ \begin{matrix} \alpha \\ jl \end{matrix} \right\} h_{\alpha i,k} - e_{jli,k} - \left\{ \begin{matrix} \alpha \\ jk \end{matrix} \right\} e_{\alpha li} - \left\{ \begin{matrix} \alpha \\ lk \end{matrix} \right\} e_{j\alpha i} - \left\{ \begin{matrix} \alpha \\ ik \end{matrix} \right\} e_{jl\alpha}. \end{aligned}$$

Also, from (5) and (8) we have

$$\begin{aligned} \left\{ \begin{matrix} \alpha \\ jk \end{matrix} \right\}' [il, \alpha]' &= \left\{ \begin{matrix} \alpha \\ jk \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ il \end{matrix} \right\} h_{\alpha\beta} - \left\{ \begin{matrix} \alpha \\ il \end{matrix} \right\} e_{jk\alpha} \\ &\quad - \left\{ \begin{matrix} \alpha \\ jk \end{matrix} \right\} e_{il\alpha} + h^{\alpha\beta} e_{jk\beta} e_{il\alpha}, \end{aligned}$$

and from (4) and (6), we deduce that

$$h_{\alpha i,k} = -2e_{\alpha i,k} = -(e_{ik\alpha} + e_{\alpha ki}).$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial x^k}[jl, i]' + \left\{ \begin{matrix} \alpha \\ jk \end{matrix} \right\}' [il, \alpha]' &= \left(\frac{\partial}{\partial x^k} \left\{ \begin{matrix} \beta \\ jl \end{matrix} \right\} + \left\{ \begin{matrix} \alpha \\ jl \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ \alpha k \end{matrix} \right\} \right) h_{\beta i} \\ &\quad + A_{ijkl} - e_{jli,k} + h^{\alpha\beta} e_{jk\beta} e_{jl\alpha}, \end{aligned}$$

where

$$A_{ijkl} \equiv \left(\begin{Bmatrix} \alpha \\ jl \end{Bmatrix} \begin{Bmatrix} \beta \\ ik \end{Bmatrix} + \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} \begin{Bmatrix} \beta \\ jl \end{Bmatrix} \right) h_{\alpha\beta} + \begin{Bmatrix} \alpha \\ jl \end{Bmatrix} h_{\alpha i,k} + \begin{Bmatrix} \alpha \\ jk \end{Bmatrix} h_{\alpha i,l} - \begin{Bmatrix} \alpha \\ lk \end{Bmatrix} e_{j\alpha i} - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} e_{jl\alpha} - \begin{Bmatrix} \alpha \\ il \end{Bmatrix} e_{jk\alpha}.$$

Clearly, the system A_{ijkl} is symmetric with respect to the third and fourth indices. If we impose the condition that the Riemann tensor R'_{ijkl} is a zero tensor, we get

$$R'_{ijkl} = \frac{\partial}{\partial x^k}[jl, i]' - \frac{\partial}{\partial x^l}[jk, i]' + \begin{Bmatrix} \alpha \\ jk \end{Bmatrix}' [il, \alpha]' = \begin{Bmatrix} \alpha \\ jl \end{Bmatrix}' [ik, \alpha]' = 0;$$

and the use of the foregoing formulas yields

$$R'_{ijkl} = h_{\beta i} R^{\beta}_{jkl} + e_{ijkl} + h^{\alpha\beta}(e_{jk\beta}e_{il\alpha} - e_{jl\beta}e_{ik\alpha}),$$

since, by (6),

$$e_{jki,l} - e_{jli,k} = e_{ik,jl} + e_{jl,ik} - e_{li,jk} - e_{jk,li} \equiv e_{ijkl}.$$

However, we have $R^{\beta}_{jkl} = 0$, and hence the equations satisfied by the strain tensor are:

$$e_{ijkl} + h^{\alpha\beta}(e_{jk\beta}e_{il\alpha} - e_{jl\beta}e_{ik\alpha}) = 0.$$

If the second order terms are neglected, these expressions reduce to the usual form for the equations of compatibility,

$$e_{ijkl} \equiv e_{ik,jl} + e_{jl,ik} - e_{li,jk} - e_{jk,li} = 0,$$

familiar in the theory of infinitesimal deformations.

SOME ELEMENTARY IDENTITIES OF ORDERED ABELIAN SETS

A. L. FOSTER, University of California, Berkeley

1. Introduction. By slightly generalizing and abstracting the relations between exponents which occur in the representation of the endomorphisms of an Abelian group as a (pseudo) matrix ring, one finds as a by-product a number of elementary ordering identities which seem to be of some independent interest. It is the purpose of this note to call attention to several of these, which are faintly reminiscent of four point ordering theorems of projective geometry (Theorem 5).

2. Results. An ordered Abelian set $S=(S, <, +)$ is an ordered set $(S, <)$

$$(i) \qquad \left.\begin{array}{l} a < b,\ a = b, \text{ or } b < a \text{ (exactly one)} \\ a < b \quad \text{and} \quad b < c \qquad \text{imply } a < c \end{array}\right\} a, b, c \in S,$$

such that S is a subset of an Abelian group $(G, +)$,

$$(ii) \qquad S \subseteq G,$$

and where

(*iii*) $$a, b \in S \quad \text{and} \quad a \leqq b \qquad \text{imply } b - a \in S.$$

In an ordered Abelian set $(S, <, +)$, $(S, +)$ need not be closed with respect to $+$, nor need $<$ be defined throughout G. From (*iii*), however, one has

(2.1) $$0 \in S.$$

In the special case where $S = G$ (= "ordered Abelian group"), (*iii*) is evidently redundant; an ordered Abelian group is then simply an Abelian group $(G, +)$ whose elements have been ordered in any way (*i*).

For example, for given m, the set of all integers $0 \leqq x \leqq m$ is an ordered Abelian set with respect to ordinary $+$, and $<$ = natural order. Again the subset 0, 3, 6, 9 of the additive group of whole numbers is an ordered Abelian set if $<$ refer to the natural order, but not if $<$ is determined by

$$0 < 6 < 3 < 9.$$

Again, for example, the natural numbers 0, ± 3, 6, 9, 12, $\cdots$, ordered by

$$0 < -3 < 3 < 9 < 6 < 12 < 15 < 18 < \cdots,$$

form an ordered Abelian set as do also 0, ± 3, -6, -9, -12, $\cdots$, ordered by

$$0 < 3 < -3 < -6 < -9 < -12 < \cdots.$$

Among the elementary functions definable in an ordered Abelian set $S = (S, <, +)$, for all a, b, c of S, we consider

(2.2) $$|a, b| = \begin{cases} b - a & \text{if } a \leqq b, \\ a - b & \text{if } b \leqq a; \end{cases}$$

(2.3) $$/a, b/ = \begin{cases} b - a & \text{if } a \leqq b, \\ 0 & \text{if } b < a; \end{cases}$$

(2.4) $[a, b, c] = b'$, where $a' \leqq b' \leqq c'$, and $a', b', c' = a, b, c$, in some order.

It is seen that $[a, b, c]$ is uniquely defined by (2.4), that each of the three functions (2.2)–(2.4) has S as its range, and that (2.2) and (2.4) are symmetric in all arguments,

(2.5) $$\begin{aligned} |a, b| &= |b, a|, \\ [a, b, c] &= [a, c, b] = [b, a, c] = \textit{etc.} \end{aligned}$$

Relative to a given ordered Abelian set $(S, <, +)$ one may call $|a, b|$ the "absolute range" between a and b, $/a, b/$ the "forward range" from a to b, and $[a, b, c]$, the "middle" of the set a, b, c.

THEOREM 1. *In any ordered Abelian set we have*

(2.6) $$/a, b/ + /b, c/ - /a, c/ = |b, [b, c, a]|.$$

The proof of (2.6) is seen by direct verification for each of the possible relative orderings of a, b, c. If

$$
\begin{aligned}
&a \leqq b \leqq c \quad \text{both sides of (2.6) reduce to} \quad 0,\\
&a \leqq c \leqq b \quad \text{both sides of (2.6) reduce to} \quad b - c,\\
&b \leqq a \leqq c \quad \text{both sides of (2.6) reduce to} \quad a - b,\\
&b \leqq c \leqq a \quad \text{both sides of (2.6) reduce to} \quad c - b,\\
&c \leqq a \leqq b \quad \text{both sides of (2.6) reduce to} \quad b - a,\\
&c \leqq b \leqq a \quad \text{both sides of (2.6) reduce to} \quad 0.
\end{aligned} \tag{2.7}
$$

In view of the fact that S is in general not closed with respect to $+$ (or with respect to $-$), the following corollary is of interest:

COROLLARY. *In an ordered Abelian set, $/a, b/+/b, c/-/a, c/$ is an element of S for all a, b, c in S.*

As in Theorem 1, we have the following result:

THEOREM 2. *In any ordered Abelian set we have*

$$|x, y| + |y, z| - |x, z| = 2|y, [y, z, x]|. \tag{2.8}$$

(*Here $2t = t+t$*).

If one abbreviates

$$
\begin{aligned}
/xyz \cdots st/ &= /x, y/ + /y, z/ + \cdots + /s, t/\\
|xyz \cdots st| &= |x, y| + |y, z| + \cdots + |s, t|,
\end{aligned} \tag{2.9}
$$

by induction one obtains the following results:

THEOREM 3. *In any ordered Abelian set we have*

$$
\begin{aligned}
/xyz \cdots uvw/ = /x, w/ &+ |y, [x, y, z]| + |z, [z, t, x]|\\
&+ \cdots + |u, [u, v, x]| + |v, [v, w, x]|.
\end{aligned} \tag{2.10}
$$

THEOREM 4. *In any ordered Abelian set we have*

$$
\begin{aligned}
|xyz \cdots uvw| = |x, w| &+ 2|y, [y, z, x]| + 2|z, [z, t, x]|\\
&+ \cdots + 2|u, [u, v, x]| + 2|v, [v, w, x]|.
\end{aligned} \tag{2.11}
$$

Again by direct verification of the six possible arrangements of a, b, c one proves:

THEOREM 5. *In an ordered Abelian set we have*

$$|a, [a, b, c]| + |b, [b, c, d]| = |b, [b, a, d]| + |a, [a, d, c]|. \tag{2.12}$$

If the left of (2.12) is denoted by $\Pi(a, b; c, d)$ then (2.12) may be written

$$\Pi(a, b; c, d) = \Pi(b, a; d, c).$$

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College,

All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pa.

MATHEMATICS "AU NATUREL"

N. A. COURT, University of Oklahoma

In a recent article of this MONTHLY (Artificial Mathematics, vol. 57, No. 5, May, 1950, pp. 329–330) Professor F. H. Miller takes sharp issue with writers of textbooks, say, in differential equations or the calculus who present their solutions and proofs in an arbitrary, *deus ex machina* fashion, without letting the reader in on the secret which led to those solutions.

The transgressing authors are indicted on three counts: (1) Such a procedure on the part of the said authors tends to undermine the faith of college teachers of mathematics that the science they profess is based on logic and that the solutions they offer are arrived at in a rational way. (2) Those artificial ways have a discouraging effect upon the students who begin to doubt their ability to do mathematical thinking of their own if such "crystal ball methods" are an integral part of it. (3) Such unmotivated mathematical arguments are unnecessary and unjustified. They can readily be avoided if a little care and thought is devoted to the situation.

To take the last point first, there is very little doubt that the solutions objected to could be dressed up so that they would look "natural" and would appeal to our sense of "reasonableness." But if that neat appearance of theirs is meant to suggest that this was the path followed by their discoverers, it would be sadly misleading. Mathematical truths, solutions of mathematical problems, are seldom arrived at in such a matter-of-fact fashion. The proofs offered for them by their discoverers themselves have in most cases very little to do with the form in which these results were first conceived. The great masters of mathematical thought have been quite outspoken about this state of affairs (see Jacques Hadamard, *An Essay on the Psychology of Invention in the Mathematical Field*, Princeton University Press, 1945). Is being hit on the head by an apple falling off a tree a strictly logical approach to the invention of the theory of gravitation?

But there is no need to climb to such dizzy heights, or even to go as high as differential equations, to discover "artificial" elements in mathematical proofs. Almost any proof in plane geometry offers convincing evidence of it. Take the theorem of the sum of the angles of a triangle (Euclid, Book I, prop. 32). After a line has been drawn through the vertex of the triangle parallel to the opposite side, the proof is very simple and very "natural." But "where on earth does that parallel line come from?" Of all the things one might do, how does one know that the parallel line is the thing which will do the job?

We might as well tell our students that solving problems involves an element of inventiveness and that they will have to supply it if they are to be successful in their mathematical endeavors. Perhaps by the time they are ready to study differential equations it is not too early to shock them with this disturbing and challenging piece of news.

And while we are divulging secrets, we might perhaps let our students in on another one we have been guarding from them with zeal worthy of a better cause. In the problems we propose to our students we are very careful to supply all the data relevant to their solution, and we are just as careful to keep out anything that is irrelevant thereto. We might as well tell them that this is artificial, that it is done in order to make their work easier. In problems they may actually be confronted with that may not be the case. In fact, the most difficult part of their task may be precisely the separation of what is relevant from what is not. We might even go so far as to offer them problems where such a discrimination would have to be resorted to. Perhaps it would be advisable to start on this course much earlier. Suppose we amuse ourselves by asking a seventh grader to work a problem like the following.

One end of a string 4.5 ft. long is attached to an end of a diagonal of a rectangular white piece of paper 1/16 of an inch thick; the two sides of the rectangle passing through that point are 3 in. and 4 in. long. The other end of that same string is fixed to the point of intersection of the diagonals of a rectangular ceiling painted green, in a room that is 8 ft. high. Find the length of the diagonal of the paper and of the diagonal of the ceiling.

So much about the students. As to the college teachers of mathematics themselves, they can find in their subject so many solid virtues with which to bolster up their professional enthusiasm that they can, without much regret, dispense with one of questionable validity.

THE GENERAL SINE AND COSINE CURVES

E. L. EAGLE, University of Tennessee

In many textbooks on analytic geometry, and trigonometry, the equations of the general sine and cosine curves are given expression, canonically, in the form:

$$y = k \sin (bx + c)$$
$$y = k \cos (bx + c).$$

However, there is an advantage to writing such equations in the following form:

$$y = a \sin n(x - \alpha)$$
$$y = a \cos n(x - \alpha).$$

Then, a will be the amplitude and $2\pi/n$ the period (as usual), but the lag or lead will be indicated, definitely, by the point $(\alpha, 0)$. For example: Given

$y=2/3 \sin (3x+\pi/3)$, we convert to canonical form:

$$y = \frac{2}{3} \sin 3\left(x - \left[-\frac{\pi}{9}\right]\right).$$

Therefore, the amplitude is $2/3$ and the period is $2\pi/3$; and, to sketch the curve, we may conveniently start a cycle at $(-\pi/9, 0)$.

This approach removes any uncertainty, when the curve is out of phase, as to whether we have a lag or a lead.

It is gratifying to note that the taking of α from behind a minus sign is a procedure familiar to the student for he has already done the same thing in dealing with the circle, parabola, ellipse, and hyperbola.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 941. *Proposed by P. A. Piza, San Juan, Puerto Rico*

Find positive integers a, b, c such that $a^2+b^2+3c^2=(a+b+c)^2$.

E 942. *Proposed by L. A. Ringenberg, Eastern Illinois State College*

In the sequence of powers of 2: 2, 4, 8, 16, 32, $\cdots$, the units digits form a sequence of period four. What periodic properties do the other digits have?

E 943. *Proposed by S. H. Gould, Purdue University*

By analogy with the motion of the planets, it seems natural to assume for any central motion that, at least if the particle never passes through the center, (a) the distance of the particle from the center is a maximum only when the velocity of the particle is a minimum, (b) the velocity is a maximum only when the distance is a minimum. Prove (a) and give a counterexample of (b).

E 944. *Proposed by E. R. Bowersox, Chicago, Ill.*

A truncated cone with one end open is to be formed from a circular sheet, cutting out a central portion for the base, and using the remaining annular ring for the curved area. Solve for the truncated cone of maximum volume.

E 945. *Proposed by Leo Moser, University of North Carolina*

If all the faces of a convex polyhedron have central symmetry show that there are at least eight vertices where exactly three edges meet. (The cube has exactly eight such vertices.)

SOLUTIONS

Odd and Even Derangements

E 907 [1950, 184]. *Proposed by C. D. Olds, San Jose State College*

A derangement of n ordered letters is a permutation of the n letters which leaves no letter fixed. Find formulas which will give the number of odd and the number of even derangements of n ordered letters.

Solution by S. T. Thompson, Tacoma, Washington. Let O_n, E_n, and D_n denote the numbers of odd, even, and all derangements of the n letters. One easily finds $O_1=E_1=0$, $O_2=1$, $E_2=0$. We now determine a recursion formula for O_n, $n>2$. Starting with the original arrangement of the n letters, transpose the first and kth letters. There will then be E_{n-2} derangements of the remaining $n-2$ letters which lead to odd derangements of the entire set. Since k may be any one of the integers $2, 3, \cdots, n$, there are $(n-1)E_{n-2}$ odd derangements of this type. Again, starting with the original arrangement of the n letters, consider the first letter followed by any one of the E_{n-1} even derangements of the last $n-1$ letters. Now transpose the letters in the first and kth positions, and we obtain an odd derangement of all n letters. There are $(n-1)E_{n-1}$ odd derangements of this type. The two types of odd derangements considered are mutually exclusive and exhaustive. Therefore

$$O_n = (n-1)(E_{n-1} + E_{n-2}), \qquad n > 2.$$

Similarly

$$E_n = (n-1)(O_{n-1} + O_{n-2}), \qquad n > 2.$$

From these results one now easily establishes for all positive n, by mathematical induction, that

$$O_n - E_n = (-)^n(n-1). \tag{1}$$

But it is known that

$$O_n + E_n = D_n = n!\sum_{i=0}^{n}(-)^i/i!.$$

Therefore

$$O_n = \{D_n + (-)^n(n-1)\}/2,$$
$$E_n = \{D_n - (-)^n(n-1)\}/2.$$

Also solved by D. H. Browne, C. F. Pinzka, James Singer, G. W. Walker, and the proposer.

The Proposer obtained the relation (1) by considering the determinant $\Delta(n)$ of order n all of whose elements are equal to 1, except those of the main diagonal, which equal 0. The value of the determinant is $(-)^{n-1}(n-1)$. Now let j_1, j_2, $\cdots$, j_n be any permutation of the integers 1, 2, $\cdots$, n. Then we can expand $\Delta(n)$ in the form

$$\Delta(n) = \sum \pm a_{1j_1}a_{2j_2} \cdots a_{nj_n}.$$

Now any term in this sum which does not correspond to a derangement $j_1, j_2, \cdots, j_n$ of the n integers 1, 2, $\cdots$, n contains an element from the main diagonal and so is zero. The value of a term which does correspond to a derangement will be $+1$ or -1 according as the permutation is even or odd. Thus the value of $\Delta(n)$ is $E_n - O_n$, and we obtain relation (1).

A Property of 1/2

E 908 [1950, 184]. *Proposed by R. J. Koch, Tulane University*

Show that $r=1/2$ is the only rational value of r for which the geometric series $\sum ar^n$, $a \neq 0$, has the property that there exists a term equal to the sum of all terms from some point on, and that when $r=1/2$ this property holds for every term.

Solution by Walter Fleming, Mankato State Teachers College. The sum after m terms, for $-1<r<1$, is $ar^m/(1-r)$. Equating this to ar^n we have

$$r^m/(1-r) = r^n.$$

For $-1<r<0$ it is necessary that $m<n$ and we can write

$$r^{n-m+1} - r^{n-m} + 1 = 0.$$

This equation has no rational roots. For $0<r<1$ it is necessary that $m>n$ and we can write

$$r^{m-n} + r - 1 = 0.$$

This equation has no rational root except if $m-n=1$, in which case $r=1/2$ is a root.

It is clear that for $r=1/2$, $r^{n+1}/(1-r)=r^n$, for all n, which means that each term is equal to the sum of all the terms after it.

In the trivial case, $r=0$, the required property holds for every term except the first.

Also solved by R. E. Crane, Ragnar Dybvik, David Ellis, J. E. Freund, Roger Lessard, P. Na Nagara, C. S. Ogilvy, W. O. Pennell, Alex Rosenberg, Azriel Rosenfeld, C. M. Sandwick, O. D. Smith, Charles Solky, O. E. Stanaitis, Alfred Sylvester, C. W. Trigg, G. W. Walker, J. W. Young, and the proposer. Many of these solutions were incomplete.

Limit of a Determinant

E 910 [1950, 184]. *Proposed by R. M. Gordon, Carnegie Institute of Technology*

If $k>0, f_i(x)$ are independent, $f_i^{(n)}(x)$ exist and are well behaved, find

$$\lim_{h\to 0} h^{-k}\begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1(x+h) & f_2(x+h) & \cdots & f_n(x+h) \\ f_1(x+2h) & f_2(x+2h) & \cdots & f_n(x+2h) \\ \cdot & \cdot & \cdots & \cdot \\ f_1[x+(n-1)h] & f_2[x+(n-1)h] & \cdots & f_n[x+(n-1)h] \end{vmatrix}.$$

Solution by O. E. Stanaitis, St. Olaf College, Minn. By adding to each row multiples of the rows above it as suggested by the difference operator

$$\Delta^m f(x) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f[x+(m-j)h]$$

we may write the given expression in the form

$$\lim h^{(n-1)n/2-k}\begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ \dfrac{\Delta f_1(x)}{h} & \dfrac{\Delta f_2(x)}{h} & \cdots & \dfrac{\Delta f_n(x)}{h} \\ \dfrac{\Delta^2 f_1(x)}{h^2} & \dfrac{\Delta^2 f_2(x)}{h^2} & \cdots & \dfrac{\Delta^2 f_n(x)}{h^2} \\ \cdot & \cdot & \cdots & \cdot \\ \dfrac{\Delta^{n-1} f_1(x)}{h^{n-1}} & \dfrac{\Delta^{n-1} f_2(x)}{h^{n-1}} & \cdots & \dfrac{\Delta^{n-1} f_n(x)}{h^{n-1}} \end{vmatrix}.$$

The limit of the determinant is the well known Wronskian, which, according to the conditions under consideration, is finite and different from zero. It follows that the limit of the expression is the Wronskian for $k=(n-1)n/2$, equals zero for $k<(n-1)n/2$, and does not exist for $k>(n-1)n/2$.

Also solved by Kurt Eisemann, Roger Lessard, Norman Miller, and the proposer.

Miller stated that for $k=(n-1)n/2$, the theorem in question is a special case of a more general theorem in which $x, x+h, x+2h, \cdots, x+(n-1)h$ are replaced by $x_1, x_2, \cdots, x_n$ and in which the denominator h^k is replaced by a Vandermonde determinant. This more general theorem, again, arises from a generalized theorem of mean value of which the case $n=3$ is given in Goursat-Hedrick, *Mathematical Analysis*, vol. 1, p. 10. References to the more general theorems and proofs are as follows: H. A. Schwarz, *Collected Works*, vol. 2, p. 296; T. J. Stieltjes, *Collected Works*, p. 105; N. Miller, *Trans. of the Royal Soc. of Canada*, 1931, p. 195.

Miller also mentioned, as an allied problem, that of showing that the determinant

$$H = \begin{vmatrix} 1 & x & x^2 & \cdots & x^n \\ 1 & x+h & (x+h)^2 & \cdots & (x+h)^n \\ 1 & x+2h & (x+2h)^2 & \cdots & (x+2h)^n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & x+nh & (x+nh)^2 & \cdots & (x+nh)^n \end{vmatrix}$$

is independent of x. This may be done by showing that $dH/dx \equiv 0$. The value of the determinant, found by taking $x=0$, is

$$1!\,2!\,3!\cdots n!\,h^{n(n+1)/2}.$$

The special case, $k=3$, $n=3$, of the given problem, is found in Widder, *Advanced Calculus*, ex. 13, p. 220.

Professor Umbugio's Expedition

E 911 [1950, 259]. *Proposed by G. B. Huff, University of Georgia*

The following item appeared in the April 1, 1913, issue of the Erewhon Daily Howler: "Science lovers of Guayazuela will be disappointed to learn that the antarctic expedition of Professor Euclide Paracelso Bombasto Umbugio, which was planned to start the first of next month and last through November, has been temporarily postponed. In making his characteristically thorough preparations, the eminent numerologist discovered that June 13 of this year falls on Friday and has decided that he will postpone the trip until a year when no Friday the thirteenth occurs in the seven months of the expedition. The Professor, an authority on the doctrine of probability, assured the press that the delay will be of short duration. He explained that the probability of an auspicious month is 6/7 and that hence the probability of seven consecutive auspicious months is $(6/7)^7$ or a little more than 2/5. From this the genius infers that the probability of postponing the trip next year is less than 3/5 and the probability that it will be postponed for nine consecutive years is $(3/5)^9$ or less than 1/100."

Of course, we now know that the Professor has never made his trip, though four times nine years have passed. Is the Professor having a tremendous run of bad luck?

Solution by C. F. Pinzka, Xavier University, Cincinnati. Beginning with May 13 of any year, the days elapsing until May 13, June 13, $\cdots$, November 13 are congruent modulo 7 to 0, 3, 5, 1, 4, 6, 2, respectively. Since these numbers form a complete residue system, one of these dates must fall on a Friday.

Or, a glance at any calendar will show that the 13th day of May to November, inclusive, is always on a different day of the week, whence one of these dates must always fall on a Friday.

Also solved by Martin Berman, A. D. Bradley, G. M. Dillon, Ragnar Dybvik, Emil Grosswald, E. S. Keeping, J. R. F. Kent, Roger Lessard, Marie A. Moore, C. S. Ogilvy, Bart Park, Azriel Rosenfeld, R. M. Simons, G. W. Walker, A. Wilansky, and R. V. Andree's freshman mathematics class at the University of Oklahoma.

The Professor's probability argument falls down, of course, because calendar months are not independent of each other. It is interesting that the thirteenth of a month is more likely to be Friday than any other day of the week. See E 36 [1933, 607].

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4415. *Proposed by R. P. Boas, Jr., and W. K. Hayman, Brown University*

Find all the values of α and β for which the series

$$\sum_{n=1}^{\infty} n^{\alpha} \sin (n^{\beta})$$

converges.

4416. *Proposed by D. J. Newman, New York University*

Let $a_1, a_2, a_3, \cdots$ be a sequence of integers such that $\sum_{n=1}^{\infty} 1/a_n$ diverges. Show that almost all integers have a factor in common with some a_n.

4417. *Proposed by Victor Thébault, Tennie, Sarthe, France*

(1) The necessary and sufficient condition that an orthocentric tetrahedron $T \equiv ABCD$ can be bordered by isosceles tetrahedrons $T_a \equiv A_1BCD$, $T_b \equiv B_1CDA$, $T_c \equiv C_1DAB$, $T_d \equiv D_1ABC$ is that all the face angles of T be acute. (2) The tetrahedrons T and $T_1 \equiv A_1B_1C_1D_1$ are orthologic. (3) The lines AA_1, BB_1, CC_1, DD_1 are four hyperbolic lines. (4) The sum of the squares of the areas of the faces of T_a, T_b, T_c, T_d is equal to the sum of the squares of the products of the opposite edges of T.

4418. *Proposed by A. C. Aitken, Edinburgh, Scotland*

Evaluate:

$$\begin{vmatrix} \frac{1}{1!} & -1 & \cdots & \cdots & \cdots \\ \frac{2^2}{2!} & \frac{1}{1!} & -2 & \cdots & \cdots \\ \frac{3^3}{3!} & \frac{2^2}{2!} & \frac{1}{1!} & -3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{n^n}{n!} & \frac{(n-1)^{n-1}}{(n-1)!} & \frac{(n-2)^{n-2}}{(n-2)!} & \cdots & \frac{1}{1!} \end{vmatrix}.$$

4419. *Proposed by L. G. Jones, University of Oregon, and Alex Rosenberg, University of Toronto.*

In a paper, Remarks on Divisors of Zero, this MONTHLY [1942, 286–295], N. H. McCoy stated and proved the following theorem: Let $f(x)$ be an element of $R[x]$, where R is a commutative ring. Then $f(x)$ is a divisor of zero in $R[x]$ if and only if there exists a nonzero element c of R such that $cf(x)=0$.

Show that, if R is not assumed to be commutative, the theorem no longer holds.

SOLUTIONS

Area Having Lowest Centroid

4336 [1949, 186]. *Proposed by Orrin Frink, Pennsylvania State College*

Find the arc of fixed length $l>2a$, lying below the x-axis and joining the points $(-a, 0)$ and $(a, 0)$, which includes between itself and the x-axis an area of lowest possible center of gravity. This will be the form actually assumed by a weightless flexible arc supported at its ends if it is holding water.

Solution by Frank Herlihy, Comstock, Michigan. Let:

R = tension in the cable, which will be constant throughout its length since it is itself massless and the liquid cannot exert any shear on it;
ρ = radius of curvature of the cable;
σ = mass per unit area of the liquid;
g = gravitational constant;
and let y be measured downward from the x-axis.

Then the pressure at depth y below the axis is σgy. If the cable is in equilibrium the pressure σgy and the radius ρ at the point at distance y below the x-axis will determine $R=-\sigma gy\rho$ or

$$\sigma g y \frac{(1 + |\, dy/dx\,|^2)^{3/2}}{d^2y/dx^2} = -R.$$

With $R/\sigma g = a^2/4$, an integration gives

$$y^2 = \tfrac{1}{2}a^2(1 + |\, dy/dx\,|^2)^{-1/2} + c = h^2 - a^2 \sin^2 \tfrac{1}{2}\theta,$$

where $\theta = \arctan(dy/dx)$, and $h = (c + \frac{1}{2}a^2)^{1/2}$ is the maximum value of y.

In order that y shall vanish at the supports, $a^2 > h^2$. Let $h^2 = k^2a^2$ where $k^2 < 1$, and let $k \sin\phi = -\sin\frac{1}{2}\theta$. Then $y = h\cos\phi$ and since

$$\sin\theta = dy/ds = -h(\sin\phi)(d\phi/ds)$$

we shall have

$$\frac{ds}{d\phi} = -h\frac{\sin\phi}{\sin\theta} = \frac{a}{2}\frac{1}{\sqrt{1 - k^2\sin^2\phi}}.$$

Thus

$$s = \frac{a}{2}\int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} = \tfrac{1}{2}aF(k, \phi).$$

Similarly

$$\frac{dx}{d\phi} = -h\frac{\sin\phi}{\sin\theta}\cos\theta = \frac{a}{2}\frac{1 - 2k^2\sin^2\phi}{\sqrt{1 - k^2\sin^2\phi}},$$

$$x = a\int_0^{\phi}\sqrt{1 - k^2\sin^2\phi}\,d\phi - \frac{a}{2}\int_0^{\phi}\frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}$$

$$= aE(k, \phi) - \tfrac{1}{2}aF(k, \phi).$$

If α is the half span of the problem, then

$$\alpha = aE(k, \tfrac{1}{2}\pi) - \tfrac{1}{2}aF(k, \tfrac{1}{2}\pi)$$

since $y = h\cos\phi$ implies $\phi = \frac{1}{2}\pi$ when $y = 0$. Also $\frac{1}{2}l = \frac{1}{2}aF(k, \frac{1}{2}\pi)$.

If k and a are fixed, the curve is given in the parameter ϕ by

$$y = h\cos\phi, \qquad x = aE(k, \phi) - \tfrac{1}{2}aF(k, \phi).$$

This solution is merely Euler's solution of the Elastica, identical except for the values of k and a. See MacMillan, *Statics and Dynamics of a Particle*, pp. 196–200, where also curves are drawn for various values of k (including both the case where there are inflections, which is the present problem, and the non-inflectional case where $k^2 \geqq 1$).

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

A Short Course in Differential Equations. By E. D. Rainville. New York, The Macmillan Company, 1949. 9+210 pages.

Upon reflection on the objective of the book which is being reviewed, one concludes that it was designed as a text for students who need a more complete set of recipes for the solution of the commonly occuring ordinary differential equations than he would find in the usual calculus book. The student reader, however, is not expected to have reached the stage where the term "theorem" belongs in a mathematics text. When a result is stated for the reader, he frequently finds a footnote indicating other hypotheses or the location of a precise statement and proof.

The standard material of a first differential equations course is included with the exception that neither power nor Fourier series solutions are included nor is any treatment given of systems of equations. The highlight of the presentation is the inclusion of a large number of examples in each section, so that the student may see in rather complete detail the procedure suitable for the particular equation being treated. However, one may question the advisability of the inclusion of several examples with no variation in technique whatsoever in some sections.

The book is well supplied with exercises and answers are included for virtually all the problems. The space devoted to applications is given to a more detailed study of the problems of the vibrating string and the electric circuit rather than a more superficial treatment of numerous applications.

PAUL CIVIN

An Introduction to College Geometry. By E. H. Taylor and G. C. Bartoo. New York, The Macmillan Company, 1949. 6+143 pages. 112 figures. $3.15.

The book is a revision of one which appeared in 1939, and of which the reviewer could find no mention in this MONTHLY. It is designed for prospective teachers of secondary mathematics who have had no synthetic geometry since high school; also to stimulate students interested in geometry to study the discoveries made in this subject since Euclid.

No knowledge of mathematics is presupposed save that from the usual course of plane geometry in high school; no attempt is made to stress the logical foundations of the subject, nor any use made of imaginary elements in algebra. However, on page 108, the cross ratio of a pencil of lines is defined in terms of the sines of the angles. Historical notes are scattered throughout the text wherever appropriate.

The first four chapters deal with directed lines and angles, points at infinity, similar figures and loci. Chapter five is devoted to the properties of the triangle,

but nothing more modern than the Theorems of Ceva, Pascal and Desargues, the nine-point circle and the Simson line. The remaining chapters are concerned with harmonic ranges and pencils, pole and polar as to a circle, orthogonal and coaxal circles, cross ratio and the principle of duality.

Numerous exercises of various degrees of difficulty appear in each chapter. The appendix lists 108 theorems from elementary plane geometry for ready reference in the text and 22 constructions the reader should know. Here may be found 70 additional exercises and 50 construction problems if the student desires supplementary work.

Within the scope of the book it is well written and should serve as a good introduction to a second course in the modern geometry of the triangle and circle. The press-work and the proof-reading seem excellent. One slip occurs on page 59 where Menelaus is credited as living around 100 B.C. instead of 100 A.D.

J. R. MUSSELMAN

NEW BOOKS RECEIVED

From Euclid to Eddington, the Tarner Lectures, 1947. By Sir E. Whittaker. Cambridge University Press, 1949. 10+212 pp.

Analysis and Design of Experiments. By H. B. Mann. New York, Dover, 1949. 10+198 pp. $2.95.

La Geometrie Contemporaine. ("Que sais-je?," no. 401.) By A. Delachet. Paris, Presses Universitaires de France, 1950. 128 pp.

An Index of Nomograms. By D. P. Adams. New York, Wiley, 1950. 10+174 pp. $4.00.

College Algebra. By H. A. Bender. New York, Pitman, 1950. 12+451 pp. $3.50.

Leçons sur la Théorie des Fonctions. Fourth Edition. By E. Borel. Paris, Gauthier-Villars, 1950. 12+295 pp.

Experimental Designs. By W. Cochran and G. Cox. New York, Wiley, 1950. 10+454 pp. $5.75.

Monografia sobre los Números. By F. J. Duarte. Caracas, Tipografia Americana, 1949. 246 pp.

Premiers principes de géométrie moderne. Third Edition. By E. Duporq. Paris, Gauthier-Villars, 1949. 173 pp.

Plane Trigonometry. By Gordon Fuller. New York, McGraw-Hill, 1950. 13+270 pp. $2.75.

Éléments de Physique Moderne Théorique. By Georges Guinier. Paris, Bordas, 1949. 159 pp.

Gelöste und Ungelöste Mathematische Probleme aus Alter und Neuer Zeit. By Heinrich Tietze. Biederstein Publishers, Munich, 1949. Volume I: xx+256 pp.; Volume II: iv+305 pp. Paper Bound 18 DM, Linen Bound 25 DM (German Marks).

Science is a Sacred Cow. By Anthony Standen. New York, E. P. Dutton and Company, 1950. 221 pp. $2.75.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1949–50

Newtonian Society, Lehigh University

The *Newtonian Society* of Lehigh University is composed of sophomores and juniors who have demonstrated outstanding ability in mathematics during their freshman year.

During the academic year 1949–50, the society devoted several meetings to reorganization. The following informal talks were presented at the remaining meetings:

Some applications of number theory, by Prof. G. W. Petrie III

Stellar astronomy, by Prof. R. N. Van Arnam

Numerical analysis, by Prof. V. W. Latshaw

Homogeneous functions and homogeneous differential equations, by Prof. J. W. Mettler.

In February the constitution was changed to provide for the election of officers in February instead of in September. It was also decided to obtain certificates of membership for all members.

The present officers are: President, J. Burk McNamara; Vice-President and Treasurer, Frederick Kieshauer; Secretary, Alan Koppes; Faculty Advisors, Prof. J. O. Chellevold, Prof. G. W. Petrie III (fall semester).

Kappa Mu Epsilon, Hofstra College

The *New York Alpha* Chapter of *Kappa Mu Epsilon* held regular monthly meetings at which the following talks were presented:

Mathematics in the schools of England, by Geoffrey Charlesworth

The four-color problem, by Prof. L. F. Ollmann

Verb tense determined symbolically, by William Beller

Complex roots of polynomials determined graphically, by Prof. H. F. Fehr of Columbia Teachers College

Non-Euclidean geometry, by Perry Watts

Lissajous's figures, by Otto Juelich

Biometry, by Jane Brandt.

In addition to the regular meetings, the chapter held a Christmas party and square dance, a picnic and an initiation banquet at which Prof. J. E. Eaton of Queens College spoke on *The nature of applied mathematics.*

John Richard Prussen was the winner of the award offered to the best student of first year mathematics.

The chapter was represented by five faculty and five student members at the *Kappa Mu Epsilon* luncheon held in New York City at the time of the Christ-

mas meeting of the M.A.A.

The national officers of *Kappa Mu Epsilon* met on the Hofstra campus during the Christmas holidays.

During the past semester, the chapter sponsored student help sections. *Kappa Mu Epsilon* members gave their time to help those who came at the assigned periods.

Prof. L. F. Ollmann, chairman of the department, and national treasurer of *Kappa Mu Epsilon*, installed the *Pennsylvania Alpha* Chapter of *Kappa Mu Epsilon* at Westminster College, New Wilmington, Pennsylvania on May 17th.

The officers for 1950–51 are: President, Peter Hinrichs; Vice-President, Richard Jaeger; Secretary, Emily Schuppel; Treasurer, Peter Marshall; Historian, Jane Brandt; Corresponding Secretary, Lysle Marshall; Faculty Sponsor E. Marie Hove.

Mathematics Club, Case Institute of Technology

The activities of the *Mathematics Club* of the Case Institute of Technology for the year 1949–50 were entirely those of lectures by both students and members of the faculty. These lectures were:

The method of least squares, by Dr. F. C. Leone

The problem of trisecting an angle, by Irwin Wieder

The Königsberg bridge problem, by Dr. C. Saltzer

Conformal mapping, by George Millburn.

The new officers are: President, James T. Morse; Vice-President, Ernst Prasse.

Pi Mu Epsilon, Bucknell University

The papers presented to the *Pennsylvania Beta* Chapter of *Pi Mu Epsilon* for the academic year 1949–50 were:

Binomial theorem, by James Bortner

Group theory, by Donald Ohl

Practical equation solution, by Edwin Irland

Applications of the finite calculus, by Evan Boden.

Other activities included a business meeting, a social, and a picnic. Average attendance at each meeting was 30.

The officers for 1950–51 are: Director, John W. Sprout; Vice-Director, F. R. Stauffer; Secretary, Charlotte Stratton; Treasurer, R. E. Jones, Jr.

Kappa Mu Epsilon, Alabama College

The *Alabama Gamma* Chapter held regular meetings from September through May. In October a party was given for all students in the college taking mathematics. In March nine new members were initiated and a banquet followed the initiation ceremony.

The following papers were presented at other meetings:

Number is everything, by Joyce Moncrief

The number system, by Annie Falls.

Officers elected for the year 1950–51 are: President, Joyce Moncrief; Vice-President, Betty Ellis; Secretary, Gay Penn; Treasurer, Annie Falls; Program Chairman, Betty Crow; Faculty Sponsor, Mamie Braswell; Corresponding Secretary, Rosa Lea Jackson.

Kappa Mu Epsilon, Central College (Fayette, Missouri)

The *Missouri Epsilon* Chapter of *Kappa Mu Epsilon* at Central College enjoyed the following papers during 1949–50:

Newton and Leibniz, by Paul Calvert

Hamilton and quaternions, by Mark Barton

The kinetic theory of gasses, by Merle Cartwright

An introduction to modern geometry, by Dr. Floyd Helton

Cantor's theory of infinity, by Niels Nielson

A brief history of the development of the engineering profession, by Prof. C. Denny

History of the number system, by David Bouldin

Gauss, Prince of Mathematics, by Norman Drissell

Galileo, the Star Gazer, by Eva Gilbertson

The duodecimal system of counting, by J. P. Karnes

Methods of summability of infinite series, by Dr. Paul Burcham, Chmn. of Department of Mathematics, University of Missouri

Evariste Galois, by Mary J. Groce

Calculating prodigies, by Robert Christian

Peaucellier's cell, by Mary Cronbaugh.

At the final meeting of the year, a picnic dinner for members and guests was held at the home of the Sponsor, Dr. Floyd Helton.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

THE ELEVENTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The eleventh annual William Lowell Putnam Mathematical Competition will be held on Saturday, March 31, 1951. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to undergraduate students in universities and colleges of the United States and Canada who have not received

a degree. The examination will consist of two parts of three hours each. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from Professor L. E. Bush 112 Albertus Magnus Hall, College of St. Thomas, St. Paul 1, Minnesota, by a postcard request. All applications must be filed with the Director not later than March 1, 1951. If three candidates are presented from a college or university, they are to constitute a team; if more than three are presented from any one college or university, the team of three must be named on the application. Fewer than three from one college or university may compete as individuals.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to this rule may be made by the Director in cases where it would entail unusual inconvenience to a contestant. Sealed copies of the examinations will be sent to the supervisor of the examination in time for the examination day and are not to be opened before the hour set.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are $400, $300, $200 and $100, in the order of their rank. In addition, there will be prizes of $40, $30, $20 and $10 awarded to the members of these teams according to the rank of the team; a prize of $50 to each of the five highest contestants and a prize of $20 to each of the succeeding five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four winning teams and to several individuals next in order after the ten individual winners. For further encouragement of the Competition, there will be awarded at Harvard University (or at Radcliffe College in the case of a woman) an annual $1800 William Lowell Putnam Prize Scholarship to one of the first five contestants, this to be available either immediately or on the completion of the student's undergraduate work.

Reports on the ten previous competitions and examination questions will be found in this MONTHLY for May, 1938, 1939, 1940, 1941, 1942, October, 1946, August-September, 1947, December, 1948, and August-September, 1949, 1950.

COMMITTEE ON THE MATHEMATICAL TRAINING OF SOCIAL SCIENTISTS

At the summer meeting of the Association at the University of Colorado on August 29–30, 1949 there was held a Symposium on the Mathematical Training of Social Scientists (this MONTHLY, vol. 56, 1949, p. 658). The Symposium was sponsored by the Mathematical Association of America, the Institute of Mathematical Statistics and the Econometric Society. A summary of the statements made by the participants in the Symposium has been printed in Econometrica, vol. 18, 1950, pp. 193–205.

At the conclusion of the Symposium it was voted that the officers of the participating societies be requested to appoint a joint committee to study the need for better mathematical training of social scientists and the ways and means of improving their mathematical preparation.

In accordance with this resolution a Committee on the Mathematical Training of Social Scientists has been established under the chairmanship of Professor W. G. Madow of the University of Illinois. Twelve organizations interested in the problem are represented on the committee. The representatives of the Mathematical Association of America are Professors F. L. Griffin and E. P. Northrop. Members interested in the work of this committee are urged to communicate with our representatives on the committee.

MATHEMATICAL COLLOQUIUM AT THE UNIVERSITY OF ST. ANDREWS

A Mathematical Colloquium under the auspices of the Edinburgh Mathematical Society will be held at the University of St. Andrews on July 18–28, 1951. The Colloquium will consist of three or four short courses of lectures on topics of general interest, supplemented by several individual lectures on more specialized subjects. Professor H. S. M. Coxeter of the University of Toronto and Professor J. L. Synge of Dublin Institute for Advanced Studies will give courses. Other courses are being arranged. As accommodation in the University residence halls may be limited, application for membership should be made as soon as possible to the Colloquium Secretary, Dr. D. E. Rutherford, United College, St. Andrews, Scotland. Further information can be obtained from the Secretary.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held at the University of Florida, Gainesville, Florida, on Friday and Saturday, April 7–8, 1950. Professor F. A. Lewis, Chairman of the Section, presided at the Friday afternoon and Saturday morning meetings of the subsections, which were presided over by Professors C. G. Phipps, George Sauté, R. J. Levit, T. M. Simpson, F. W. Kokomoor, C. L. Seebeck, Jr., L. A. Dye, and W. A. Gager. Professor C. G. Phipps presided on Friday evening at the dinner.

There were about two hundred and fifty present, including the following one hundred and eleven members of the Association: W. H. Badgley, Jr., W. S. Beckwith, R. G. Blake, F. C. Bolser, Floyd Bowling, A. T. Brauer, N. R.

Bryan, R. K. Butz, T. C. Carson, J. D. Charron, C. E. Clark, M. J. Cleveland, Reginal Cobb, A. C. Cohen, Jr., J. B. Coleman, R. W. Cowan, R. R. Croxton, J. C. Currie, R. D. Depew, N. E. Dodson, R. D. Doner, L. A. Dye, J. C. Eaves, G. B. Findley, R. B. Folsom, Tomlinson Fort, W. A. Gager, D. B. Goodner, S. T. Gormsen, E. H. Hadlock, B. F. Hadnot, G. B. Huff, W. R. Hutcherson, Rosa L. Jackson, A. J. Killebrew, C. E. Kirkwood, Jr., F. W. Kokomoor, J. W. LaGrone, G. B. Lang, J. W. Lasley, Jr., R. J. Levit, F. A. Lewis, M. E. McCarty, W. G. McGavock, S. W. McInnis, J. D. Mancil, Nathaniel Macon, C. F. Martin, J. E. Martin, R. H. Mason, W. N. Mebane, Jr., H. A. Meyer, Herman Meyer, E. P. Miles, Jr., W. L. Miser, J. A. Morales, J. C. Morelock, H. W. Morrow, Jr., K. W. Morrow, W. B. Moye, T. F. Mulcrone, T. A. Newton, A. J. Owens, W. V. Parker, P. B. Patterson, W. D. Peeples, Jr., Lillian G. Perkins, P. B. Peyton, Jr., C. G. Phipps, Z. M. Pirenian, P. A. Piza, M. D. Prince, F. M. Rankin, Ellen F. Rasor, B. P. Reinsch, G. E. Reves, T. L. Reynolds, M. C. Rhodes, J. M. Robertson, A. J. Robinson, H. A. Robinson, L. V. Robinson, George Sauté, P. C. Scott, C. L. Seebeck, Jr., E. B. Shanks, D. C. Sheldon, T. M. Simpson, C. B. Smith, W. R. Smythe, Jr., W. S. Snyder, W. H. Spragens, Jr., F. W. Stallard, E. L. Stanley, P. M. Swingle, J. M. Thomas, S. L. Thompson, H. S. Thurston, E. A. Voorhees, Jr., B. T. Wade, T. L. Wade, Jr., D. T. Walker, F. A. Wallace, Betty R. Weber, W. W. Weber, Ernest Williams, W. H. Wilson, Herbert Wolf, G. N. Wollan, J. W. Young, R. W. Young.

At the business session the following officers were elected: Chairman, C. G. Phipps, University of Florida; Vice-Chairman, B. G. Clark, Vanderbilt University; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. The Section voted to hold its next meeting on March 16–17, 1951, at Vanderbilt University and George Peabody College.

The program consisted of the following papers:

1. *A note on problem number 4288 in the American Mathematical Monthly, January 1950*, by Professor C. G. Phipps, University of Florida.

This problem was proposed in the March 1948 issue of this MONTHLY. Two purported solutions were published in the January 1950 issue. These solutions appear to contain several misinterpretations. The present note explained the derivation of the probabilities of the events to which the probability 13/41 applies. The analysis was based on a corrected diagram.

2. *Partial fractions*, by Professor J. M. Thomas, Duke University.

It was shown that synthetic division can be used to develop the theory completely and to calculate the coefficients systematically.

3. *Simultaneous linear equations*, by Professor L. A. Dye, The Citadel.

This paper presents an orderly, economical method of solving any system of simultaneous linear equations possessing a solution. A check on the computations is provided at each step and inconsistent equations are easily recognized as such.

4. *A criterion for a common root of k algebraic equations*, by Professor A. T. Brauer, University of North Carolina.

This paper has appeared in this MONTHLY for May, 1950.

5. *Graph paper as a teaching aid*, by Professor Floyd Bowling, Lincoln Memorial University.

The author demonstrated two types of graph paper which he designed, one for trigonometric curves, and the other for three dimensions. The latter is useful in solid geometry, solid analytics and some phases of engineering drawing.

6. *The linear congruence group modulo n*, by Professor F. A. Lewis, University of Alabama.

The speaker discussed the results of a study of $GLH(m, n)$, the group of linear homogeneous transformations on m variables whose coefficients range over the integers $0, 1, \cdots, n-1$ in such a way that the determinant of each transformation is relatively prime to n. Properties of $GLH(m, n)$ were developed by the general linear homogeneous group on m variables with coefficients in the $GF[p^n]$.

7. *The brachistochrone problem*, by W. R. Smythe, Jr., Rollins College.

In the first part of this paper, the path of minimum time of descent between two fixed points was discussed. In the second part, the path from a point to a curve, including Barnett's construction for the focal point, was given.

8. *Mathematics in art and nature*, by Sarah I. Hancock, Agnes Scott College, introduced by the Secretary.

In this paper, the author discussed certain curves, numbers, and geometric forms which appear in art and nature.

9. *Notes on certain unsolved problems*, by E. A. Voorhees, Jr., Vanderbilt University.

In this paper the explanation, progress, and by-products of the investigation of certain of the more elementary unsolved problems, including some of the so-called "Hilbert problems," were discussed.

10. *A prime-representing function*, by D. T. Walker, University of Georgia.

A theorem given by W. H. Mills in June 1947 *Bulletin of the American Mathematical Society* was discussed and extended. Between the cube of an integer and that of the next higher integer less one there is always at least one prime. From this a bounded monotone increasing sequence of real numbers was constructed.

11. *Some theorems in matrix algebra*, by Professor T. L. Wade, Florida State University.

With the use of the delta and epsilon permutation symbols (numerical tensors) and identities on them, explicit proofs are given of the theorem "if A is a matrix with n columns and B is a matrix with n rows, then any m-rowed determinant of the matrix $C=AB$ is equal to the sum of terms, each of which is the product of an m-rowed determinant of A and an m-rowed determinant of B," and related theorems.

12. *A generalization of the field of values of a matrix*, by Professor J. C. Currie, Georgia Institute of Technology.

For a given matrix A, and a positive definite Hermitian matrix S, the set of complex numbers $xAS\bar{x}'$, where x ranges over the set of vectors such that $x\,S\,\bar{x}'=1$, is called the S-field of values

of A. The nature of this S-field, its relation to the ordinary field of values (for $S=\mathrm{I}$), and some results of its application to the problem of limits for the characteristic roots of A were discussed.

13. *A method of construction of a properly primitive positive ternary quadratic form*, by Professor E. H. Hadlock, University of Florida.

The author gave an example illustrating a method of construction of a properly primitive positive ternary quadratic form when the invariants r and delta and the associated progression were known.

14. *A new interpolation formula*, by Professor C. L. Seebeck, Jr., University of Alabama.

The formula involves m derivatives evaluated at $n+1$ points, and reduces to Lagrange's formula when $m=0$. It is

$$f(x) = \sum_{k=0}^{n} \Big[\prod_{k \neq i} (x - x_i) S_k \Big/ \prod_{k \neq i} (x_k - x_i) \Big] + R,$$

$$S_k = \sum_{j=0}^{m} \frac{(m+n-j)!}{(m+n)!} {}_mC_j f^{[j]}(x_k).$$

15. *An algebraic geometry formula*, by J. C. Morelock, University of Florida.

The formula referred to may be used for writing the terms of any homogeneous polynomial of prime degree in four variables. The method permits a classification of the terms of the polynomial into sets where each set is invariant under the same special linear transformation. The method also affords a new way to count the number of terms in the most general polynomial of prime degree.

16. *Geometric aspects of the euclidean algorithm in imaginary quadratic domains*, by Professor H. S. Thurston, University of Alabama.

Professor Thurston demonstrated with diagrams the existence or non-existence of the euclidean algorithm in quadratic domains $R_a[\sqrt{m}]$, where $m<0$.

17. *Plane analytic geometry of the complex domain*, by B. F. Hadnot, University of Georgia.

By use of a one to one correspondence between the field of complex numbers and points oriented by the rectangular coordinate system, Mr. Hadnot outlined a study of straight lines and conics with as little reference as possible to the plane analytics of the real domain.

18. *Effect of a rigid elliptic disk on the stress distribution in an orthotropic plate*, by A. J. Owens, University of Florida.

An analysis of the stress distribution in an orthotropic plate was made from a study of a stress function of two complex variables. The results were applied to a Sitka spruce plate in a state of plane stress.

19. *The Monte Carlo method*, by Professor W. S. Snyder, University of Tennessee.

The Monte Carlo method of solving a problem consists in devising a mathematical model involving a stochastic variable whose expected value is the answer to the problem, and in evaluating the expected value by sampling. In many cases the mathematical model is at once suggested by the problem, but the method is proving adaptable to such problems as the inversion of matrices, solution of integral equations, evaluation of integrals, and determination of eigenvalues and eigenfunctions of certain partial differential equations. The difficulties of sampling are often less serious than

those encountered in attempting a solution by the customary algebraic or analytic procedures. Present experience seems too limited to characterize the class of problems where the method may be used, or to give very precise indications of its relative efficiency.

20. *A grade crossing quadratic*, by Doctor J. B. Coleman, formerly with the South Carolina Highway Department.

Where visibility is limited at a grade crossing the safe speed of approach will depend upon the distance at which visibility occurs, the time reaction of the driver, and the rate of deceleration. Finding the safe speed involves the solution of a quadratic whose coefficients have physical significance, the importance of which may not be realized by the average driver.

21. *A simple perfect point becomes a triple point*, by Professor W. R. Hutcherson, University of Florida.

Curves of order 21 determined by cubic and seventh degree surfaces were studied. These curves were invariant under a certain transformation. A simple perfect invariant point on the seventh degree surface became a triple point on the cubic curve in the tangent plane.

22. *Applications of summation by parts in the theory of summability of infinite series*, by Professor Tomlinson Fort, University of Georgia.

In this paper the author established certain tests for the summability of series by a triangular matrix analogous to well-known tests for convergence.

23. *Some remarks on the Monte Carlo method*, by Professor A. C. Cohen, Jr., University of Georgia.

A discussion of the statistical approach in the numerical evaluation of n-fold integrals was given. When n is large, for example when $n=10$, ordinary methods of numerical integration require an abnormal length of time to complete even when modern high-speed electronic computing machines are employed. By use of the Monte Carlo method, such computations can be quickly completed with sufficient accuracy for most practical purposes.

24. *Some plane curves drawn by a method suggested by a common construction of an ellipse*, by Professor W. L. Miser, Vanderbilt University.

The common method of constructing an ellipse from two concentric circles and the line $y=mx$ suggested the problem of constructing a curve which comes from replacing the two circles by two general conics. An equation of degree eight in x and y is found. Some simple examples are given, among them being the witch, the pseudo-witch, the serpentine, and other simple curves.

25. *Continuity concepts in college algebra*, by Professor J. D. Mancill, University of Alabama.

The need for the treatment of the concept of continuity of the real number system in college algebra is stressed, particularly as a prerequisite for calculus. Methods and means of presenting concepts of continuity in college algebra are demonstrated.

26. *On the diophantine problems that arise in teaching*, by Professor G. B. Huff, University of Georgia.

Most teachers of elementary mathematics are familiar with the fact that diophantine problems occur in constructing "nice" numerical problems. It was shown that for many of these problems, satisfactory solutions may be obtained by reducing the problem to one that can be solved by factoring a determinate integer.

27. *Electronic differentiating and integrating circuits and their use as teaching aids*, by M. D. Prince, Georgia Institute of Technology.

This speaker described the operation and use of an electronic device which performs differentiation and integration of periodic functions. The technique employed permits direct observation of the input and output functions by means of a standard cathode-ray oscilloscope which traces out the curves on its screen.

28. *Newton and his Principia Mathematica*, by Professor W. A. Gager, University of Florida.

In this descriptive paper attention was called to certain highlights in the *Principia* such as in Book I, laws of motion and fluxion; in II, motion of bodies in resisting media; and in III, Newton's "system of the world." Behind a great book there must exist a great personality. Many interesting phases of Newton's life were discussed.

29. *The use of group diagrams in the theory of finite groups*, by W. D. Peeples, Jr., University of Georgia.

The author considered the construction of group diagrams when the generators and defining relations are known, and he illustrated same by solving a particular problem.

30. *A study of a certain transformation in the complex plane*, by Polly A. Harris, Agnes Scott College, introduced by the Secretary.

This paper consisted of a study of the properties of the function $w=z/(z^2-1)$ when the function is considered as a transformation between the z and w planes.

31. *Kummer's test for convergence obtained by the Abel-Dini theorem*, by T. A. Newton, University of Georgia.

The Abel-Dini theorem and the Pringsheim modification were used to obtain a generalization of the comparison test of the second kind for infinite series. The result was Kummer's test for convergence.

32. *The evaluation of certain definite integrals and some applications to heat flow*, by M. D. Findley, University of Florida, introduced by the Secretary.

By contour integration in the complex plane, integrals of the type

$$\int_0^\infty \sinh ax \cos bx(\sinh mx \cosh nx)^{-1}dx$$

were evaluated. An application of the integral was given where the temperature distribution along one edge of an infinite strip was represented by B sech kx and the other edge was kept at zero.

33. *A proof of a theorem from Bromwich's "Infinite Series,"* by G. N. Wollan, North Georgia College.

A theorem stating a necessary and sufficient condition under which alternating series of a certain type can be multiplied by the Cauchy rule was proved directly from the definition of convergence.

34. *Iterated sums of powers of the binomial coefficients*, by Professor E. B. Shanks, Vanderbilt University.

For arbitrary positive integers i, k, n, p, $(i \leqq n)$, let

$$s_i^k(n, p) = \sum_{j=1}^{n} s_i^k(j, p-1) \quad \text{where} \quad s_i^k(n, 0) = \binom{n}{i}^k.$$

It is shown that

$$s_i^k(n, p) = \sum_{j=1}^{ik-i+1} A_j \binom{r+ik-i+p-j+1}{ik+p}$$

where $A_1 = 1$ and $A_j = A_{ik-j+2}$ for $j = 1, 2, \cdots, ik-i+1$. The A_j are positive integers independent of n and p, whose values are given by a determinant of order j. This result is a two parameter generalization of Bernoulli's "Summa Potestatum."

35. *A mechanical system related to a continued fraction*, by Leo Moser, University of North Carolina, by title.

The oscillations of a certain mechanical system were shown to have a close connection with the continued fraction expansion of a certain quadratic surd. The possibility of obtaining a mechanical system similar in structure to a given continued fraction was discussed.

36. *On series of Walsh functions*, by Professor W. H. Spragens, Jr., Florida State University.

Professor Spragens extended a result of Walsh on series of certain eigenfunctions which he proved to be equiconvergent with Fourier sine series for functions in L^2. It was shown that this result may be extended to functions in L. Extensions to double series were indicated.

37. *A generalization of the remainder theorem*, by Professor R. J. Levit, University of Georgia.

The familiar theorem, "If a polynomial $p(x)$ irreducible over a field K has a root in common with a polynomial $f(x)$ over K, then $p(x)$ divides $f(x)$," is a generalization of the factor theorem. In this paper a corresponding generalization of the remainder theorem was discussed.

38. *Powers of triangular numbers*, by Mr. P. A. Piza, San Juan, Puerto Rico.

This paper, privately printed in English, will appear in French in *Mathesis*. The author presented many interesting properties of the powers of triangular numbers, Fermat coefficients, and tetrahedral numbers.

39. *A new generalization of Legendre's differential equation*, by Professor R. W. Cowan, University of Florida.

The differential equation is taken in a form containing two parameters, and is solved by the method of Frobenius. For certain particular values of the parameters the general solution reduces to polynomials. It is shown that the polynomials are orthogonal over the interval $(-1, 1)$ with respect to a certain weight function.

40. *A further generalization of the Leibnitz theorem*, by Professor L. V. Robinson, University of South Carolina.

When D in $F(D)$ of the Leibnitz theorem is replaced by X defined as the sum of terms $g_k D_k$, where g_k is a function of n independent variables x_i, and D_k is the partial differential operator of x_k, it is shown that a more general theorem is thus obtained.

41. *Calculus concepts in connection with the quadratic function*, by Professor N. R. Bryan, Clemson College.

Professor Bryan suggested how continuity may be maintained in the teaching of quadratic functions in algebra and in calculus.

42. *Extended integration by parts*, by B. F. Hadnot, University of Georgia.

By repeated integration by parts, Taylor's formula and certain indefinite integrals of a given type of determinant were obtained.

43. *On Akilov's Theorem*, by Professor D. B. Goodner, Florida State University.

Akilov (*Doklady Akad. Nauk SSSR* (N.S.) 57(1947), 643–46) proved the theorem by using Kantorovich's extension theorem. This speaker presented a direct proof of Akilov's theorem and certain ramifications of the theorem. Certain spaces having property P were characterized.

44. *Discussion of classroom notes and junior college mathematics.*

This discussion was led by Professor W. A. Gager, University of Florida.

H. A. ROBINSON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December 30, 1950.

Joint Meeting with American Society for Engineering Education, Michigan State College, East Lansing, June 25–26, 1951.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHANY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May, 1951.

ILLINOIS, University of Illinois, Urbana, May 11–12, 1951.

INDIANA, May 5, 1951.

IOWA, Wartburg College, Waverly, April 20–21, 1951.

KANSAS

KENTUCKY, Eastern Kentucky State College, Richmond, April 28, 1951.

LOUISIANA-MISSISSIPPI, Mississippi State College, State College, February 16–17, 1951.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Catholic University, Washington, D. C., December 9, 1950.

METROPOLITAN NEW YORK, Spring, 1951.

MICHIGAN, Michigan State College, East Lansing, March 24, 1951.

MINNESOTA, College of St. Benedict, St. Joseph, April 28, 1951.

MISSOURI, Central College, Fayette, Spring, 1951.

NEBRASKA, University of Nebraska, Lincoln, May 5, 1951.

NORTHERN CALIFORNIA, University of San Francisco, January 27, 1951.

OHIO, Columbus, April 21, 1951.

OKLAHOMA

PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.

PHILADELPHIA

ROCKY MOUNTAIN, Colorado State College of Education, Greeley, April, 1951.

SOUTHEASTERN, Vanderbilt University and Peabody College, Nashville, Tennessee, March 16–17, 1951.

SOUTHERN CALIFORNIA, Whittier College, Whittier, March 10, 1951.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1951.

TEXAS, Southern Methodist University, Dallas, Spring, 1951.

UPPER NEW YORK STATE, Hamilton College, Clinton, May 5, 1951.

WISCONSIN, Carroll College, Waukesha, May, 1951.

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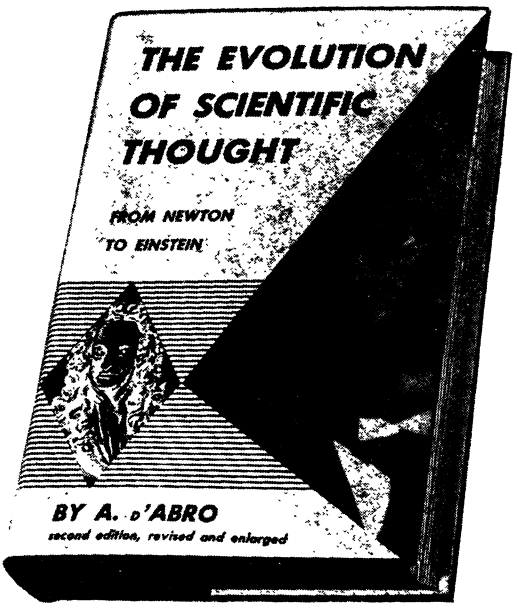
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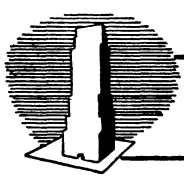
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